Super Cartan geometry and (loop) quantum supergravity

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Section 1

Introduction

Introduction

- Want to provide mathematically rigorous approach toward **geometric supergravity**
- Aim at possible applications in the context of (Loop) quantum supergravity
- $\rightarrow$ Use and develop tools in supergeometry
- **Main issue**: pullback of superfields to the underlying spacetime manifold purely commutative (bosonic) $\rightarrow$ no fermionic d.o.f.!
- **Resolution**: Study *relative (parametrized) supermanifolds*!

In this talk

Supergeometry:

- Review: Supermanifold theory (BKL, RdW & MS)
- Relative supermanifolds
- Super connections and their parallel transport

Supergravity:

- Supergravity via super Cartan geometry
- 1-parameter family generalization of MacDowell-Mansouri SUGRA actions
- Extension to $\mathcal{N} > 1$!
- Boundary theory
- Special properties of self-dual theory

Section 2

(Relative) Supermanifolds

Supermanifolds

Various different approaches:

- **Berezin-Kostant-Leites**(algebro-geometric): "Definition via function sheaves". But: what are the points?
- **Rogers-DeWitt**(concrete): "Start with topological space of points". But: Too many points, ambiguities
- **Molotkov** '84 and **Sachse** '08(categorial): Shows BKL and RdW are two sides of the same coin.

algebro-geom.: Observation: Structure of a smooth manifold M completely encoded in a suitable ring of functions on it
$\rightarrow$ describe M as locally ringed space $(|M|, \mathcal{O}_M)$ s.t.

- $|M|$ paracompact topological Hausdorff space
- $\mathcal{O}_M$ abstract sheaf of local rings on $|M|$
- M is locally Euclidean $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$

Supermanifolds

Definition: Supermanifold [Carmeli+Fioresi+Caston '11]

Supermanifold $\mathcal{M}$ is a locally ringed space $(M, \mathcal{O}_\mathcal{M})$ s.t.

- M paracompact topological Hausdorff space
- $\mathcal{O}_\mathcal{M}$ abstract sheaf of local super rings on M
- $\mathcal{M}$ locally looks like flat superspace $\mathbb{R}^{m|n} = (\mathbb{R}^m, C^\infty_{\mathbb{R}^m} \otimes \bigwedge \mathbb{R}^n)$

- $\rightarrow$ locally, functions of the form:

$$f(x,\theta) = f_0(x) + f_i(x)\theta^i + \ldots + f_n(x)\theta^1 \cdots \theta^n$$

- $\mathcal{J} := \mathcal{O}_{\underline{1}} + \langle \mathcal{O}_{\underline{1}}^2 \rangle$ (nilpotent sub ideal) $\rightarrow \mathcal{M}_{\underline{0}} := (M, \mathcal{O}_\mathcal{M}/\mathcal{J})$ ordinary manifold (body)
- **Super Lie groups** as group objects in this category **SMan**.

Supermanifolds

Supermanifold $\mathcal{M}$ yields *functor of points* $\mathcal{M} : \mathbf{SMan}^{\mathrm{op}} \to \mathbf{Set}$

$$\mathcal{T} \mapsto \mathcal{M}(\mathcal{T}) := \mathrm{Hom}_{\mathbf{SMan}}(\mathcal{T}, \mathcal{M}) \quad (\mathcal{T}\text{-point})$$
$$(f : \mathcal{T} \to \mathcal{T}') \mapsto (\mathcal{M}(f) : g \mapsto g \circ f)$$

- Restrict to *superpoints* $\mathcal{T} \cong (\{*\}, \Lambda)$ ($\to$ Grassmann algebras)

$$\mathcal{M}(\Lambda) \cong \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{M}), \Lambda)$$

- Contains *real spectrum* $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) = \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{M}), \mathbb{R})$
- Topological space via *Zariski* or *Gelfand topology*
- Equip $\mathcal{M}(\Lambda)$ with coarsest topology s.t. $\mathcal{M}(\Lambda) \to \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ is continuous $\to$ **DeWitt-topology**

Supermanifolds

- $\mathcal{M}(\Lambda)$ structure of topological manifold →*Rogers-DeWitt supermanifold*
- ⇒ yields functor
$$\mathbf{Gr} \to \mathbf{Top},\ \Lambda \mapsto \mathcal{M}(\Lambda)$$
- → *Supermanifold in the sense of Molotkov and Sachse*
- Atarting point for the construction of **infinite-dimensional supermanifolds** requiring $\mathcal{M}(\Lambda)$ to be a Banach [M '84, S '08] or Fréchet supermanifold [Schütt '19]
- → Groups of super diffeomorphisms and supersymmetry transformations

Fermionic d.o.f. and parametrization

- **General issue:** pullback of superfields to the body M of a supermanifold $\mathcal{M}$ are purely commutative (bosonic)

$$f|_{\theta=0} \equiv f_0 \in C^\infty(M)$$

$\rightarrow$ **no fermionic degrees of freedom** on the underlying spacetime manifold!

