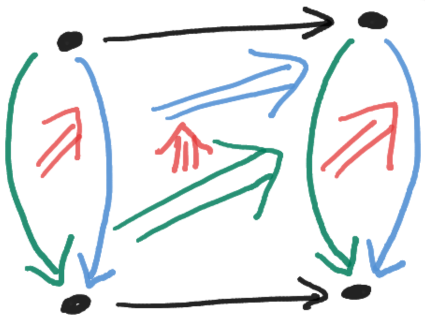
Non-semisimple WRT as fully extended non-compact Relative TQFTs



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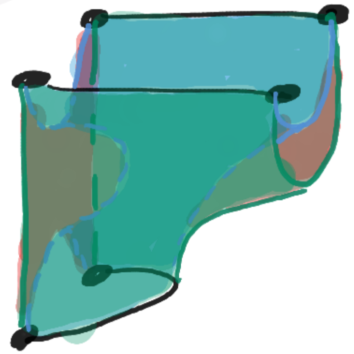
Lisbon TQFT seminar



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Overview:

$\mathcal{V}$ (non-semisimple) modular	3D (2+1)		4D (3+1)
Explicit constructions of TQFTs low dimensional topology	Witten Reshetikhin Turaev DGGPR	Turaev Viro $\mathcal{V} = Z(\mathcal{C})$ CGPV	Crane Yetter CGHP
Classification of TQFTs by the Cobordism Hypothesis Higher Algebra	TODAY ANOMALY	Douglas Schommer-Pries Snyder $\mathcal{C}$ is 3-dualizable in Tens	Brochier Jordan Snyder $\mathcal{V}$ is 4-dualizable in BrTens BJSS?

extends to the point
Bärenz–Kirillov

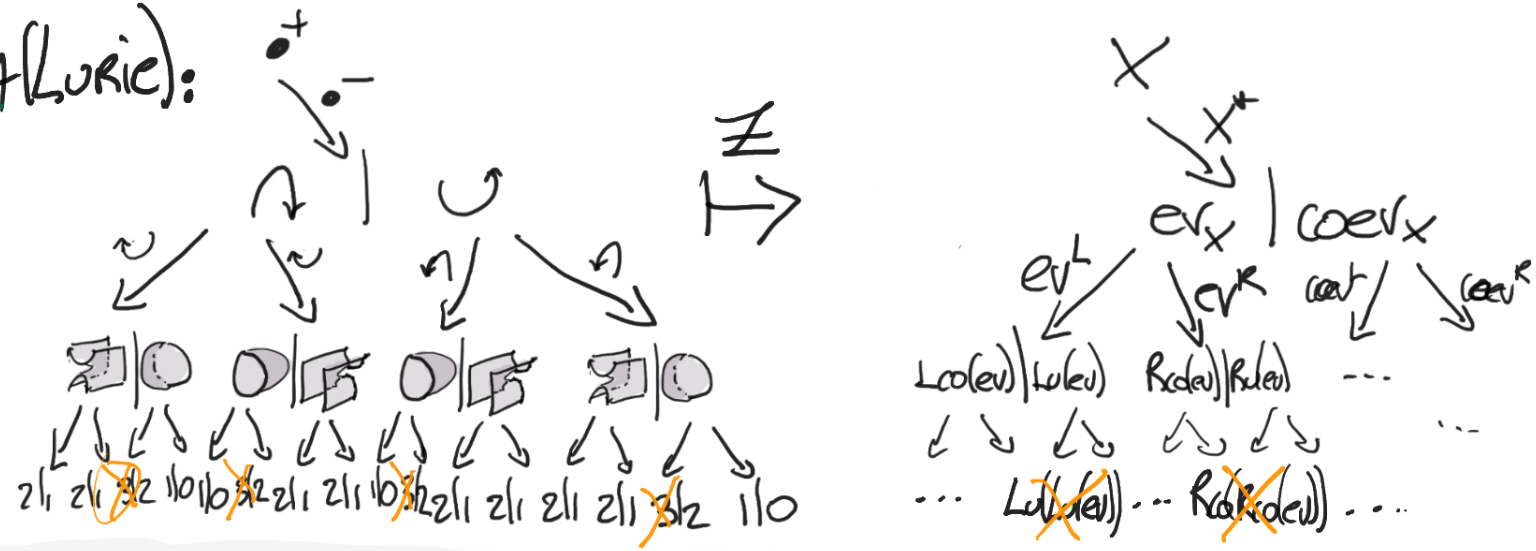
Tens = { tensor categories, bimodule categories, functors, natural transformations }

