A representation of the String-2-Group

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2-groups and 2-group representations

2-group $\Gamma = \begin{pmatrix} \Gamma_1 \\ \downarrow\downarrow \\ \Gamma_0 \end{pmatrix}$

monoidal groupoid with all objects invertible

Examples

a) A abelian group, $\Gamma = BA := \begin{pmatrix} A \\ \downarrow\downarrow \\ * \end{pmatrix}$ „delooping"

b) G a group, $\Gamma = G_{dis} = \begin{pmatrix} G \\ \downarrow\downarrow \\ G \end{pmatrix}$ „discrete" – only identity morphisms

2-group extension:

$$1 \longrightarrow B\pi_1\Gamma \longrightarrow \Gamma \longrightarrow \pi_0\Gamma_{dis} \longrightarrow 1$$

$\pi_0\Gamma :=$ Iso classes of objects

$\pi_1\Gamma :=$ Automorphisms of unit object

$k_\Gamma \in H^3(B\pi_0\Gamma, \pi_1\Gamma)$ „k-invariant"

classification Baez-Lauda

2-groups play the rôle of groups in Higher Gauge Theory

2-groups have representations on... 2-vector spaces!

$2Vect_k := Alg_k$

bicategory of algebras, bimodules, and intertwiners

unital, assoc.

Composition is the relative tensor product of bimodules

Isomorphism = Morita equivalence

Common recognition principle for 2-vector spaces:

$$End_{2Vect_k}(\mathbb{1}) \cong Vect_k$$

$A \in 2Vect_k$

$AUT(A) := Aut_{2Vect_k}(A) =$ mon. category of invertible A-A-bimodules

2-group extension:

$$1 \longrightarrow BZ(A^\times) \longrightarrow AUT(A) \longrightarrow Pic(A)_{dis} \longrightarrow 1$$

Definition: A representation of a 2-group Γ is an algebra A and a monoidal functor $R: \Gamma \longrightarrow AUT(A)$.

$$\pi_0 R: \pi_0\Gamma \longrightarrow \pi_0 AUT(A)$$
$$\pi_0\Gamma = G \longrightarrow Pic(A)$$
$$\pi_1 R: \quad U \longrightarrow Z(A^\times) = \mathbb{C}^\times$$

We want to consider **smooth** 2-groups and **smooth** representations!

→ require that Γ_0, Γ_1 are Lie groups, all structure smooth.

→ consider smooth replacement for $AUT(A)$.

Crossed module of Lie groups:

$$A^\times \longrightarrow Aut(A) \circlearrowright A^\times$$

inner autom.

evaluation

$$AUT^\infty(A) := \begin{pmatrix} A^\times \ltimes Aut(A) \\ \downarrow\downarrow \\ Aut(A) \end{pmatrix}$$

strict Lie 2-group

$(u, \varphi) \overset{s}{\mapsto} \varphi$, $(u, \varphi) \overset{t}{\mapsto} i(u) \circ \varphi$

Every invertible A-A-bimodule is of the form A_φ, $\varphi \in Aut(A)$
$\Leftrightarrow Aut(A) \twoheadrightarrow Pic(A)$

Lemma: 1.) A Picard-surjective $\Rightarrow AUT^\infty(A) \cong AUT(A)$.

2.) Every finite-dimensional Algebra over C is Morita equivalent to a Picard-surjective one.

3.) A Picard-surjective and central simple $\Rightarrow AUT^\infty(A) = BK^\times$.

Definition: A smooth representation of a Lie 2-group Γ is an algebra A and a smooth monoidal functor $R: \Gamma \longrightarrow AUT^\infty(A)$.

The String-2-group

Universal central extension:

$$1 \longrightarrow U(1) \longrightarrow \widetilde{\Omega Spin(d)} \longrightarrow \Omega Spin(d) \longrightarrow 1$$

based loop group

+ fusion product

exists uniquely, described in terms of:
* Transgression
* Mickelsson model

composition law for:

$$String(d) := \begin{pmatrix} \widetilde{\Omega Spin(d)} \\ \downarrow\downarrow \\ P_e Spin(d) \end{pmatrix}$$

$\widetilde{\Omega Spin(d)} \xrightarrow{UCE} \Omega Spin(d) \rightrightarrows P_e Spin(d)$

splitting:

π

0

source   target

What is the string 2-group good for?

