Varying the non-semisimple Crane-Yetter theory over the character stack

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Varying the non-semisimple Crane-Yetter theory over the character stack

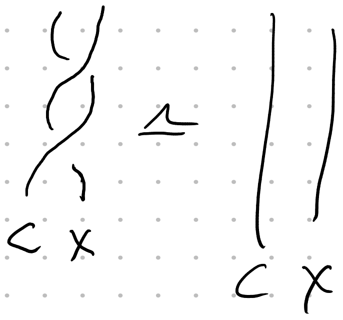
(✮) Algebraic result: the 1-morphism $\mathrm{Rep}\,G \xrightarrow{\mathrm{Rep}_q G} \mathrm{Rep}\,G$ in the 5-category SymTens is invertible.

Plan
- motivation WRT/CY
- varying over char. stack
- alg. result + (✮)

Witten-Reshetikhin-Turaev (WRT) and Crane-Yetter (CY)

Fix $\mathcal{C}$ a $\left\{\begin{matrix}\text{finite}\\ \text{semisimple}\\ \text{modular}\end{matrix}\right\}$ tensor category (linear $/\mathbb{C}$)

so $\mathcal{C}$ is ... ribbon $\rightsquigarrow$ has a graphical calculus



finite + modular $\iff$ $Z_2(\mathcal{C}) \simeq \mathrm{Vect}$

$$Z_2(\mathcal{C}) = \{ c \in \mathcal{C} : \forall x \in \mathcal{C},\ \sigma_{c,x}\sigma_{x,c} = id_{c\otimes x} \}$$

<u>Example</u>: Rep u_q at root of 1, $\mathcal{C}$ a particular subquotient.

RT construction: invariants of framed links $RT(L) \in \mathbb{C}$

Well-behaved under Kirby moves...

invariants of closed 3-manifolds $WRT(M_L) \in \mathbb{C}$

(via surgery)

Can try to form a TQFT $Z : \mathrm{Bord}_{3,2}^{\mathrm{or},\sigma} \longrightarrow \mathrm{Vect}$

- functoriality fails up to scalars (anomaly) MCG $\curvearrowright$ projective reps
- can fix by choosing extra data

E.g. for M^3: choose W^4, $\partial W^4 \simeq M^3$, etc...

Formalised as a relative TQFT [Freed – Teleman, Johnson-Freyd – Scheimbauer]

Given n-d thy α, an $(n-1)$-d thy rel α is

$$\mathbb{1} \xrightarrow{F} \alpha$$

E.g. $n=4$, have:

$$\mathbb{C} = \mathbb{1}(M^3) \xrightarrow{F(M^3)} \alpha(M^3) \xrightarrow{\alpha(W^4)} \mathbb{C} = \alpha(\emptyset)$$

defines F up to scalars if α is invertible, i.e. $\alpha(M^3)$ is 1-d.

Under the Cobordism/Tangle Hypothesis [Baez–Dolan, Lurie (sketch), others...]

$$\left\{ \begin{array}{c} \text{Framed, fully} \\ \text{extended TQFTs} \\ Z : \mathrm{Bord}_n^{fr} \to \mathcal{T} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{Fully dualizable} \\ \text{objects} \\ A \in \mathcal{T} \end{array} \right\}$$

$$Z \longmapsto Z(pt)$$

Invertible theory — Invertible object $A \in \mathcal{T}$

homomorphism of theories $Z^A \to Z^B$ — sufficiently dualizable $A \to B$ in $\mathcal{T}$

Example: $\mathcal{T} =$ BrTens "Morita 4-category"

[Brochier Jordan Safronov Snyder]

$\mathrm{Rep}_q \in$ BrTens is invertible.

Example $\mathcal{T} = \mathrm{Vect}$

V dualizable: $V^\vee$

$ev : V^\vee \otimes V \to \mathbb{C}$

$coev : \mathbb{C} \to V \otimes V^\vee$.

$\Leftrightarrow$ V is f.d.

V invertible: dualizable and ev, coev are isomorphisms

$\Leftrightarrow$ $\dim V = 1$.

What about non-SS WRT? Hard to compute.