BrTens = { braided tensor cocomplete cat., bimodule algebras, bimodules, functors, natural transformations }

Non-compact Cobordism Hypothesis:

$$\left\{\begin{array}{l}\text{Non-compact fully extended} \\ \text{framed } n\text{-TQFTs } Z: \mathrm{Bord}_n^{fr} \to \mathcal{C}\end{array}\right\} \xrightarrow[\mathrm{ev}_{pt}]{\sim} \left\{\begin{array}{l}\text{Non-compact- } n\text{-dualizable} \\ \text{object } X \in \mathcal{C}\end{array}\right\}^{\sim}$$

Idea (Lurie):



Non-compact: Top-dimensional bordisms have outgoing boundary in every c.c.

$\rightsquigarrow$ Disallow $B^n: S^{n-1} \to \emptyset$

Can we get WRT from the Cobordism Hypothesis?

⚠ WRT needs cobordisms with extra structure

Freed–Teleman

WRT relative to CY

Walker

(3)

$$\begin{tikzcd} M \arrow[r, "W"] \arrow[d, squiggly, "``\partial\text{''}"'] & \emptyset \arrow[d, squiggly, "CY"] & \\ \mathbb{C} \arrow[r, "\emptyset"] \arrow[rr, bend right, "WRT"'] & Sk(M) \arrow[r, "CY(W)"] & \mathbb{C} \end{tikzcd}$$

(2)

$$\begin{tikzcd} \Sigma \arrow[r, "H"] \arrow[d, squiggly, "\partial"'] & \emptyset \arrow[d, squiggly, "CY"] & \\ Vect \arrow[r, "\partial(\Sigma)"] \arrow[rr, bend right, "WRT"'] & CY(\Sigma) \arrow[r, "CY(H)"] & Vect \end{tikzcd}$$

(0)

$$\begin{matrix} +\bullet \\ -\bullet \end{matrix} \quad \xrightarrow{\ \partial\ } \quad \xrightarrow{\ CY\ } \quad WRT(S^0)$$

⚠ WRT does not extend to the point

⚠ DGGPR only defined on admissible cobordisms → Non-compact cobordism Hypothesis

Big picture:

CY fully extended 4-TQFT $\overset{\text{Cobordism Hypothesis}}{\longleftrightarrow}$ $CY(\overset{+}{\bullet}) = \mathcal{V} \in BrTens$

$\underline{\text{Thm}}$ (BJSSS): $\mathcal{V}$ is 4-dualizable

∂ fully extended relative 3-TQFT $\overset{CH}{\longleftrightarrow}$ $\partial(\overset{+}{\bullet}) =$ unit inclusion $\eta : Vect \to \mathcal{V}$
$=_{Vect} \mathcal{V}_{\mathcal{V}}$

$\underline{\text{Theorem}}$ (H.) η is non-c- 3-dualizable

in the non-semisimple case

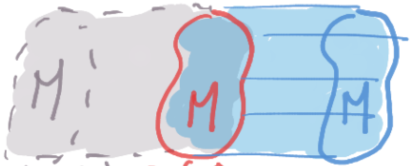
in the sense of SS

$\underline{\text{Conjecture}}$: $\mathcal{V} + \eta \;\underset{BJSSS + H}{\overset{CH}{\longleftrightarrow}}\; Z \simeq CY + \partial \xrightarrow{\text{compose}} WRT$

(I) Relative TQFTs

Freed–Teleman: boundary condition for Z:

"transformation" $R : \mathrm{Triv} \Rrightarrow Z|_{n-1}$



$$\mathrm{Triv}(M) \xrightarrow{\partial(M)} Z(M)$$

Johnson-Freyd – Scheimbauer's model:

oplax

$$\begin{tikzcd} \mathrm{Bord}_{n-1} \arrow[r] \arrow[rr, bend left, "\mathrm{Triv}"] \arrow[rr, bend right, "Z"'] & \mathcal{C}^{\to} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{C} \end{tikzcd}$$

$R :$

Arrow category:

$\mathcal{C}^{\to} = \Big\{$

- arrow "f"
- oplax square "h" $f \to f'$
- ...

$$\begin{tikzcd} s(f) \arrow[r, "f^\#"] \arrow[d, "s(h)"'] & t(f) \arrow[d, "t(h)"] \arrow[dl, Rightarrow, "h^\#"'] \\ s(f') \arrow[r, "f'^\#"'] & t(f') \end{tikzcd}$$

Triv $\quad \partial \quad Z$

RK: $R \xleftrightarrow{\text{CH}} R(\bullet^+) : \mathrm{Triv}(\bullet^+) \to Z(\bullet^+)$

$(n-1)$-dualizable in $\mathcal{C}^{\to}$

Main Result: We can use the Cobordism Hypothesis on η!

Theorem (H.) $\mathcal{V} \in \mathrm{BrTens}$, enough compact projectives, $\eta = {}_{\mathrm{Vect}}\mathcal{V}_{\mathcal{V}} \in \mathrm{BrTens}^{\rightarrow}$, then:

η 3-dualizable $\iff$ $\mathcal{V}$ cp-rigid with cp unit

η non-compact-3-dualizable $\iff$ $\mathcal{V}$ cp-rigid

$\mathcal{V}$ (non-semisimple) modular:
- semisimple: η is 3-dualizable
- non-ss: η is NOT 3-d but it is n-c-3-d.

(II) Rebuilding WRT:

$\nu + \eta \xrightarrow{C-H} \mathcal{Z}: \mathrm{Bord}_4 \to \mathcal{C}$ $+$ $\mathcal{R}: \mathrm{Bord}_3 \to \mathcal{C}^{\to}$

BJSS + H + orientation issues ⚠

$$\mathrm{Bord}_4^{\to\emptyset} \; (M \xrightarrow{W} \emptyset) \xrightarrow{\mathcal{Z}^{\to 1}} \mathcal{C}^{\to 1} \; (\mathcal{Z}(M) \xrightarrow{\mathcal{Z}(W)} 1)$$

$$\mathrm{Bord}_4 \xrightarrow{\mathcal{R}} \mathcal{C}^{1\to} \; (1 \xrightarrow{\mathcal{R}(M)} \mathcal{Z}(M))$$

$$\xrightarrow{\text{compose}} \mathcal{C}^{1\to 1}_{WRT}$$

$$\begin{cases} \mathcal{Z}^{\to 1} : \mathcal{Z}|_{n-1} \Rightarrow \mathrm{Triv} \\ \mathcal{R} : \mathrm{Triv} \Rightarrow \mathcal{Z}|_{n-1} \end{cases}$$

compose

Anomalous theory

$\Omega \mathrm{BrTens} = \mathrm{Tens}$

$\mathcal{A}: \mathrm{Triv} \Rightarrow \mathrm{Triv}$

$\mathrm{End}(1)$

$\mathcal{A}: \mathrm{Bord}_4^{\to\emptyset,\, oddopp} \to \Omega\mathcal{C}$

obj: $S^0 \xrightarrow{C} \emptyset$

1-m:  , $\Gamma \xrightarrow{\Sigma} \emptyset$

2-m: $\Sigma \xrightarrow{H} \emptyset$

3-m: $M \xrightarrow{W} \emptyset$

WRT as an anomalous theory:

$$\mathrm{Cob}^{\text{filled}}_{321} = \left\{ \begin{array}{l} \text{objects: } \bigsqcup_n S^1 \xrightarrow{\sqcup_n D^2} \emptyset \\[2em] \text{1-m: } \begin{tikzcd} \bigsqcup_n S^1 \arrow[r, "\sqcup_n D^2"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\Nearrow \text{handlebody}"] & \emptyset \arrow[d, equal] \\ \bigsqcup_m S^1 \arrow[r, "\sqcup_m D^2"'] & \emptyset \end{tikzcd} \\[2em] \text{2-m: } \begin{tikzcd} \Sigma \arrow[r, "H"] \arrow[d, "M"'] \arrow[dr, phantom, "\Nearrow W"] & \emptyset \arrow[d, equal] \\ \Sigma' \arrow[r, "H'"'] & \emptyset \end{tikzcd} \end{array} \right. \overset{\mathcal{A}}{\longmapsto} \left\{ \begin{array}{l} \text{obj: categories} \\[2em] \text{1-m: functions} \\[2em] \text{2-m: nat tr.} \end{array} \right.$$