E.g., one can consider principal String(d)-2-bundles.

(e.g. Saemann et al.)

Theorem Nikolaus-KW '12
X a spin manifold.
X string $\Leftrightarrow$ Frame bundle lifts to a principal String(d)-2-bundle.

$\frac{1}{2}p_1(X) = 0$

Remarks

1.) Central 2-group extension:

$$1 \longrightarrow BU(1) \longrightarrow String(d) \longrightarrow Spin(d)_{dis} \longrightarrow 1$$

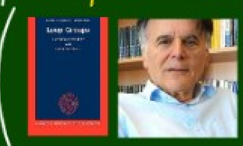
2.) Isomorphic to BCSS-model

3.) k-invariant is $1 \in H^4(BSpin(d), \mathbb{Z})$

Baez Crans Schreiber Stevenson

We want to construct a representation of the string 2-group!

⇝ use that $\widetilde{\Omega Spin(d)}$ has a well-established theory of positive energy representations



Pressley-Segal '86

Step 1: $V := \Gamma_{L^2}(\$ \otimes \mathbb{C}^d) \geq L$

real Hilbert space

odd spinor bundle of S^1

Atiyah-Patodi-Singer-Lagrangian of positive Eigenspaces of $\not{D}: V \to V$

„odd spinors on the circle"

Step 2: Fock space $\mathcal{F} := \overline{\Lambda^\infty L} \circlearrowleft Cl(V)$ Clifford algebra

irreducible module

C^*-algebra

will not be our algebra A

Step 3: $g \in O(V) \rightsquigarrow$ Bogoliubov automorphism $\theta_g \in Aut(Cl(V))$

$U \in U(\mathcal{F})$ implements g: $\boxed{\theta_g(a) = UaU^*}$ $\forall a \in Cl(V)$

Step 4: Form central extension:

$$1 \longrightarrow U(1) \longrightarrow Imp_L(V) \longrightarrow O_L(V) \longrightarrow 1$$

$Imp_L(V) \subseteq U(\mathcal{F})$

(Araki '87, Pressley-Segal '86)

„restricted" orthogonal group

Step 5: Observe:

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & Imp_L(V) \arrow[r] & O_L(V) \arrow[r] & 1 \\
& & & \Omega SO(d) \arrow[u, "\text{act point wise}"'] & \\
1 \arrow[r] & U(1) \arrow[r] \arrow[uu, equal] & \widetilde{\Omega Spin(d)} \arrow[r] \arrow[uu, "\text{This is our PER}"] & \Omega Spin(d) \arrow[r] \arrow[u, "\text{project}"'] & 1
\end{tikzcd}$$

with Peter and Matthias

A representation of the string 2-group
arxiv:2206.09797

The insidious bicategory of algebra bundles
arxiv:2204.03900

2-vector bundles
arxiv:2106.12198

with Peter

Connes fusion of spinors on loop space
arxiv:2012.08142

Smooth Fock bundles, and spinor bundles on loop space
J. Diff. Geom., to appear
arxiv:2009.00333

Fusion of implementers for spinors on the circle
Adv. Math., to appear
arxiv:1905.00222

(Associated) 2-vector bundles

What is the purpose of our representation
$R: \mathrm{String}(d) \longrightarrow \mathrm{AUT}^{cls,*}(A)$?

One can associate to a string structure a 2-vector bundle, the stringor bundle.

Composition difficult!

algebra bundles, bimodule bundles, etc.

Recall: $2\mathrm{Vect}_k := \mathrm{Alg}_k$

Thus, $2\mathrm{VectBdl}_k(X) := \mathrm{AlgBdl}_k(X)$?

not a good definition: no gluing!

