Defect Skein Theory

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Lisbon TQFT's Seminar

June 21st 2023

Plan

- Stratified Factorization Homology
- Research Intermission
- Skeins with Defects

Stratified Spaces

Def: A Labelled stratified space is

$$\begin{tikzcd} X \arrow[r, "f"] & P \arrow[r] & L \end{tikzcd}$$

Paracompact Hausdorff

conf. map

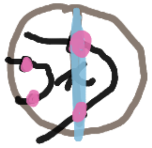
Poset

Closed: $P_{\leq a}$.

finite set

morphisms:

$$\begin{tikzcd} X_1 \arrow[r, "h"] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"] \\ P_1 \arrow[r, "h'"] \arrow[dr] \arrow[ur, phantom, "\circlearrowleft"] & P_2 \arrow[d] \\ & L \end{tikzcd}$$



embedding if h, each $h|_{f_1^{-1}(a)}$ is. an embedding.

stratified isotopy: $h_0 \sim h_1$

$$\begin{tikzcd} X \times [0,1] \arrow[r, "H"] \arrow[d] & Y \arrow[d] \\ P \arrow[r, "h_0' = h_1'"] & Q \end{tikzcd}$$

$H|_0 = h_0$

$H|_1 = h_1$

"Defect" → {1 ≤ 2}

Examples:
- $L \subset M$ submanifold

- Finite graphs:

Our Context: $L = \{A, B, C\}$

Def Let $\mathbb{S}$urf be the 2-category:

$\mathbb{D}$isk $\subset$ full subcategory gen by

objects: oriented decorated surfaces:

$\mathbb{S} = \mathbb{S}_A \sqcup_{\Gamma_B} \mathbb{S}_C$

→ locally look like

Monoidal: Disjoint union $\sqcup$, ϕ is unit.

1-morphisms: stratified embeddings

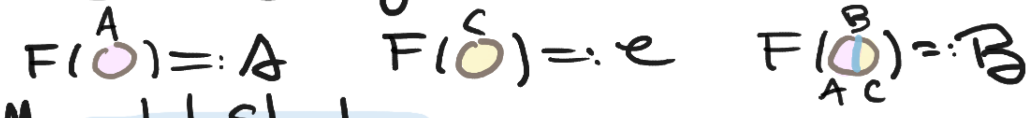
2-morphisms: stratified isotopies.

Stratified Disk Algebras

Def A disk algebra is a symmetric monoidal 2-functor $\mathbb{D}\mathrm{isk}^{\sqcup} \xrightarrow{F} \mathcal{V}^{\boxtimes}$

Cat
- linear cats
- functors
- nat. trans.

What's the algebraic structure?



Monoidal Structure:

$-\otimes-$

$B \boxtimes B \to B$

$F(\emptyset) \to F(\ominus)$

not for

Braiding:

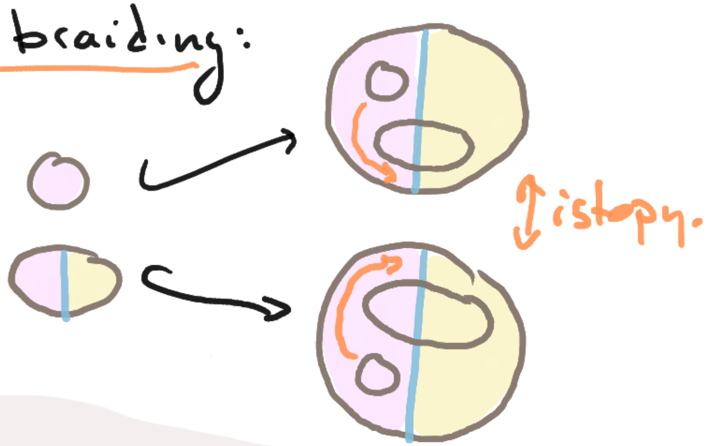
$\rightsquigarrow$ $\mathcal{C}$: $V\otimes W \to W \otimes V$

Central Structure:

- action:

$A \boxtimes B \boxtimes \mathcal{C} \to B$

- half braiding:



All together:

$\mathbb{D}\mathrm{isk} \xrightarrow{F} \mathrm{Cat} \leftrightarrow \left(\begin{array}{l} A, \mathcal{C} \text{ braided}, B \text{ monoidal} \\ A \boxtimes \overset{\text{braided op.}}{\mathcal{C}} \to Z_{Dr}(B) \end{array} \right)$

braided op.