- Resolution: Add **parametrization supermanifold** $\mathcal{S}$ and consider parametrized superfields

$$f : \mathcal{S} \times \mathcal{M} \rightarrow \mathbb{R}^{1|1}$$

- $f|_M \in C^\infty(M) \otimes \mathcal{O}(\mathcal{S}) \rightarrow$ fermionic d.o.f. enocoded in parametrzation
- $\mathcal{S}$ *a priori* completely arbitrary $\rightarrow$ have to require that physical fields transform covariantly under change of parametrization $\lambda : \mathcal{S} \rightarrow \mathcal{S}'$

Relative supermanifolds

[Deligne '99, Hack et al '15, Keßler+Jost+Tolksdorf '17, K+Sheshmani+Yau '20, KE '20+'21]

Definition: Relative supermanifold

Category $\mathbf{SMan}_{/\mathcal{S}}$ of $\mathcal{S}$-*relative supermanifolds* with tuples $\mathcal{M}_{/\mathcal{S}} := (\mathcal{S} \times \mathcal{M}, \mathrm{pr}_{\mathcal{S}})$ as objects and morphisms $\phi : \mathcal{M}_{/\mathcal{S}} \to \mathcal{N}_{/\mathcal{S}}$ s.t.

$$\begin{array}{ccc} \mathcal{S} \times \mathcal{M} & \xrightarrow{\phi} & \mathcal{S} \times \mathcal{N} \\ & {\scriptstyle \mathrm{pr}_{\mathcal{S}}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{\mathcal{S}}} & \\ & \mathcal{S} & \end{array}$$

commutes.

- Morphism $\lambda : \mathcal{S} \to \mathcal{S}'$ (*change of parametrization*) induces map

$$\lambda^* : \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}'}}(\mathcal{M}_{/\mathcal{S}'}, \mathcal{N}_{/\mathcal{S}'}) \to \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}}}(\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}})$$

commuting with compositions $\lambda^*(\phi \circ \psi) = \lambda^*(\phi) \circ \lambda^*(\psi)$

Relative super fiber bundles

Definition: Relative principal super fiber bundle

$$\begin{array}{ccc} \mathcal{P}_{/\mathcal{S}} & \longleftarrow & \mathcal{G} \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{M}_{/\mathcal{S}} & & \end{array}$$

- projection: $\pi : \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ surjective morphism of $\mathcal{S}$-relative supermanifolds
- $\mathcal{G}$-right action: $\Phi : \mathcal{P}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{P}_{/\mathcal{S}}$ satisfying $\mathrm{pr}_{\mathcal{S}} \circ \Phi = \mathrm{pr}_{\mathcal{S}} \times \mathrm{id}_{\mathcal{G}}$ and
  1. $\pi \circ \Phi = \pi$
  2. $\Phi \circ (\Phi \times \mathrm{id}) = \Phi \circ (\mathrm{id} \times \mu_{\mathcal{G}})$

Relative super connections

Definition: Relative super connection 1-form [KE '20+'21]

An even $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_{\underline{0}}$ on a $\mathcal{S}$-relative principal bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \xrightarrow{\pi} \mathcal{M}_{/\mathcal{S}}$ is called a *super connection 1-form* if

1. $\langle \widetilde{X} | \mathcal{A} \rangle = X$, $\forall X \in \mathfrak{g} \quad (\widetilde{X} := \mathbb{1} \otimes X \circ \Phi^*)$
2. $\Phi_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ and $L_{\widetilde{X}} \omega = -\mathrm{ad}_X \circ \omega \quad (g \in |\mathcal{G}|, X \in \mathfrak{g})$

Here:

$\Phi_g : \mathcal{P}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$ isomorphism of $\mathcal{S}$-relative supermanifolds induced by the pullback morphism $\Phi_g^\sharp := (\mathbb{1} \otimes \mathrm{ev}_g) \circ \Phi^\sharp$

Super parallel transport

$\rightarrow$ Super connection 1-form $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}})$ induces **parallel transport map** along parametrized paths $\gamma : \mathcal{S} \times [0,1] \rightarrow \mathcal{M}$

$$\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma} : \Gamma(\gamma_0^*\mathcal{P}) \rightarrow \Gamma(\gamma_1^*\mathcal{P})$$

Proposition [KE '20+'21]

$\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}$ natural under reparametrization:

$$\begin{array}{ccc} \Gamma(f^*\mathcal{P}) & \xrightarrow{\mathscr{P}^{\mathcal{A}}_{\mathcal{S}',\gamma}} & \Gamma(g^*\mathcal{P}) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ \Gamma((f \circ \lambda)^*\mathcal{P}) & \xrightarrow{\mathscr{P}^{\lambda^*\mathcal{A}}_{\mathcal{S},\lambda^*\gamma}} & \Gamma((g \circ \lambda)^*\mathcal{P}) \end{array}$$

for any parametrized path $\gamma : f \rightarrow g$ on $\mathcal{M}_{/\mathcal{S}}$

Super parallel transport

Super Wilson loop observable

$$W_\gamma[\mathcal{A}] = \mathrm{str}\left(g_\gamma[\omega]\cdot\mathcal{P}\exp\left(-\oint_\gamma \mathrm{Ad}_{g_\gamma[\omega]^{-1}}\psi^{(\tilde{s})}\right)\right) : \mathcal{S}\to\mathcal{G}$$