Solution: stackify using plus construction.

can be generalized

Definition: A 2-vector bundle over X is an open cover $\{U_\alpha\}$ of X and:

1.) algebra bundles $\mathcal{A}_\alpha$ over U_α
2.) $\mathcal{A}_\alpha$-$\mathcal{A}_\beta$-bimodule bundles $\mathcal{M}_{\alpha\beta}$ over $U_\alpha \cap U_\beta$
3.) Intertwining isomorphisms on $U_\alpha \cap U_\beta \cap U_\gamma$
$\mathcal{M}_{\alpha\beta} \otimes_{\mathcal{A}_\beta} \mathcal{M}_{\beta\gamma} \cong \mathcal{M}_{\alpha\gamma}$

satisfying an associativity condition

Examples: 1.) Every (line) bundle gerbe is a 2-vector bundle ($A = \mathbb{C}$)
2.) Every algebra bundle is a 2-vector bundle ($\mathcal{A}_\alpha = \mathcal{A}|_{U_\alpha}$)

$\Rightarrow$ bundle gerbes and algebra bundles in one setting!

Theorem (Kristel-Ludewig-KW)

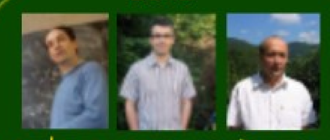
$A\text{-}2\mathrm{VectBdl}(X) \cong H^1(X, \mathrm{AUT}^\infty(A))$

fixed Morita class „rank" A

Giraud's non-abelian cohomology

use Key Lemma

2005



$\Gamma = \mathrm{AUT}^\infty(A)$

Recall: „non-abelian" Γ-bundle gerbe:

1.) an open cover $\{U_\alpha\}$
2.) principal Γ-bundles $P_{\alpha\beta}$ over $U_\alpha \cap U_\beta$
3.) bundle isomorphisms
$P_{\alpha\beta} \otimes P_{\beta\gamma} \cong P_{\alpha\gamma}$

$A^\times$-principal bundle with anchor map $P_{\alpha\beta} \longrightarrow \mathrm{Aut}(A)$

„twisted" tensor product

anchor-preserving bundle morphism

Look again at a finite-dimensional, strict Lie 2-group Γ, and a smooth representation on a finite-dimensional algebra A,
$R: \Gamma \longrightarrow \mathrm{AUT}^\infty(A)$.

$\Gamma\text{-bundle gerbe} \xmapsto{R_*} \mathrm{AUT}^\infty(A)\text{-bundle gerbe} \xmapsto{\text{associate}} \text{2-vector bundle}$

Key Lemma: There is a morphism of monoidal stacks:
Principal $\mathrm{AUT}^\infty(A)$-bundles $\longrightarrow$ A-A-bimodule bundles
$P \longmapsto (P \times A)/_{A^\times}$

twisted tensor product

relative tensor product

Generalization to von Neumann algebras:
Principal $\mathrm{Aut}^{*,cls}(A)$-bundles $\longrightarrow$ A-A-bimodule bundles

Connes fusion

2-vector bundle association generalizes!

More things left to do/check

Summary:

Γ-bundle gerbe over X	+	Continuous representation	=	von Neumann 2-vector bundle over X
$\mathcal{G}$		$R: \Gamma \to \mathrm{Aut}^{*,cls}(A)$		$\mathcal{G} \times_\Gamma A$
String structure on a string manifold		our representation of String(d)		stringor bundle

Nice analogy to spin group, spinor representations, and spinor bundles!

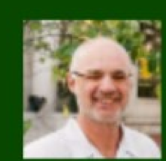
The stringor bundle

Theorem (Kristel-Ludewig-KW '22)

The transgression of the stringor bundle to the Loop space LX yields:

1.) The spinor bundle on loop space (Killingback-Witten '86, Ambler '12)
2.) Its Connes fusion product (Stolz-Teichner '06, Kristel-KW '20)

Unifies two approaches to String Geometry:
1.) Higher structure on X
2.) Geometry/analysis on LX



The „stringor" representation

Solution: complete to von Neumann algebra:

$A := \mathcal{C}\ell(V_+)'' \subseteq \mathcal{C}\ell(V)'' = B(\mathcal{F})$

type III_1 factor

type I factor

1.) $A^{op} = \mathcal{C}\ell(V_-)''$
2.) $B(\mathcal{F}) = \mathcal{C}\ell(V)'' = A \otimes A^{op} \Rightarrow \mathcal{F}$ is an A-A-bimodule, in fact, a standard form.
3.) $\mathrm{Aut}(A)$ has a nice topology (u-topology) and a canonical implementation
$i: \mathrm{Aut}(A) \longrightarrow U(\mathcal{F})$
s.t. $\theta(a) = i(\theta)\, a\, i(\theta)^*$

Haagerup '75

$$\begin{tikzcd} & U(\mathcal{F}) \ni u \circ i(\varphi) \\ \widetilde{\Omega \mathrm{Spin}}(d) \arrow[ur, "\mathrm{PER}"] \arrow[r, dashed] & U(A) \ltimes \mathrm{Aut}(A) \ni (u,\varphi) \arrow[u, "\text{top. emb.}"'] \end{tikzcd}$$

Main Lemma (Kristel-Ludewig-KW '22)
The PER factors, and results into a continuous representation
$\mathrm{String}(d) \longrightarrow \mathrm{AUT}^{*,cls}(A)$.

Side remark: $2\mathrm{Hilb} := \mathrm{vNAlg}$

bicategory, with Connes fusion as composition

$\mathrm{End}_{\mathrm{vNAlg}}(\mathbb{1}) \cong \mathrm{Hilb}$

For a representation of String(d), we need an algebra A and a group homomorphism

$P_e \mathrm{Spin}(d) \longrightarrow \mathrm{Aut}(A)$

on level of objects
$\mathrm{String}(d) \longrightarrow \mathrm{AUT}^\infty(A)$

Idea: a path is one half of a loop!

$V_\pm \subseteq V$ spinors with support on $(0,\pi)$ and $(\pi, 2\pi)$
V_+ V_-

$P_e\mathrm{Spin}(d) \xrightarrow{\text{act point-wise}} O(V_+) \xrightarrow[\theta]{\text{Bogoliubov}} \mathrm{Aut}(\mathcal{C}\ell(V_+))$

Seems as if we could choose $A := \mathcal{C}\ell(V_+)$, however...

We have to amend this to a full representation!

$$\begin{tikzcd} & & U(\mathcal{F}) \\ \left(\widetilde{\Omega \mathrm{Spin}}(d) \rightrightarrows P_e\mathrm{Spin}(d) \right) \arrow[urr, "\mathrm{PER}"] \arrow[rr, "\checkmark", "\text{Bogoliubov}"'] \arrow[d, equal] & & \left(U(A) \ltimes \mathrm{Aut}(A) \rightrightarrows \mathrm{Aut}(A) \right) \arrow[u, "?"] \arrow[d, equal] \\ \mathrm{String}(d) \arrow[rr] & & \mathrm{AUT}^{cls,*}(A) \end{tikzcd}$$

? no chance!

replace $A^\times$ by $U(A)$
replace ∞ by cls

We want to construct Fermionic string theory as a smooth, fully extended, functorial field theory!

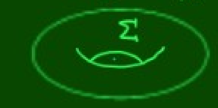
$Z: 2\mathrm{Bord}^{spin}(X) \longrightarrow 2\mathrm{VectBdl}$

Bordisms equipped with a map to a string manifold X, fibred over manifolds

2-stack of 2-vectorbundles

Bunke '10

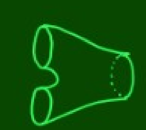
$T \times \Sigma \longrightarrow X$



$\longmapsto$ Fermionic path integral ✓

$T \times S^1 \longrightarrow X$

$\longmapsto$ Pullback of spinor bundle on loop space along $T \to LX$ ✓



$\longmapsto$ Fusion ✓

$\longmapsto$ Parallel transport (no connection yet)

$T \to X$

$\longmapsto$ Pullback of stringor bundle (guess)