Skein modules and 4d TQFTs

Pavel Safronov

University of Edinburgh

September 2022

Chern–Simons TQFT

Let G be a connected compact simple Lie group and $k \in \mathbf{Z}$ the level. Witten has described the Chern–Simons theory as one of the first examples of a TQFT. This was formalized by Reshetikhin–Turaev as follows:

- There is a semisimple modular tensor category (semisimple MTC) $\mathcal{C}(G, q)$, where $q = \exp(\pi i/k)$, constructed from representations of the quantum group $\mathrm{U}_q(\mathfrak{g})$.
- Let $\mathrm{Bord}^{sgn}_{3,2,1}$ be the 2-category whose objects are closed 1-manifolds, 1-morphisms are 2-dimensional bordisms and 2-morphisms are 3-dimensional bordisms between bordisms, all equipped with orientation and an *extra geometric structure*.
- Let $2\mathrm{Vect}$ be the 2-category of linear categories, functors and natural transformations.
- Given a semisimple modular tensor category $\mathcal{C}$, there is an extended 3-2-1 TQFT

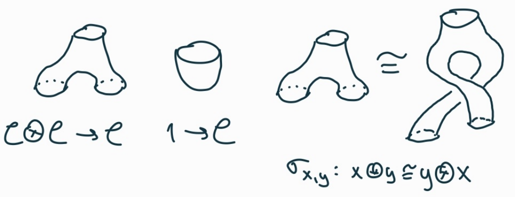
  $$Z\colon \mathrm{Bord}^{sgn}_{3,2,1} \longrightarrow 2\mathrm{Vect}$$

  which sends $S^1 \mapsto \mathcal{C}$.

Consider the opposite question. Suppose

$$Z\colon \mathrm{Bord}^{sgn}_{3,2,1} \longrightarrow 2\mathrm{Vect}$$

is an extended 3-2-1 TQFT. What structure is there on the category $\mathcal{C} = Z(S^1)$?



Theorem (Bartlett–Douglas–Schommer-Pries–Vicary)

$Z(S^1)$ is a semisimple modular tensor category.

Framing anomaly

In a d-dimensional QFT the partition function $Z(M)$ on a closed d-manifold M is a number $Z(M) \in \mathbf{C}$. *Anomaly*: it is an element of a 1-dimensional vector space $Z(M) \in \mathcal{H}(M)$. We can think of $\mathcal{H}(M)$ as the space of states in a $(d+1)$-dimensional QFT, the **anomaly QFT**.

- The Chern–Simons TQFT, as a theory of *oriented* cobordisms, is anomalous. The corresponding anomaly TQFT is the 4d Crane–Yetter TQFT.
- The 4d Crane–Yetter TQFT is **invertible**: all partition functions are nonzero, all spaces of states are 1-dimensional etc.
- Given an extra structure (bounding 4-manifold, p_1-structure, ...) on the 2- or the 3-manifold, the Crane–Yetter theory can be trivialized.

Fully extended Crane–Yetter TQFT

Let BrTens be the symmetric monoidal 4-category as follows:

- Its objects are k-linear braided monoidal categories.
- 1-morphisms from $\mathcal{C}$ to $\mathcal{D}$ are $\mathcal{C} \otimes \mathcal{D}^{\sigma\mathrm{op}}$-monoidal categories.
- 2-morphisms from $\mathcal{A}\colon \mathcal{C} \to \mathcal{D}$ to $\mathcal{B}\colon \mathcal{C} \to \mathcal{D}$ are $\mathcal{C} \otimes \mathcal{D}^{\sigma\mathrm{op}}$-centered $(\mathcal{A}, \mathcal{B})$-bimodule categories.
- ...

See the works of Johnson-Freyd–Scheimbauer, Brochier–Jordan–Snyder for more details.

Conjecture

Let $\mathcal{C}$ be a semisimple modular tensor category. There is a 4-dimensional fully extended TQFT

$$Z_{\mathcal{C}}\colon \mathrm{Bord}^{or}_{4,3,2,1,0} \longrightarrow \mathrm{BrTens}$$

which sends $\mathrm{pt} \mapsto \mathcal{C}$ and which restricts to the Crane–Yetter TQFT in dimensions 3 and 4.

Theorem (Brochier–Jordan–Safronov–Snyder)

Let $\mathcal{C}$ be a modular tensor category (not necessarily semisimple). There is an invertible 4-dimensional fully extended TQFT

$$Z_{\mathcal{C}}\colon \mathrm{Bord}^{fr}_{4,3,2,1,0} \longrightarrow \mathrm{BrTens}$$

which sends $\mathrm{pt} \mapsto \mathcal{C}$.

Partially defined 4d TQFTs

Let $q \in \mathbf{C}^\times$ be a quantum parameter. There is a ribbon category $\mathrm{Rep}_q(G)$ of representations of the quantum group. For q a root of unity the semisimple modular tensor category $\mathcal{C}(G, q)$ is a certain subquotient of $\mathrm{Rep}_q(G)$.