- $\mathcal{A} = \omega + \psi$ and $\gamma : [0,1] \to M \subset \mathcal{M}$
- $g_\gamma[\omega]$: parallel transport map induced by ω

Proposition

- $W_\gamma[\mathcal{A}] : \mathcal{S}\to\mathcal{G}$ element in $\mathcal{G}(\mathcal{S})$ ($\mathcal{S}$-point of $\mathcal{G}$)
- natural under reparametrization $\lambda^* W_\gamma[\mathcal{A}] = W_\gamma[\lambda^*\mathcal{A}]$
- invariant under gauge transformations

for $\mathcal{S} = \{*\} \to W_\gamma[\mathcal{A}] \equiv W_\gamma[\omega]$

Universal parametrization and pAQFT

Ex: Consider $\mathcal{S}$-parametrized field theory described by a gauge field $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_{\underline{0}}$

- $\to$ configuration space: $\mathcal{C}_{\mathcal{S}} = \Omega^1(\mathcal{M}_{/\mathcal{S}}, \mathrm{Ad}(\mathcal{P}_{/\mathcal{S}}))_{\underline{0}}$

$$\mathcal{C}_{\mathcal{S}} \cong (F \otimes \mathcal{O}(\mathcal{S}))_{\underline{0}}$$

  $F := \Omega^1(\mathcal{M}, \mathcal{P} \times_{\mathrm{Ad}} \mathfrak{g})$: ∞-dimensional super vector space

- Change of parametrization $\lambda : \mathcal{S} \to \mathcal{S}'$ induces pullback

$$\lambda^* : \mathcal{C}_{\mathcal{S}'} \to \mathcal{C}_{\mathcal{S}}$$

Q: Does there exist some *universal parametrization* $\mathbb{S}$ such that any $\Phi \in \mathcal{C}_{\mathcal{S}}$ with $\mathcal{S}$ finite can be obtained via pullback? [Schmitt '97, KE '21]

Universal parametrization and pAQFT

Regard $\mathcal{S}$ as Molotkov-Sachse supermanifold $\mathcal{S}: \mathbf{Gr} \to \mathbf{Top}$

- for $\Phi \in \mathcal{C}_{\mathcal{S}}$ and $s \in \mathcal{S}(\Lambda)$: $\Phi(s) \in \mathbb{S}(\Lambda) := (F \otimes \Lambda)_{\underline{0}}$

$$\Rightarrow \mathcal{C}_{\mathcal{S}} = (\mathcal{SC}^{\infty}(\mathcal{S}) \otimes \mathbb{S})_{\underline{0}} \cong \mathcal{SC}^{\infty}(\mathcal{S}, \mathbb{S})$$

- can identify $\Phi \in \mathcal{C}_{\mathcal{S}}$ with morphism $\mu_{\Phi} : \mathcal{S} \to \mathbb{S}$ (*classifying morphism*)
- $\to$ $\Phi = \mu_{\Phi}^{*}(x_{\mathbb{S}})$ with $x_{\mathbb{S}} := \mathrm{id}_{\mathbb{S}}$ (*fundamental coordinate*)
- for $p \in M$ yields evaluation functional

$$\Psi_{\mu}^{\alpha}(p) := \mathrm{pr}_{\alpha} \circ \langle \partial_{\mu}|_{p} | x_{\mathbb{S}}(\cdot) \rangle : F_{\underline{1}} \to \mathbb{R},\ \psi \mapsto \psi_{\mu}^{\alpha}(p)$$

- $\to$ fermionic fields in pAQFT! [Rejzner '11]

Section 3

Gravity as Cartan geometry

Klein geometry

F. Klein: "Classify geometry of space via group symmetries".

Example: Minkowski spacetime $\mathbb{M} = (\mathbb{R}^{1,3}, \eta)$

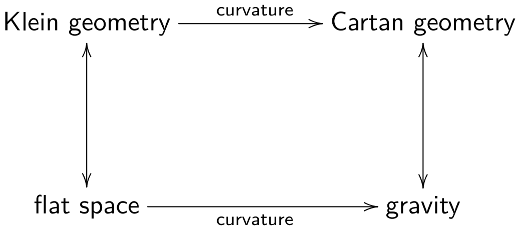
- isometry group $\mathrm{ISO}(\mathbb{R}^{1,3}) = \mathbb{R}^{1,3} \ltimes \mathrm{SO}_0(1,3)$
- event $p \in \mathbb{M}$: $G_p = \mathrm{SO}_0(1,3)$ (isotropy subgroup)

$$\mathrm{ISO}(\mathbb{R}^{1,3})/\mathrm{SO}_0(1,3) \cong \mathbb{M}$$

Definition

A *Klein geometry* is a pair (G, H) where G is a Lie group and $H \subseteq G$ a closed subgroup such that G/H is connected.

Cartan geometry



Cartan geometry

Definition: Cartan geometry

A **Cartan geometry** modeled on a Klein geometry (G, H) is a principal H-bundle

$$\begin{array}{ccc} P & \xleftarrow{r} & H \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

together with a **Cartan connection** $A \in \Omega^1(P, \mathfrak{g})$ s.t.