Drinfeld center

Factorization Homology

Turns disk algebras $\mathbb{D}isk \xrightarrow{F} \mathcal{V}$

$\wr$

homology theories $\mathcal{S}urf \xrightarrow{\int_- F} \mathcal{V}$

Idea:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{S}urf, \mathcal{V}) \arrow[r, "\text{restriction}"] & \mathrm{Fun}(\mathbb{D}isk, \mathcal{V}) \arrow[l, bend left, "\int_- F"] \end{tikzcd}$$

$F \mapsto \int_- F$ left adjoint

Called "Left Kan Extension"

This is a standard construction

(Categories for the Working Mathematician X.3)

but in general a big colimit. ↙ all objects in $\mathbb{D}isk$ and all embeddings.

Thm [Ayala-Francis-Tanaka 2017]

Let $H: \mathcal{S}urf \to \mathcal{V}$ be a sym. monoidal 2-functor. If H satisfies

(1) Excision:

$\Sigma \simeq \Sigma_1 \cup_{\Gamma} \Sigma_2 \Rightarrow H(\Sigma) \simeq H(\Sigma_1) \underset{H(\Gamma \times [0,1])}{\otimes} H(\Sigma_2)$

(2) Continuity:

$W_0 \subset W_1 \subset \cdots \subset X$ with $\cup W_i = W \Rightarrow \mathrm{colim}(H(W_0) \to \cdots) = H(W)$

then $\int_- H|_{\mathbb{D}isk} = H(-)$

So: To get a skeiny description of $\int_- F$ we want to use skein categories to build a disk algebra

Intermission: Research Overview / Outlook

Borel

$\mathcal{A} = \mathrm{Rep}_q G$, $\mathcal{B} = \mathrm{Rep}_q B$, $\mathcal{C} = \mathrm{Rep}_q T$ ← $B/[B,B]$ or maximal torus.

FG $\mathcal{O}_q(\mathcal{X}) \hookrightarrow \mathcal{T}_q$ ← $\boxed{\text{quantum tori}}$

- Stratified FH extends quantum cluster coordinates Character Variety

  gluing properties ↓

  ⇝ Char. Stack

  Framing data: Full Flag ⟷ Borel reduction

$F_0 \subset F_1 \subset \cdots \subset F_n$

Jt. in prog. w/ David Jordan

- A skein description prescribes what happens on 3-manifolds:

  Application: Quantizing the A-polynomial, relevant for AJ conjecture.

  $\hookleftarrow (S^3 \setminus K : T^2 \to \emptyset$ $\quad$ $\hookrightarrow \hat{A} \cdot \hat{J} = 0$

- Extend to stratified cobordisms: $\Sigma_1 \xrightarrow{M^3} \Sigma_2 \rightsquigarrow \mathrm{SkCat}(\Sigma_1) \times \mathrm{SkCat}(\Sigma_2)^{op} \to \mathrm{Vect}$

  Application: Categorical / Generalized description of the quantum trace map

  ↖ Jt. in progress w/ Benjamin Haïoun. $\quad$ Skeins ↔ Clusters.

Ribbons and Skeins

Def A **ribbon category** $\mathcal{A}$ has:

- monoidal structure $\otimes, \mathbb{1}$
- braiding $V\otimes W \xrightarrow{c_{V,W}} W\otimes V$
- duality $V^*\otimes V \xrightarrow{ev} \mathbb{1}$
  coev: $\mathbb{1} \longrightarrow V\otimes V^*$
- twist $V \xrightarrow{\theta_V} V$

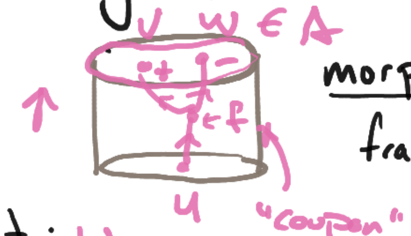
+ compatibility conditions

famous: Yang Baxter

follow from isotopy.