Hennings-Lyubashenko, m-traces, ...

current state-of-the-art: [Blanchet-Costantino-Geer-Patureau-Mirand]

$$CGP^{\mathrm{Rep}\,u_q}(M^3) \in \mathbb{C}$$

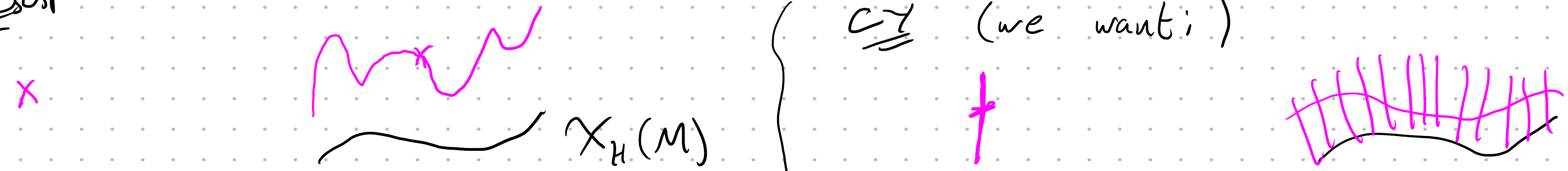
$$CGP^{\mathrm{Rep}\,u_q^H}(M^3) \in \mathcal{O}(X_H(M^3)) \qquad H \leq G \text{ Cartan}$$

where $X_G(M) = \mathrm{Hom}(\pi_1(M), G)/\!/G$ GIT

$= \{\text{flat } G\text{-connections on } M\}$ $\qquad \mathrm{Ch}_G(M) \to X_G(M)$

stack quotient $\rightsquigarrow \mathrm{Ch}_G(M)$ character stack

WRT/CGP $\qquad\qquad$ (we want:)



$X_H(M)$ $\qquad$ $\mathrm{Ch}_G(M)$

$\mathrm{Rep}\,u_q$ $\qquad$ $\mathrm{Rep}\,u_q^H$ (bigger) $\qquad$ $\mathrm{Rep}\,u_q$ $\qquad$ something bigger

A sheaf on $Ch_G(M)$

Fix G reductive, q a good root of 1, $q^\ell = 1$.

$$1 \longrightarrow (u_q) \longrightarrow U_q G \xrightarrow{Fr} UG \longrightarrow 1$$

small quantum subalg gen. by $K_i^{\pm}, E_i, F_i$ mod $K_i^{\ell} = 1$

big / Lusztig quantum group

classical

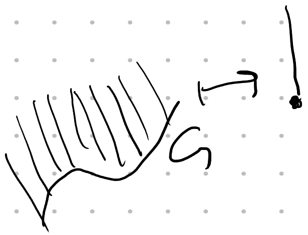
generators: $K_i^{\pm} \longmapsto 0$

$E_i^{(r)} \longmapsto \begin{cases} E_i^{(r/\ell)} & \ell \mid r \\ 0 & \text{o/w} \end{cases}$

$F_i^{(r)} \longmapsto \dots$

$E_i^{r} = [r]!\, E_i^{(r)}$

so $E_i^{\ell} = 0$ etc.



$O_q \otimes_{O(G)} 1$

$\cong$

$$1 \longleftarrow o_q \longleftarrow O_q \xleftarrow{Fr^\vee} O(G) \longleftarrow 1$$

$$\mathrm{Vect} \longleftarrow \mathrm{Rep}\, u_q \longleftarrow \mathrm{Rep}_q G \xleftarrow{Fr^*} \mathrm{Rep}\, G \longleftarrow \mathrm{Vect}$$

$\cong$

$\mathrm{Rep}_q G \otimes_{\mathrm{Rep} G} \mathrm{Vect}$ [Negron]

$V_q(\ell\lambda) \longleftarrow V(\lambda)$

[Ben-Zvi–Francis–Nadler]

$Z^{\mathrm{Rep} G} : M^3 \longmapsto QCoh(Ch_G(M))$

Def (sketch): The 5-category SymTens has:

objects: symmetric tensor categories $\mathcal{A}, \mathcal{B}, \ldots$

1-morphisms: braided tensor categories as bimodules $\mathcal{A} \circlearrowright \mathcal{X} \circlearrowleft \mathcal{B}$

$$\longleftrightarrow \Big(\underset{\text{br. tens.}}{\mathcal{X}} , \mathcal{A} \otimes \mathcal{B}^{op} \xrightarrow[\text{sym. tens.}]{} Z_2(\mathcal{X}) \Big)$$

composition: ${}_{\mathcal{A}}\mathcal{X}_{\mathcal{B}}, {}_{\mathcal{B}}\mathcal{Y}_{\mathcal{C}}$, $\mathcal{Y} \circ \mathcal{X} = \mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$.