1. $\mathrm{pr}_{\mathfrak{h}} \circ A$ defines ordinary gauge field
2. **Cartan condition (CC)**: $A: \ T_pP \to \mathfrak{g}$ isomorphism $\forall p \in P$

If (CC) not satisfied $\to$ **generalized Cartan connection**

Cartan geometry

Theorem

$$\textbf{\textit{gen. Cartan}} \textit{ on } P \stackrel{1:1}{\longleftrightarrow} \textbf{\textit{Ehresmann}} \textit{ on } P[G] := P \times_H G$$

$$\widehat{A} \circ \pi_*(X_p, Y_g) = \mathrm{Ad}_{g^{-1}} \langle X_p | A_p \rangle + \langle Y_g | \theta_{\mathrm{MC}} \rangle$$

with $\pi : P \times G \to P[G]$.

Cartan geometry

Theorem

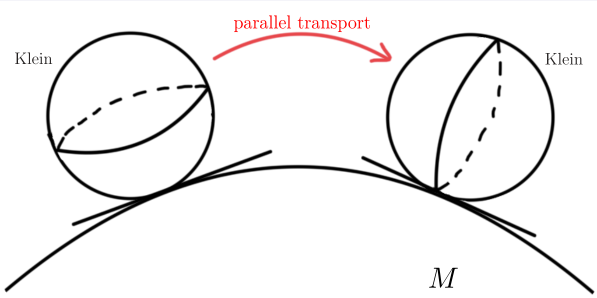
$$\textit{\textbf{gen. Cartan}} \text{ on } P \overset{1:1}{\longleftrightarrow} \textit{\textbf{Ehresmann}} \text{ on } P[G] := P \times_H G$$

$$\widehat{A} \circ \pi_*(X_p, Y_g) = \mathrm{Ad}_{g^{-1}} \langle X_p | A_p \rangle + \langle Y_g | \theta_{\mathrm{MC}} \rangle$$

with $\pi : P \times G \to P[G]$.

Parallel transport

$\Rightarrow$ $\widehat{A}$ induces parallel transport on $P[G] \times_G G/H$



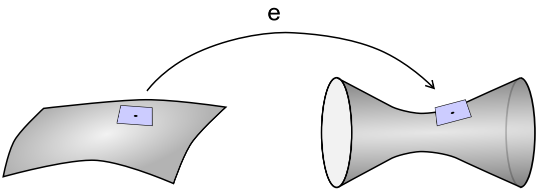
Gravity as Cartan geometry

Example: Cartan geometry modeled on **AdS$_4$** $(\mathrm{SO}(2,3), \mathrm{SO}_0(1,3))$

Cartan connection

$$A = \mathrm{pr}_{\mathbb{R}^{1,3}} \circ A + \mathrm{pr}_{\mathfrak{so}(1,3)} \circ A =: e + \omega$$

- ω: **Lorentz-connection**, e: **soldering form** (co-frame)



Gravity as Cartan geometry

$\rightarrow$ Can use A to formulate action principle

Here: AdS_4: $\mathfrak{so}(2,3) \cong \mathfrak{sp}(4) \supset \mathfrak{so}(1,3)$, basis $M_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J]$

Definition

$$\mathcal{P}_\infty := \frac{i}{2}\gamma_5 : \mathfrak{so}(1,3) \rightarrow \mathfrak{so}(1,3)$$

$\rightarrow$ yields inner product on $\mathfrak{sp}(4)$:

$$\langle \cdot, \cdot \rangle_\infty := \mathrm{tr}(\cdot \, \mathbf{0} \oplus \mathcal{P}_\infty \, \cdot)$$

MacDowell-Mansouri action

$$S_{\mathrm{MM}}[A] = \int_M \langle F[A] \wedge F[A] \rangle_\infty = S_{\mathrm{grav}} + \textit{boundary term}$$

$F[A] = \mathrm{d}A + \frac{1}{2}[A \wedge A]$: Cartan curvature

Holst action for Einstein gravity

Holst action S^{β}_{Holst}: 1-parameter family of deformed actions of first-order Einstein gravity

Holst action [Holst '96]

$$S^{\beta}_{\text{Holst}} = \frac{1}{4\kappa}\int_M \epsilon_{IJKL} e^I \wedge e^J \wedge F[\omega]^{KL} + \frac{2}{\beta} e^I \wedge e^J \wedge F[\omega]_{IJ}, \quad \beta: \text{ Immirzi}$$

$I, J, \ldots$: $\mathrm{SO}(1,3)$

- Yield **same** EOM as Einstein gravity!
- In canonical formulation $\rightarrow$ yields canonical phase space described in terms of $\mathrm{SU}(2)$**-gauge field** $A^{\beta} = \Gamma + \beta K$ and canonically conjugate momentum (*Ashtekar-Barbero variables*)
- Starting point for canonical quantization program inspired by lattice gauge theory $\rightarrow$ Loop Quantum Gravity (LQG)

Holst as Mac-Dowell-Mansouri

Introduce β-deformed inner product!