Example: Category of Ribbon Graphs: $Rib_{\mathcal{A}}$

objects:

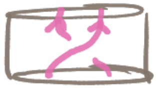


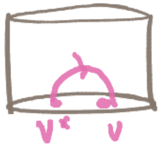
morphisms:

framed oriented colored graphs. "ribbon graphs" $/\sim$ isotopy

product: $\sqcup$,

$\mathbb{1} =$ 



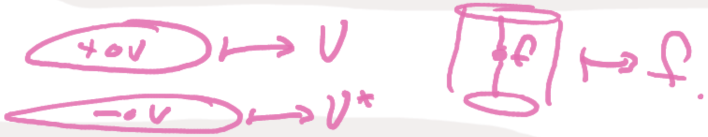


Thm [Reshetikhin–Turaev]

There's a unique braided functor

$$Rib_{\mathcal{A}} \xrightarrow{RT_{\mathcal{A}}} \mathcal{A}$$

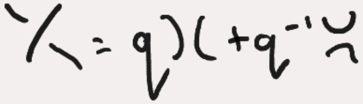
which preserves dualities + coupons



surjective!

Skein Relations:

$\Longleftrightarrow RT_{\mathcal{A}}(T_1) = RT_{\mathcal{A}}(T_2)$

note: Our beloved 

is $\mathcal{A} = Rep_q\, sl_2$

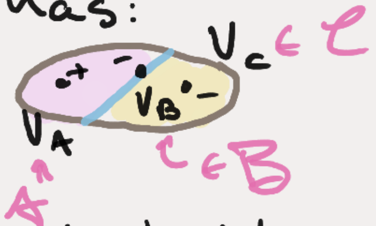
Fact: $Rib_{\mathcal{A}}/\sim \;\simeq \mathcal{A}$

Near a Defect

Def (Colored Ribbon Graphs)

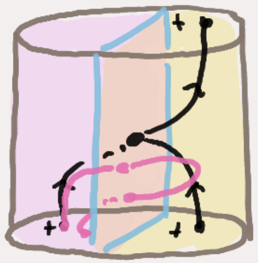
Rib_Φ has:

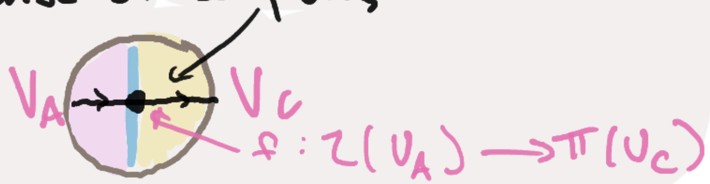
objects:



morphisms: colored ribbon graphs, now as stratified spaces:

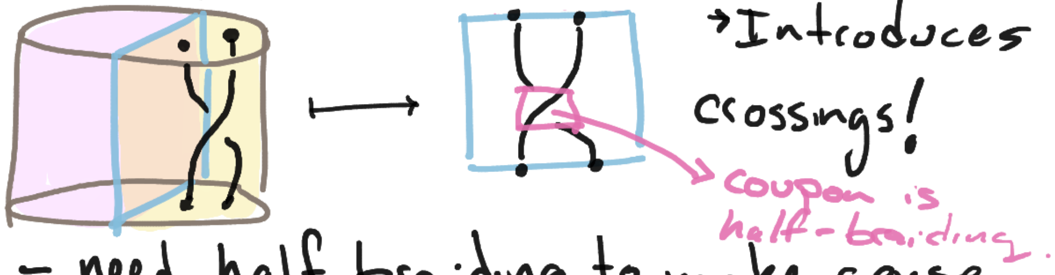
- coupons ↔ 0-dim'l strata
- we need $L \boxtimes \pi : \mathcal{A} \boxtimes \mathcal{C} \to \mathcal{B}$ to make sense of coupons