Theorem (Brochier–Jordan–Snyder)

Let $\mathcal{C}$ be a ribbon tensor category (more generally, a rigid braided tensor category). Then there is a partially defined 4d TQFT

$$Z_\mathcal{C} \colon \mathrm{Bord}^{fr}_{3,2,1,0} \longrightarrow \mathrm{BrTens}$$

which sends $\mathrm{pt} \mapsto \mathcal{C}$.

In particular, for every framed 3-manifold M there is a vector space $Z_\mathcal{C}(M)$.

Conjecture

$Z_\mathcal{C}(M) \cong \mathrm{Sk}_\mathcal{C}(M)$, the $\mathcal{C}$-labeled *skein module* of M.

Remark

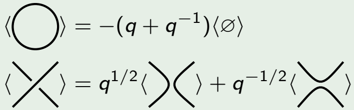
Analogous results for 3-manifolds are known thanks to the works of Scheimbauer, Cooke and Kirillov–Tham.

Skein modules

If $\mathcal{C}$ is a ribbon category and M an oriented 3-manifold, there is a vector space $\mathrm{Sk}_{\mathcal{C}}(M)$ spanned by $\mathcal{C}$-labeled ribbon graphs modulo local relations.

Example

The **Kauffman bracket skein module** is the $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-module $\mathrm{Sk}(M)$ spanned by isotopy classes of framed unoriented links in M modulo the relations



It coincides with the skein module for $\mathcal{C} = \mathrm{Rep}_q(\mathrm{SL}_2)$.

There is a 4d TQFT whose fields on an oriented 4-manifold M are given by a principal G-bundle P, a connection A and a two-form $B \in \Omega^2(M; \mathrm{coad}\, P)$. The action is

$$S = \int_M \langle B \wedge F_A \rangle.$$

It is expected that this theory can be fully extended and its 3-2-1-0 part coincides with the previous 4d TQFT for $\mathcal{C} = \mathrm{Rep}(G)$.

Example

Let M be an oriented 3-manifold. We have an isomorphism

$$\mathrm{Sk}_{\mathrm{Rep}(G)}(M) \cong \mathcal{O}(\mathrm{Loc}_G(M)),$$

the algebra of polynomial functions on the *character variety* (moduli space of flat G-bundles)

$$\mathrm{Loc}_G(M) = \mathrm{Hom}(\pi_1(M), G)/G.$$

$\mathcal{O}(\mathrm{Loc}_G(M))$ is essentially the space of states in the 4d BF theory on M.

q a root of unity

Suppose q is a *good* root of unity. Then we may define the *metaplectic dual group* $G^\vee$ (depends on G and q).

Conjecture

If q is a good root of unity, there is a line bundle $\mathcal{L}_q$ on $\mathrm{Loc}_{G^\vee}(M)$ such that

$$\mathrm{Sk}_{\mathrm{Rep}_q(G)}(M) \cong \Gamma(\mathrm{Loc}_{G^\vee}(M), \mathcal{L}_q)$$

is given by the space of sections of $\mathcal{L}_q$.

For instance, for $q = 1$ we have $G^\vee = G$ and $\mathcal{L}_q = \mathcal{O}$.

Example

For $G = \mathrm{SL}_2$ any root of unity is good. In this case $G^\vee$ is either SL_2 or $\mathrm{PGL}_2 = \mathrm{SO}_3$.

q generic

As $\mathrm{Sk}_{\mathrm{Rep}(G)}(M) \cong \mathcal{O}(\mathrm{Loc}_G(M))$, we can think of $\mathrm{Sk}_{\mathrm{Rep}_q(G)}(M)$ as the algebra of functions on the "quantum character variety" of M.

If $Z_{\mathcal{C}}$ were a fully defined 4d TQFT, the vector spaces $Z_{\mathcal{C}}(M^3)$ would be finite-dimensional.

Claim: $\mathrm{Rep}_q(G) \in \mathrm{BrTens}$ is not 4-dualizable for any q.

Nevertheless:

Theorem (Gunningham–Jordan–S)

Let M be a closed oriented 3-manifold. Then $\mathrm{Sk}_{\mathrm{Rep}_q(G)}(M)$ is finite-dimensional for generic q.

Generic skein module: cohomology of the character variety

Recall that

$$\mathrm{Sk}_{\mathrm{Rep}(G)}(M) \cong \mathcal{O}(\mathrm{Loc}_G(M)).$$

Can we give a similar formula for $\mathrm{Sk}_{\mathrm{Rep}_q(G)}(M)$ for generic q?

$\mathrm{Loc}_G(M)$ is an example of a d-critical stack: it is a space obtained (locally) as the critical locus of an action functional. In this context Brav–Bussi–Dupont–Joyce–Szendroi have defined a certain constructible complex of sheaves $\phi_{\mathrm{Loc}_G(M)}$.

Theorem (Gunningham–S, in progress)

For q generic one has an isomorphism

$$\mathrm{Sk}_{\mathrm{Rep}_q(G)}(M) \cong \mathrm{H}^0(\mathrm{Loc}_G(M), \phi_{\mathrm{Loc}_G(M)}).$$

For instance, if $\mathrm{Loc}_G(M)$ is smooth, this reduces to the usual (singular) cohomology of $\mathrm{Loc}_G(M)$.