Definition

$$\mathcal{P}_\beta := \frac{\mathbb{1} + i\beta\gamma_5}{2\beta} : \mathfrak{so}(1,3) \to \mathfrak{so}(1,3), \quad \beta : \textit{Immirzi}$$

$\to$ yields inner product on $\mathfrak{sp}(4)$:

$$\langle \cdot, \cdot \rangle_\beta := \mathrm{tr}(\,\cdot\, \mathbf{0} \oplus \mathcal{P}_\beta\, \cdot\,)$$

Holst-MacDowell-Mansouri action [Wise '10, KE '21]

$$S_{\text{H-MM}}[A] = \int_M \langle F[A] \wedge F[A] \rangle_\beta = S^\beta_{\text{Holst}} + \textit{boundary term}$$

Section 4

Super Cartan geometry

Super Cartan geometry

Definition: Super Cartan geometry [KE '20+'21]

A **super Cartan geometry** modeled on a super Klein geometry $(\mathcal{G},\mathcal{H})$ is a $\mathcal{S}$-relative principal super fiber bundle

$$\begin{array}{ccc} \mathcal{P}_{/\mathcal{S}} & \longleftarrow & \mathcal{H} \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{M}_{/\mathcal{S}} & & \end{array}$$

together with a **super Cartan connection** $\mathcal{A}\in\Omega^1(\mathcal{P}_{/\mathcal{S}},\mathfrak{g})_{\underline{0}}$ s.t.

1. $\mathrm{pr}_{\mathfrak{h}}\circ\mathcal{A}$ defines super connection 1-form
2. **Super Cartan condition** (SCC): $\iota_{\mathcal{P}}^*\mathcal{A}_p:\ T_p\mathcal{P}\rightarrow\mathfrak{g}$ isomorphism $\forall p\in|\mathcal{P}|$ and embeddings $\iota_{\mathcal{P}}:\ \mathcal{P}\rightarrow\mathcal{S}\times\mathcal{P}$

Super Cartan geometry

Definition

A super Cartan geometry modeled on a super Klein geometry $(\mathcal{G},\mathcal{H})$ is called

1. **reductive** if $\mathfrak{g}$ admits a decomposition of the form $\mathfrak{g}=\mathfrak{g}/\mathfrak{h}\oplus\mathfrak{h}$ such that $\mathfrak{g}/\mathfrak{h}$ is invariant w.r.t. the Adjoint action of $\mathcal{H}$.
2. **metric** if it is reductive and $\mathfrak{g}/\mathfrak{h}$ admits a smooth super metric $\mathscr{S}$ that is invariant w.r.t. the Adjoint action of $\mathcal{H}$.

By (I) and (SCC): E yields isomorphism

$$E:\,\mathfrak{X}(\mathcal{M})\xrightarrow{\sim}\Gamma(\mathcal{P}_{/\mathcal{S}}\times_{\mathrm{Ad}}\mathfrak{g}/\mathfrak{h})$$

$\rightarrow$ *super soldering form* (supervielbein)

Super Cartan geometry

- by (II) $\rightarrow$ can use E to lift $\mathscr{S}$ to a super metric g on $\mathcal{M}$

induced super metric

$$g(X,Y) := \mathscr{S}(E(X^*), E(Y^*))$$

X^*, Y^*: horizontal lifts

- $\rightarrow$ $(\mathcal{M}, g)$ defines super Riemannian manifold

Global symmetries: Killing vector fields $L_X g = 0$

- due to

$$E:\ \mathfrak{X}(\mathcal{M})_{\underline{1}} \xrightarrow{\sim} \Gamma(\mathcal{P} \times_{\mathrm{Ad}} \mathfrak{g}_{\underline{1}})$$

  can identify **odd** Killing vector fields X with spinors $\epsilon := \langle X|E \rangle$ ($\rightarrow$ *Killing spinors*)

Section 5

Supergravity & boundary theory

Supergravity

AdS (Holst-)Supergravity as super Cartan geometry modeled on $(\mathrm{OSp}(\mathcal{N}|4), \mathrm{Spin}^+(1,3))$ [D'Auria-Fré-Regge '80, D'Auria-Fré '80, Castellani-D'Auria-Fré '91, KE '21+'22]

Super Cartan connection ($\mathcal{N}=1$)

$$\mathcal{A}: \mathfrak{X}(\mathcal{P}) \to \mathcal{O}(\mathcal{P}) \otimes \mathfrak{osp}(1|4)$$

Decomposition

$$\mathcal{A} = \underbrace{\mathrm{pr}_{\mathfrak{g}_1} \circ \mathcal{A}}_{\psi} + \underbrace{\mathrm{pr}_{\mathbb{R}^{1,3}} \circ \mathcal{A}}_{e} + \underbrace{\mathrm{pr}_{\mathfrak{spin}(1,3)} \circ \mathcal{A}}_{\omega}$$

- e: co-frame ($\to$ metric)
- ω: spin connection
- ψ: (spin-3/2) Rarita-Schwinger field

Supergravity and LQG

Introduce deformed inner product on $\mathfrak{osp}(1|4)$ [KE+HS '21]:

- $\mathfrak{osp}(1|4) \cong \mathfrak{sp}(4) \oplus S_{\mathbb{R}}$
- $\mathfrak{sp}(4)$, $S_{\mathbb{R}}$ naturally carry structure of Clifford modules

Definition [KE '21]

$$\mathbf{P}_\beta := \mathbf{0} \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta : \mathfrak{osp}(1|4) \to \mathfrak{osp}(1|4)$$

with

$$\mathcal{P}_\beta := \frac{\mathbb{1} + i\beta\gamma_5}{2\beta} \qquad \beta : \textit{Immirzi}$$

$\to$ induces inner product on $\mathfrak{osp}(1|4)$:

$$\langle \cdot, \cdot \rangle_\beta := \mathrm{str}(\,\cdot\,\mathbf{P}_\beta\,\cdot\,)$$

Supergravity and LQG

Super Holst-MacDowell-Mansouri action [KE+HS '21]

$$S_{\text{sH-MM}}[\mathcal{A}] = \int_{\mathcal{M}_{\underline{0}}} \iota^* \langle F[\mathcal{A}] \wedge F[\mathcal{A}] \rangle_\beta$$

$\iota : \mathcal{M}_{\underline{0}} \hookrightarrow \mathcal{M}$ (embedding)

Curvature

- $F[\mathcal{A}]^I = \Theta^{(\omega)I} - \frac{1}{4}\bar{\psi} \wedge \gamma^I \psi$
- $F[\mathcal{A}]^{IJ} = F[\omega]^{IJ} + \frac{1}{L^2}\Sigma^{IJ} - \frac{1}{4L}\bar{\psi} \wedge \gamma^{IJ}\psi$
- $F[\mathcal{A}]^\alpha = D^{(\omega)}\psi^\alpha - \frac{1}{2L}e^I(\gamma_I)^\alpha{}_\beta \wedge \psi^\beta$

- $\rightarrow$ Yields Holst action of $D = 4$, $\mathcal{N} = 1$ AdS-SUGRA + *bdy terms*

Boundary theory

Q: What is special about *bdy terms* contained in sH-MM-action?

Boundary action: Most general ansatz

$$\mathscr{L}_{\text{bdy}} = C_1 F(\omega)^{IJ} \wedge F(\omega)^{KL} \epsilon_{IJKL} + C_2 F(\omega)^{IJ} \wedge F(\omega)_{IJ} + C_3 \mathrm{d}(\bar{\psi} \wedge \gamma_* D^{(\omega)} \psi) + C_4 \mathrm{d}(\bar{\psi} \wedge D^{(\omega)} \psi)$$

- C_1, C_2, C_3, C_4 arbitrary constant coefficients

- $\rightarrow$ Require that SUSY of the full theory $\mathscr{L}_{\text{full}} = \mathscr{L}_{\text{bulk}} + \mathscr{L}_{\text{bdy}}$ is preserved at boundary [Andrianopoli+D'Auria '14, A+D'A+Cerchiai +Trigiante '18]

Boundary theory

- **Castellani-D'Auria-Fré**: SUSY-trafos as **infinitesimal superdiffeomorphisms** X such that $\iota_X e^I = 0$

$\Rightarrow$ Imposes boundary condition

$$\iota_X \mathscr{L}_{\text{full}}|_{\partial M} = 0$$

- $\rightarrow$ Turns out that coefficients C_i **uniquely** fixed by this requirement!

Boundary action [KE+HS '21]

$$S_{\text{bdy}}(\mathcal{A}) = \frac{L^2}{\kappa}\int_{\partial\mathcal{M}_{\underline{0}}} \left\langle \omega \wedge \mathrm{d}\omega + \frac{1}{3}\omega \wedge [\omega \wedge \omega]\right\rangle_{\beta} + \langle \psi \wedge D^{(\omega)}\psi\rangle_{\beta}$$

- $\rightarrow$ **Exactly** reproduces boundary term in sH-MM action!

Extended SUGRA and boundary theory

What about **extended** SUSY? $\rightarrow$ Consider $\mathcal{N} = 2$ [KE+HS '21]

Super Cartan connection

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \Psi^\alpha_r Q^r_\alpha + \hat{A} T$$

- $r = 1, 2$: R-symmetry index ($\rightarrow$ gauge group $\mathrm{SO}(2) \cong \mathrm{U}(1)$)
- $\hat{A}$: $\mathrm{U}(1)$ gauge field (*graviphoton*)

- make most general ansatz of boundary term compatible with local SUSY
- $\rightarrow$ full action $S_{\text{full}}(\mathcal{A})$ acquires MacDowell-Mansouri form! [Andrianopoli et al. '14+'21, KE+HS '21]

Extended SUGRA and boundary theory

super Holst-MacDowell-Mansouri action ($\mathcal{N}=2$) [KE+HS '21]

$$S^{\mathcal{N}=2}_{\text{sH-MM}}[\mathcal{A}] = \int_{\mathcal{M}_{\underline{0}}} \iota^*\mathrm{str}(F[\mathcal{A}] \wedge \mathbf{P}_\beta F[\mathcal{A}])$$

Definition

$$\mathbf{P}_\beta : \Omega^2(M, \mathfrak{osp}(2|4)) \to \Omega^2(M, \mathfrak{osp}(2|4))$$

$$\mathbf{P}_\beta := \mathbf{0} \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta \oplus \frac{1}{2\beta}(1+\beta\star)$$

$\to$ yields θ-topological term in $\mathrm{U}(1)$-sector

$\to$ β literally has interpretation as θ-**ambiguity**!