Constructing $\mathrm{Rib}_\Phi \xrightarrow{RT_\Phi} \mathcal{B}$:

① Project onto the defect



- need half braiding to make sense of them:

② Evaluate Planar graphs

- need $\mathcal{B}$ to be pivotal for evaluation to exist

Prop (B.) This produces a well defined functor

The Skein Picture:

We've built a skein-theoretic description of a disk algebra:

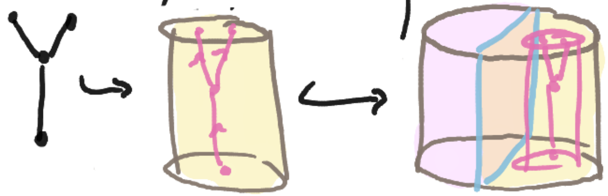
$$\mathbb{D}\mathrm{isk} \xrightarrow{F} \mathrm{Cat}$$

$\longmapsto \mathrm{Rib}_{\mathcal{A}} / \ker RT_{\mathcal{A}} \simeq \mathcal{A}$

$\longmapsto \mathrm{Rib}_{\mathcal{C}} / \ker RT_{\mathcal{C}} \simeq \mathcal{C}$

$\longmapsto \mathrm{Rib}_{\Phi} / \ker RT_{\Phi} \simeq \mathcal{B}$

Functoriality: obj^{s} + morphisms are (stratified) embeddings:



Def Fix • $(\mathcal{A}, \mathcal{C}, \mathcal{A}\boxtimes\bar{\mathcal{C}} \to Z_{D_r}(\mathcal{B}))$

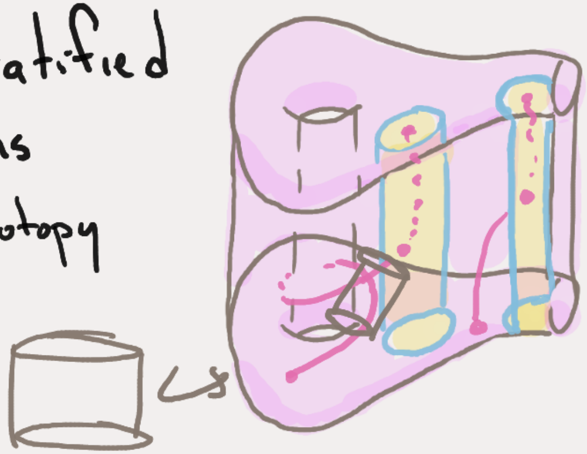
• $\mathcal{S} = \mathcal{S}_A \cup_{\Gamma} \mathcal{S}_C$ decorated surface

The skein category $\mathrm{SkCat}_{ABC}(\mathcal{S})$ has:

obj:

morphisms: stratified colored ribbon graphs upto stratified isotopy

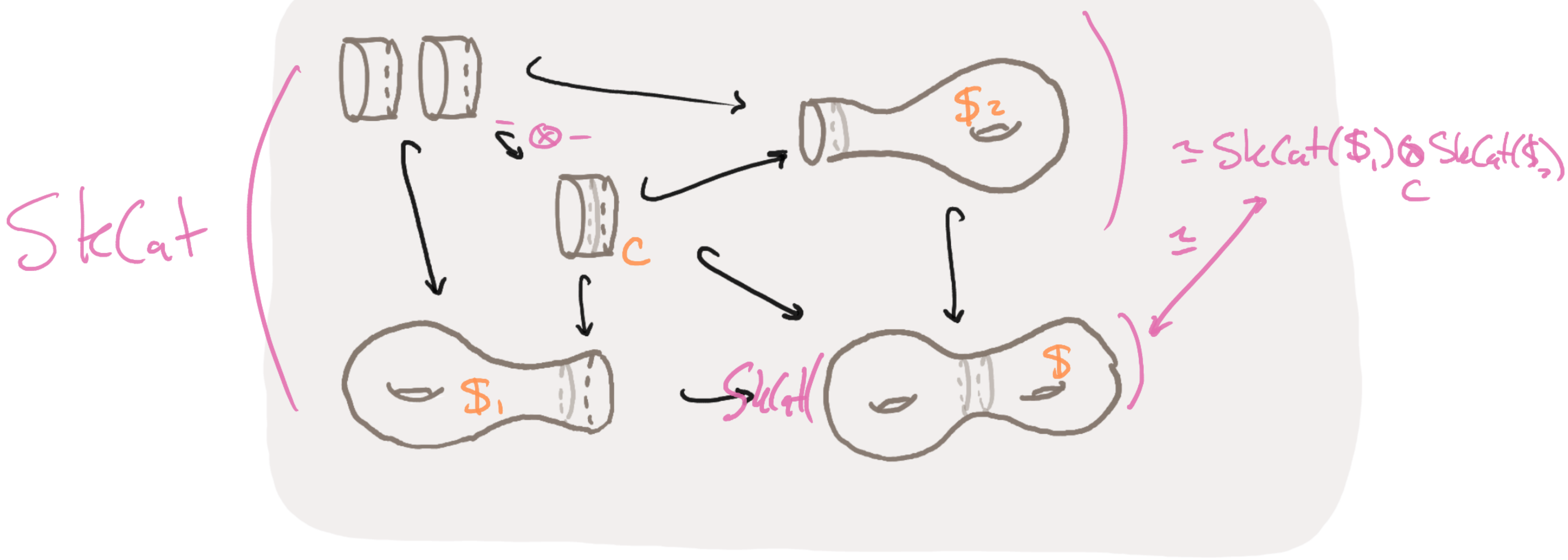
+ skein relations in any cylinder



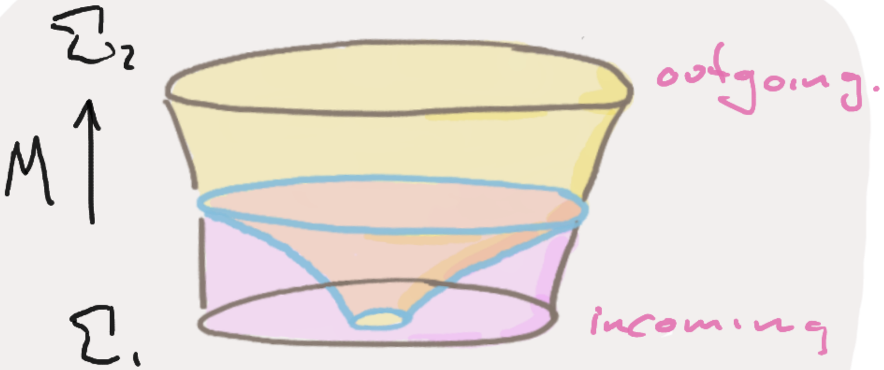
It's Factorization Homology

Thm [Cooke 2019] essentially? $\int_{\Sigma} F \simeq \widehat{SkCat}(\Sigma)$. $\left(\simeq SkCat(\Sigma_1) \otimes_{SkCat(C)} SkCat(\Sigma_2)\right)$

Idea of proof: Excision



Cobordisms



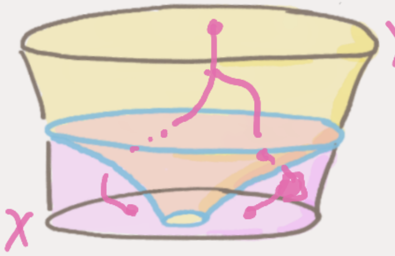
not an embedding $\Sigma_1 \hookrightarrow \Sigma_2$

Construct a bimodule:

$Sk_M(-,-): SkCat(\Sigma_1) \times SkCat(\Sigma_2)^{op} \to Vect$

GJS

$Sk_M(X,Y) = \text{span}\Big\{$



$\Big\} \Big/$ skein relations

Application:

Quantizing A-polynomial of a knot $K \subset S^3$

classically:

$$Ch^{fr}_{SL_2}(S^3 \setminus K) \longrightarrow Ch^{fr}_{SL_2}(T^2)$$

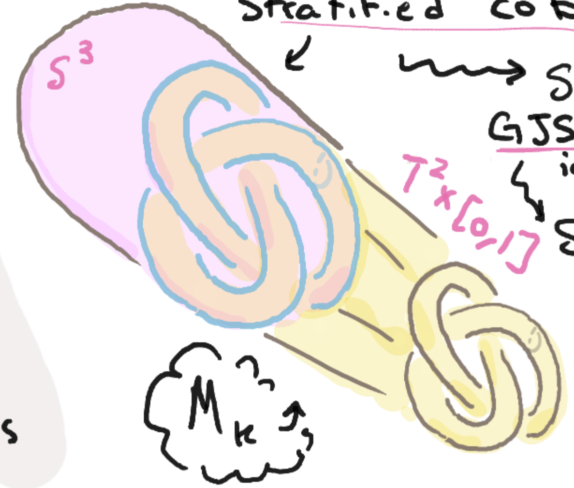
Borel reduction

$$\mathcal{O}(Ch(T^2)) \curvearrowright \mathcal{O}(Ch(S^3\setminus K))$$

$$\simeq \mathcal{O}(Ch(T^2)) \big/ \langle A \rangle$$

Defect skeins: $S^3 \setminus K$ ⇝

stratified cobordism $M_K : T^2_c \to \emptyset$



⇝ $SkCat(T^2) \longrightarrow Vect$

GJS

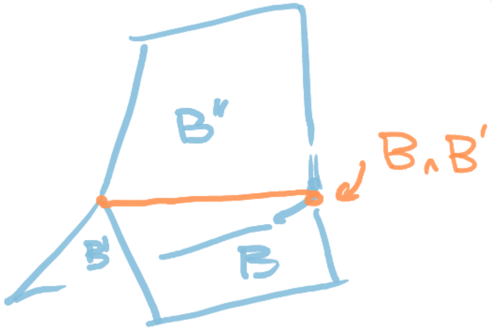
internal algebra + mod

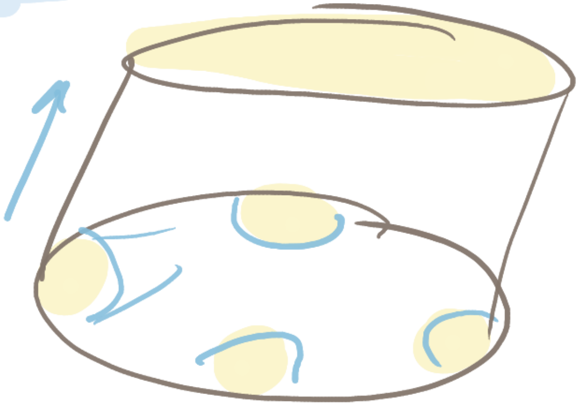
$SkAlg^{int}(T^2) \curvearrowright SkMod^{int}(M_K)$

$\simeq SkAlg^{int}(T^2) \big/ \langle A \rangle$ ★

Thank you!

Gunningham - Jordan - Safronov





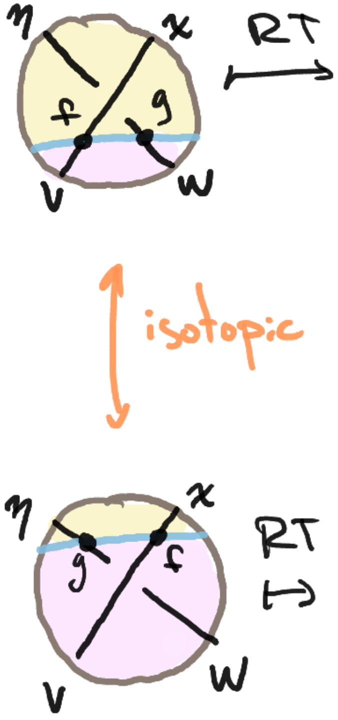
Appendix: RT_Φ is well defined

$H : \mathcal{A}\boxtimes\bar{\mathcal{C}} \longrightarrow Z_{\mathrm{Dr}}(\mathcal{B})$

$J_{a,c,b} : H(a\boxtimes c)\otimes b \xrightarrow{\simeq} b\otimes H(a\boxtimes c)$

Main point: invariant under isotopy.