2-morphisms: tensor categories as bimodules

3-morphisms: bimodule categories

4-morphisms: functors of such

5-morphisms: natural transformations of such

$$\begin{array}{ccc} \mathrm{Rep}\,G \boxtimes \mathrm{Rep}_q G & & V \boxtimes W \\ \downarrow & & \downarrow \\ \mathrm{Rep}_q G & & Fr^*(V) \otimes W \end{array}$$

Easy check: $Fr^*(\mathrm{Rep}\,G) \subseteq Z_2(\mathrm{Rep}_q G)$. Then

$$\mathrm{Rep}\,G \otimes \mathrm{Rep}\,G^{op} \xrightarrow[\text{sym. tens.}]{} \mathrm{Rep}\,G \xrightarrow{Fr^*} Z_2(\mathrm{Rep}_q G)$$ defines 1-morphism

$$\mathrm{Rep}\,G \xrightarrow{\mathrm{Rep}_q G} \mathrm{Rep}\,G$$

Thm (K): The 1-morphism $\mathrm{Rep}\,G \xrightarrow{\mathrm{Rep}_q G} \mathrm{Rep}\,G$ is invertible in SymTens.

SymTens
ob: sym. tens.
1-m: br. tens. bimod.
2-m: tens. bimod.
3-m: bimod.
4-m: functor
5-m: nat. trans

[Ben-Zvi – Francis – Nadler]: $\mathrm{Rep}\,G$ is 4-dualizable in Sym Tens, and

$$
\begin{tikzcd}
Z^{\mathrm{Rep}G}(M^3) = \mathrm{QCoh}(\mathrm{Ch}_G(M^3)) \arrow[d, "Z^{\mathrm{Rep}_q G}(M^3)", "-\otimes \mathcal{F}"'] \\
\mathrm{QCoh}(\mathrm{Ch}_G(M^3))
\end{tikzcd}
$$

Thm $\Rightarrow$ $\mathcal{F}$ is invertible, i.e. a line bundle.

$$
\begin{tikzcd}
\mathrm{Vect} \arrow[r, "1"] \arrow[dr, bend right, "1"'] & \mathrm{Rep}\,G \arrow[d, "\mathrm{Rep}_q G"] \\
\arrow[ur, phantom, "\Uparrow *"] & \mathrm{Rep}\,G
\end{tikzcd}
$$

$*$ plays role of WRT/GWP.

$\rightsquigarrow \mathrm{Hom}(\mathcal{O}, \mathcal{O}_q)$.

Call it $\mathcal{O}_q = Z^{\mathrm{Rep}_q G}(\mathcal{O})$, $\mathcal{O}$ selected by $\mathrm{Vect} \xrightarrow{1} \mathrm{Rep}\,G$.

Need $*$ dualizable – c.f. Ben Haïm

Sym Tens
ob: sym. tens.
1-m: br. tens. bimod.
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3-m: bimod.
4-m: functor
5-m: nat. trans

For 3-manifolds: $\mathrm{Rep}\,G$ is E_3 in Pr^L, a 2-cat.

$\Rightarrow$ $\mathrm{Rep}\,G$ is also E_4.

Consider

$$\begin{tikzcd} \mathrm{Rep}\,G \arrow[rr, bend left, "\mathrm{Rep}\,G"] \arrow[rr, bend right, "\mathrm{Rep}\,G"'] \arrow[rr, phantom, "\Downarrow \mathrm{Rep}_q G"] & & \mathrm{Rep}\,G \end{tikzcd}$$

an invertible 2-morphism in $\mathrm{Alg}_4(\mathrm{Pr}^L)$

Then

$$Z^{\mathrm{Rep}\,G} : M^3 \longmapsto \mathrm{QCoh}(\mathrm{Ch}_G(M))^{\otimes}$$

$$Z^{\xrightarrow{\mathrm{Rep}\,G}} : M \longmapsto \mathrm{QCoh}(\mathrm{Ch}_G(M))$$

$$Z^{\mathrm{Rep}_q G} : M \longmapsto \varphi \in \mathrm{End}_{\mathrm{QCoh}\text{-}\mathrm{QCoh}}(\mathrm{QCoh}) \simeq Z_1(\mathrm{QCoh})$$