Outlook: Embeddings & PCOs (WIP)

Super Holst-MacDowell-Mansouri action[KE+HS '21]

$$S^{\mathcal{N}}_{\text{sH-MM}}[\mathcal{A}] = \int_{\mathcal{M}_{\underline{0}}} \iota^* \mathrm{str}(F[\mathcal{A}] \wedge \mathbf{P}_\beta F[\mathcal{A}])$$

- In relative category: Embedding $\iota : \mathcal{M}_0 \hookrightarrow \mathcal{M}$ **not unique**! $(\iota_{\mathrm{can}} := \mathrm{id}_{\mathcal{S}} \times \iota_{\underline{0}}, \ldots)$

Q: Does SH-MM depend on the choice of embedding?

Outlook: Embeddings & PCOs (WIP)

Idea: Follow approach of [Grassi et al '16]and formulate action principle directly on $\mathcal{M}$ → Picture Changing Operators (PCO)

Super Holst-MacDowell-Mansouri action

$$S^{\mathcal{N}}_{\text{sH-MM}}[\mathcal{A}] = \int_{\mathcal{M}} \mathrm{str}(F[\mathcal{A}] \wedge \mathbf{P}_{\beta} F[\mathcal{A}]) \cdot \mathcal{Y}$$

- Want to interpret PCO $\mathcal{Y}$ as sections of Spencer cohomology $H^0_{\mathrm{Sp}}(\mathcal{M})$ associated to the **Spencer complex**

$$\Sigma^{4-\bullet}_{\mathrm{Sp}}(\mathcal{M}) := \mathrm{Hom}_{\mathcal{O}_S}(\Omega^{\bullet}(\mathcal{M}), \mathrm{Ber}(\mathcal{M}))$$

- Relation to PCO used in physics literature?
- What is the relation $\iota \leftrightarrow \mathcal{Y}$?
- Action independent of $\mathcal{Y}$?
- ⇒ **Solve questions on cohomological level!** (WIP) →[Huerta + Noja + KE '23]

Section 6

Chiral supergravity

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Chiral Theory

In general: $\mathbf{P}_\beta$ destroys manifest SUSY-invariance!

$\rightarrow$ This changes in **chiral theory** $(\beta = \mp i)$

Holst projection [KE+HS '21]

$$\begin{aligned}\mathbf{P}_{-i} : \mathfrak{osp}(\mathcal{N}|4)_{\mathbb{C}} &\rightarrow \mathfrak{osp}(\mathcal{N}|2)_{\mathbb{C}} \\ M_{IJ} &\mapsto T_i^+ = \frac{1}{2}(J_i + iK_i) \\ Q_\alpha &\mapsto Q_A\end{aligned}$$

Q_A: left-handed Weyl-fermion

Chiral Theory

Super Ashtekar connection [KE+HS '21]

$$\mathcal{A}^+ := \mathbf{P}_{-i}\mathcal{A} = A^{+i}T_i^+ + \psi_r^A Q_A^r + \frac{1}{2}\hat{A}_{rs}T^{rs}$$

Chiral action

$$S_{\text{sH-MM}}^{\beta=-i} = \frac{i}{\kappa}\int_M \langle \mathcal{E}\wedge F(\mathcal{A}^+)\rangle + \frac{1}{4L^2}\langle \mathcal{E}\wedge\mathcal{E}\rangle + \underbrace{S_{\text{CS}}^{\mathrm{OSp}(\mathcal{N}|2)}(\mathcal{A}^+)}_{\text{boundary term}}$$

$\mathcal{E}$: *super electric field*

SUSY-invariance: Coupling bulk $\leftrightarrow$ boundary

$$\underset{\Longleftarrow}{F(\mathcal{A}^+)} = -\frac{1}{2L^2}\underset{\Longleftarrow}{\mathcal{E}}$$

Super Ashtekar connection

$\mathcal{A}^+$ natural candidate to quantize SUGRA à la LQG

- Contains both gravity and matter d.o.f. $\rightarrow$ unified description, more fundamental way of quantizing fermions
- Substantially simplifies constraints
- **Boundary theory** described by **super Chern-Simons theory** $\rightarrow$ natural candidate to study inner boundaries in LQG ($\rightarrow$ BPS states, black holes)
- $\leftrightarrow$ Boundary theories in string theory [Mikhaylov + Witten '14]
- **Issues**: Non-compactness gauge group, reality conditions

Section 7

Outlook: SUSY BHs in LQG

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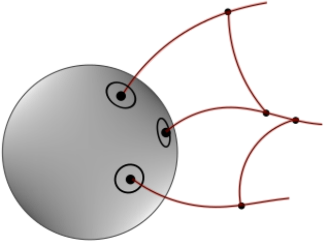
SUSY black holes in LQG