$$\cap\!\!|$$

$$\Omega\,\mathrm{Bord}_4^{\to\emptyset \text{ odd, opp}}$$

$$\Omega\,\Omega\,\mathrm{BrTens} = P_R =$$

obj: categories

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\text{unit}"] \arrow[rr, bend right] & \int_{S^1} \mathcal{V} \arrow[r, "\int_{D^2}\mathcal{V} \simeq \mathcal{V}"] & \mathrm{Vect} \end{tikzcd}$$

$$\mathcal{A}(S^1 \xrightarrow{D^2} \emptyset) = \mathcal{V}$$

1-m: functions

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\text{unit}"] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow \text{empty object}"] & \int_{\sqcup_n S^1} \mathcal{V} \arrow[r, "\mathcal{V}^{\otimes n}"] \arrow[d, "\int_\Sigma \mathcal{V}"] \arrow[dr, phantom, "\Nearrow \mathrm{SkMod}(H)"] & \mathrm{Vect} \arrow[d, equal] \\ \mathrm{Vect} \arrow[r, "\text{unit}"'] & \int_{\sqcup_m S^1} \mathcal{V} \arrow[r, "\mathcal{V}^{\otimes m}"'] & \mathrm{Vect} \end{tikzcd}$$

Ex: $\mathcal{A}($$)$: $\mathcal{V}^{\boxtimes 2} \to \mathcal{V}$ tensor product

$\mathcal{A}(\Sigma \text{ closed})$: $\mathrm{SkMod}(H, \cancel{\emptyset})$

2-m: nat tr.

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\cancel{\emptyset}"] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow \text{empty skein}"] & \int_\Sigma \mathcal{V} \arrow[r, "\mathrm{SkMod}(H)"] \arrow[d, "\mathrm{SkMod}(M)"] \arrow[dr, phantom, "\Nearrow W"] & \mathrm{Vect} \arrow[d, equal] \\ \mathrm{Vect} \arrow[r, "\cancel{\emptyset}"'] & \int_{\Sigma'} \mathcal{V} \arrow[r, "\mathrm{SkMod}(H')"'] & \mathrm{Vect} \end{tikzcd}$$

$$\mathcal{A}(M): \mathrm{SkMod}(H') \xrightarrow{W, \cancel{\emptyset}} \mathrm{SkMod}(H).$$

$\mathcal{V}$ (non-semisimple) modular

Conjecture: $\exists\ Z_{\mathcal{V}} : \mathrm{Bord}_4^{or} \to \mathrm{BrTens}$, $\mathcal{R}_{\mathcal{V}} : \mathrm{Bord}_3^{or,pc} \to \mathrm{BrTens}^{\to}$

such that: $Z_{\mathcal{V}}(\bullet^+) = \mathcal{V}$, $\quad \mathcal{R}_{\mathcal{V}}(\bullet^+) = \eta$

$t \circ \mathcal{R}_{\mathcal{V}} = Z_{\mathcal{V}}|_{\mathrm{Bord}_3}$

$\sqcup S^1 \xrightarrow{\sqcup D^2} \emptyset$

non-empty M conn'd ∂ M every c.c.

$\Sigma \xrightarrow{\text{Handlebody}} \emptyset$

$M \xrightarrow{W} \emptyset$

$\left.\right\} = \mathrm{Cob}_{321}^{\text{filled}, nc} \xrightarrow{\mathcal{A}} \Omega\Omega\mathrm{BrTens} \simeq \mathrm{Pr}_{\mathbb{R}}$

$\mathrm{Pr}_{\mathbb{R}}$ ← locally presentable linear cats.

$\mathcal{L} = \ker(i^*)$

$n = \sigma(W)$

$\sqcup S^1$

$(\Sigma, \mathcal{L} \subset H^1(\Sigma))$

$(M, n \in \mathbb{Z})$

$\left.\right\} = \widetilde{\mathrm{Cob}}_{321}^{nc} \xrightarrow[\text{DGGPR}]{\text{WRT}} \widehat{\mathrm{Cat}}_k$

$\widehat{\mathrm{Cat}}_k$ ← Cauchy-complete linear categories

Free

Conjecture

Thank you!