Strong gap theorems via Yang-Mills flow

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Based on joint work in progress with Anuk Dayaprema (UW-Madison)

Setup

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$F_A \overset{loc}{=} dA + A \wedge A$, curvature.

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- Can have $\mathscr{M} = \emptyset$!

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Reyes-Carrión '98:

$$\Lambda^2 = \overbrace{\underbrace{\mathrm{LieHol}}_{\textit{Instanton}}(g) \oplus \cdots}^{\text{“}\textit{Compatible}\text{”}} \oplus \cdots .$$

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Sibner, Sibner, and Uhlenbeck (1989) constructed Yang-Mills connections on the trivial $\mathrm{SU}(2)$-bundle on S^4 ($\mathbb{R}^4$) with arbitrarily large energy.

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Disclaimer. No (known) relation to millenum prize problem.

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Suppose that A is Yang-Mills on $P \to S^4$ and $\|F_A^+\|_{L^\infty} < \sqrt{3}$; then $F_A^+ \equiv 0$.

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Question. What is the *optimal* gap value on (the trivial bundle on) S^4? *I.e.*, what is the lowest nonzero energy of a Yang-Mills connection?

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- Does every $P \to (M^4, g)$ have an energy gap?
- More generally: is the set of critical values of $\mathcal{YM}$ discrete?
- Does every $P \to (M^4, g)$ carry at least one (non-minimal) Yang-Mills connection?

Gap in higher dimensions, cont.

Theorem (T. Huang '15 (withdrawn), Feehan '17 (corrigendum), '19)

Suppose that A is a smooth Yang-Mills connection on $P \to M^n$, compact, with $\mathcal{YM}(A) < \varepsilon_0$. Then A is flat.

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Desirable to have gap theorems which apply on bundles $P \to M^n$, $n \geq 4$, with

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Theorem (T. Taniguchi '98)

Given a quaternion-Kähler manifold M with positive scalar curvature, there exists $\delta_0 > 0$ as follows. Suppose that A is a pseudo-holomorphic, Yang-Mills connection with $\|F_A^{\mathfrak{sp}(1)}\|_{L^{n/2}} < \delta_0$; then A is an instanton.

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We can improve most of the above to strong gap theorems.

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A **strong gap theorem** states that $(*)$ gives a deformation retraction from the set of all (compatible) connections in $\mathscr{B}$ with appropriately small F^+ onto $\mathscr{M}$.

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- Good reasons to want strong gap.

Basic properties of $(*)$

- Semi-parabolicity:

$$\frac{\partial}{\partial t} A^{\alpha}_{j\beta} = \nabla^i F_{ij}{}^{\alpha}{}_{\beta}$$
$$= g^{ik}\left(\partial_i \partial_k A^{\alpha}_{j\beta} - \partial_i \partial_j A^{\alpha}_{k\beta}\right) + \partial A \# A + A \# A \# A + \Gamma \# \partial A + \Gamma \# A \# A.$$

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Donaldson's DeTurck trick: let $A(t) = A_0 + a(t)$, and solve

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Then $(\dagger)$ is gauge-equivalent to $(*)$, and

$$\frac{\partial}{\partial t} a_j = g^{ik} \left(\partial_i \partial_k a_j - \partial_i \partial_j a_k + \partial_j \partial_i a_k \right) + \partial a \# a + a \# a \# a + etc.$$
$$= g^{ik} \partial_i \partial_k a_j + etc.$$

So $(\dagger)$ is strictly parabolic $\Rightarrow$ short-time solutions exist.

Basic properties, cont.

- Global energy identity:

$$\frac{d}{dt}\mathcal{YM}(A(t)) = -\int_M |D_A^* F_A|^2 \, dV$$

$$\mathcal{YM}(A(T)) + \int_0^T \int_M |D^* F|^2 \, dV dt = \mathcal{YM}(A(0)).$$

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- $(*)$ preserves compatibility (Donaldson '85, Oliveira-Waldron '20).