Holst-MM in the presence of inner boundary: [KE '22, KE+HS '22]



- Geometric theory induces super CS on inner boundary
- Gauge group: $G = \mathrm{OSp}(\mathcal{N}|2)$

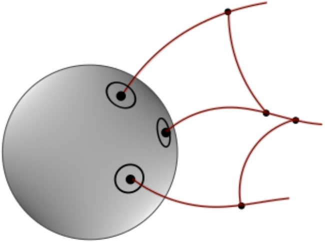
Hilbert space full theory

$$\mathcal{H}_{\gamma}^{\mathrm{full}} = \mathcal{H}_{\gamma}^{\mathrm{bulk}} \otimes \mathcal{H}_{\gamma}^{\mathrm{CS}}$$

Quantum boundary condition

$$\mathbb{1} \otimes \widehat{F}_{\underline{A}}(p) = -\frac{2\pi i}{\kappa k}\widehat{\mathcal{E}}_{\underline{A}}(p) \otimes \mathbb{1}$$

SUSY black holes in LQG



- For $\mathcal{N} = 1$: $G = \mathrm{OSp}(1|2)$
- **Issue:** G complex/non-compact $\rightarrow$ adapt strategy of [Perez et al '14, Noui et al '15]
- Consider compactification $\mathrm{UOSp}(1|2)$ and analytically continue $j \rightarrow -\frac{1}{4} + is$

$\mathrm{UOSp}(1|2)$ state counting

$$N = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, \sin^2(2\theta)\left[4 - n + \sum_{i=1}^{p} n_i d_{j_i} \frac{\tan(d_{j_i}\theta)}{\tan\theta}\right]\prod_{l=1}^{p}\left(\frac{\cos(d_{j_l}\theta)}{\cos\theta}\right)^{n_l}$$

$\rightarrow$ *analytic continuation*

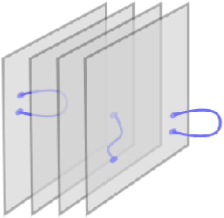
Entropy

$$S = \ln N = \frac{a}{4} + \ldots \quad \rightarrow \text{[Bekenstein '73, Hawking '75]}$$

Outlook

Outlook:

- Generalization to $\mathcal{N} \geq 3$ (*in particular:* $\mathcal{N} = 4, 8$)
- Limit $L \to \infty$ (vanishing cosmological constant) [Concha+Ravera+Rodríguez '19]
- (Charged) supersymmetric BHs $\leftrightarrow$ entropy calculations in string theory [Strominger+Vafa '96, Cardoso et al. '96, KE+HS '22]
- $\leftrightarrow$ Boundaries in string theory and **OSp**-super Chern-Simons theory [Mikhaylov+Witten '14]



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Supersymmetry & Rheomomy

Definition: Local symmetry

An infinitesimal automorphism X on $\mathcal{P}_{/\mathcal{S}}$ is called a *local symmetry* iff

$$\delta_X \mathscr{L}|_P := \iota_P^*(L_X \mathscr{L}) = \mathrm{d}\alpha \tag{1}$$

$\iota: P \hookrightarrow \mathcal{P}$: embedding

- By (SCC) $\rightarrow$ super Cartan connection provides isomorphism

$$\mathcal{A}: \mathfrak{aut}(\mathcal{P}_{/\mathcal{S}}) \xrightarrow{\sim} \mathcal{E} := \Gamma(\mathcal{P}_{/\mathcal{S}} \times_{\mathrm{Ad}} \mathfrak{g})$$

- $\mathfrak{g} = \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_{\mathbb{R}} \rightarrow$ yields decomposition

$$\mathcal{E} = \mathcal{E}^{\mathbb{R}^{1,3}} \oplus \mathcal{E}^{\mathfrak{spin}} \oplus \mathcal{E}^{\Delta_{\mathbb{R}}}$$

Supersymmetry & Rheonomy

Castellani-D'Auria-Fré:

- Supersymmetry transformations $\leftrightarrow X \in \Gamma(\mathcal{E}^{\Delta_{\mathbb{R}}})$
- in general:
$$\delta_X \mathcal{A} \equiv \iota_X R(\mathcal{A}) + D^{(\mathcal{A})}(\iota_X \mathcal{A})$$
- (1) imposes constraints on $\iota_X R(\mathcal{A}) \to$ *rheonomy conditions*

Alternatively:

Proposition [KE '21]

gen. Cartan connections $\mathcal{A} \leftrightarrow$ Ehresmann connections $\hat{\mathcal{A}}$

on $\mathcal{P}_{/\mathcal{S}}$ on $\widehat{\mathcal{P}}_{/\mathcal{S}} := \mathcal{P}_{/\mathcal{S}} \times_{\mathrm{Ad}} \mathrm{OSp}(1|4)$

- $\to$ can lift action uniquely to $\widehat{\mathcal{P}}_{/\mathcal{S}}$
- $\to$ SUSY trafos as gauge trafos $X \in \mathfrak{gau}(\widehat{\mathcal{P}}_{/\mathcal{S}})$