① Crossings can pass through defects:



$\xmapsto{RT} H(\mathrm{id}\boxtimes\sigma^{\bar{c}})\circ(f\otimes g)$

$= \sigma^{Z(\mathcal{B})}\big|_{H(\mathbb{1}\boxtimes -)} \circ f\otimes g$

$= g\otimes f \circ \sigma^{Z(\mathcal{B})}\big|_{H(-\boxtimes\mathbb{1})}$

$= ?$

$\overset{RT}{\mapsto} g\otimes f\circ H(\sigma^{A}\boxtimes \mathrm{id})$

② Stratified Reidemeister Moves:

- Part ① means this reduces to the case where strands don't pass through the defect.
- Crossings are sent to half braiding in $Z_{\mathrm{Dr}}(\mathcal{B})$: so by definition
  - Yang Baxter holds (⇒ R3)
  - invertible braiding (⇒ R2)

Appendix: Parabolic Induction

$\mathcal{A} \boxtimes \bar{\mathcal{A}} \xrightarrow{\simeq} Z_{Dc}(\mathcal{A})$

Fix G - reductive group
$\hookrightarrow B$ - Borel subgroup
$\twoheadrightarrow T = B/[B,B]$ Cartan quotient.

The parabolic induction disk algebra:

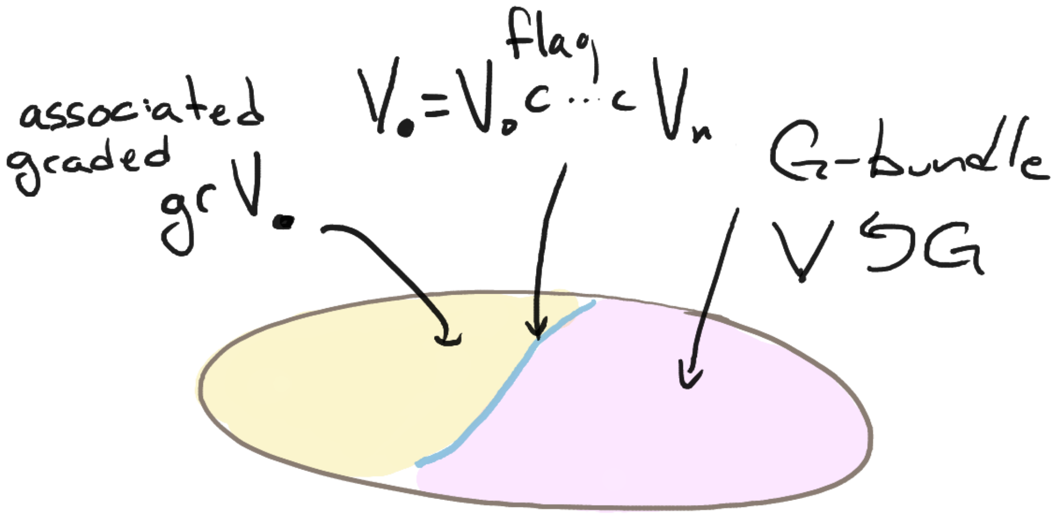
$\mathrm{Rep}_q(T \leftarrow B \rightarrow G)$:

- $\mathrm{Rep}_q G$ $(= \mathrm{Rep}\, U_q\mathfrak{g})$ $\uparrow \iota^*$
- $\mathrm{Rep}_q B$ $(\subseteq \mathrm{Rep}\, U_q\mathfrak{b})$ $\downarrow \pi^*$ $E_i \mapsto 0$
- $\mathrm{Rep}_q T$ $(= \mathrm{Rep}\, U_q\mathfrak{t})$ $K_i \mapsto k_i$

with half braiding

$$(\iota^* V \otimes \pi^* \chi) \otimes W \xrightarrow{\sigma_{(123)} R^G_{13} (R^T_{32})^{-1}} W \otimes (\iota^* V \otimes \pi^* \chi)$$

Classical Picture: Flat bundles



- Stacky extension of cluster coordinates on Character varieties