Basic properties, cont.[2]

- Pointwise curvature evolution:

$$\frac{\partial F_A}{\partial t} = D\left(\frac{\partial A}{\partial t}\right) = D\left(-D^*F\right) - D^*\overbrace{DF}^{=0}$$

by the second Bianchi Identity. This gives

$$\left(\frac{\partial}{\partial t} + \Delta_A\right) F_A = 0$$

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Split Bochner formula in 4D:

$$\left(\frac{\partial}{\partial t} + \nabla^* \nabla\right) F^+ = [\![F^+, F^+]\!] + Rm_g \# F^+.$$

Basic properties, cont.[3]

- **Hamilton's monotonicity formula** ('93): Let

$$\Phi_x(A;R) = \frac{R^{4-n}}{(4\pi)^{n/2}} \int |F_A(y)|^2 e^{-\left(\frac{d(x,y)}{2R}\right)^2} \, dV_y.$$

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- ε-**regularity** (Struwe, Chen-Shen '94): If $\Phi_x(R, t-R^2) < \varepsilon_0$, then

$$\sup_{\substack{t-R^2/2 \leq s \leq t \\ y \in B_{R/2}(x)}} |\nabla^{(k)} F(y,s)| < \frac{C_k}{R^{2+k}}.$$

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- Infinite-time convergence with singularities (a.k.a. uniqueness of Uhlenbeck limits)
- Special holonomy: separate control of curvature components.

Strong gap on the 4-sphere

Theorem 1

Let A_0 be a smooth connection on an $\mathrm{SU}(2)$-bundle $P \to S^4$, with energy

$$\mathcal{YM}(A_0) < 4\pi^2|\kappa(E)| + 8\pi^2. \tag{1}$$

If $A(t)$ solves $(*)$ with $A(0) = A_0$, then $A(t)$ converges smoothly and exponentially to an instanton on P under the flow.

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$\Rightarrow$ smooth convergence. □

Yang-Mills flow on special-holonomy manifolds

Theorem (Oliveira and Waldron, Adv. Math. '20)

Assume that M has special holonomy and $A(0)$ is compatible. For $0 \leq \gamma \leq 1$ and

$$R_1^2 - R_2^2 = \gamma^2 \left(t_2 - t_1 \right),$$

we have the following "extended" version of Hamilton's monotonicity formula:

$$\Phi_x \left(R_2, t_2 \right) \leq \Phi_x \left(R_1, t_1 \right) + C \left(1 - \gamma \right) \int_{t_1}^{t_2} \| F^+ \left(\cdot, t \right) \|_{L^\infty} \, dt,$$

where

$$F^+ = \begin{cases} F^+ & M = M^4 \\ F^7 & \mathrm{Hol}(M) = \mathrm{G}_2 \text{ or } \mathrm{Spin}(7) \\ \Lambda F & \mathrm{Hol}(M) = \mathrm{U}(n/2) \qquad \text{(type } (1,1) \text{ conn.)} \\ F^{\mathfrak{sp}(1)} & \mathrm{Hol}(M) = \mathrm{Sp}(n/4)\mathrm{Sp}(1) \qquad \text{(pseudoholomorphic conn.)}. \end{cases}$$

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Corollary

If $\|F^+(\cdot, t)\|_{L^\infty}$ remains bounded, then (YM) exists for all time. Moreover, the infinite-time singular set is calibrated.

Deformation to G_2-instantons?

For a solution of $(*)$ on a G_2-manifold, we may let

$$F_{A(t)} = \overbrace{f^7(t) \lrcorner \phi}^{F^+ = F^7} + F^{14}(t),$$

where $f^7(t) \in \Omega^1(\mathfrak{g}_P)$ and $F^{14}(t) \in \Omega^2_{14}(\mathfrak{g}_P)$. These evolve by:

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Question. Under what conditions does $\|f^7\| \leq \delta \Rightarrow \|f^7\|_{L^\infty} \leq C\delta$?

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Theorem 2

Let M be a compact quaternion-Kähler manifold. There exists a constant $\delta_0 > 0$, depending only on the geometry of M, as follows.

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Problem. Construct pseudoholomorphic connections (e.g. on $\mathbb{HP}^n$) with $\|F^{\mathfrak{sp}(1)}\|_{M^{2,4}}$ small but nonzero.

Thank you!

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Suppose $G = \mathrm{SU}(2)$ *or* $\mathrm{SU}(3)$ *and* $M = S^4$. *For* $k \geq 0$, *the moduli space* $\mathscr{M}_k$ *of charge-k instantons is path-connected.*

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The ADHM construction gives all instantons in $\mathscr{M}_k$.

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- Other base manifolds?
- Higher homotopy groups? (Atiyah-Jones conjecture)