Separation of variables and analytic Langlands correspondence

Jörg Teschner

University of Hamburg, Department of Mathematics
and DESY

Hitchin's integrable system

Let C be a Riemann surface.

Hitchin moduli space $\mathcal{M}_{\mathrm{Hit}}(C)$: Moduli space of pairs $(\mathcal{E},\varphi)$,

- $\mathcal{E}$: holomorphic $G = SL(2)$-bundle on C,
- $\varphi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K)$.

$\mathcal{M}_{\mathrm{Hit}}(C)$ has canonical Poisson structure from Serre-duality between tangent space $T\mathrm{Bun}_G \simeq H^1(C, \mathrm{End}(\mathcal{E}))$ to Bun_G at $\mathcal{E}$ and $T^*\mathrm{Bun}_G \simeq H^0(C, \mathrm{End}(\mathcal{E}) \otimes K)$.

Integrability: Let $\theta := \frac{1}{2}\mathrm{tr}(\varphi^2) \in H^0(C, K^2)$. We have

$$\theta(z) = \sum_{r=1}^{3g-3+n} H_r Q_r(z), \qquad \{H_r, H_s\} = 0.$$

$\Rightarrow$ Hamiltonians H_r define integrable structure on $\mathcal{M}_{\mathrm{Hit}}(C)$.

Fibers of $\pi : \mathcal{M}_{\mathrm{Hit}}(C) \to H^0(C, K^2)$, $\pi(E,\varphi) = \theta$ are complex tori for generic θ.

Quantisation of the Hitchin system

Theorem: (Beilinson-Drinfeld)

There exist $K_{\mathrm{Bun}}^{1/2}$, and differential operators H_r, $r = 1, \dots, 3g-3+n$ on $K_{\mathrm{Bun}}^{1/2}$, having symbols which are global functions on $T^*\mathrm{Bun}_G$ satisfying

$$[\,\mathsf{H}_r, \mathsf{H}_s\,] = 0, \qquad r, s = 1, \dots, 3g-3+n.$$

Hitchin has proven that the algebra of global functions on $T^*\mathrm{Bun}_G$ is generated by the Hamiltonians H_r.

$\Longrightarrow$ Differential operators H_r represent a **quantisation** of the Hamiltonians H_r!

Beilinson and Drinfeld furthermore show that **eigenvalues** E_r of H_r, defined by

$$\mathsf{H}_r \Psi = E_r \Psi,$$

have a canonical relation with **functions on the space** $\mathrm{Op}(C)$ **of opers** on C:

Opers

Opers for case $\mathfrak{g} = \mathfrak{sl}_2$: Pairs $\chi = (\mathcal{E}, \nabla_\chi)$

- $\mathcal{E} = \mathcal{E}_{\rm op}$: Unique up to isomorphism extension

$$0 \to K^{\frac{1}{2}} \to \mathcal{E}_{\rm op} \to K^{-\frac{1}{2}} \to 0. \qquad \left[\text{transition functions } \sim \begin{pmatrix} \lambda_{\imath\jmath} & \partial_{z_\jmath}\lambda_{\imath\jmath} \\ 0 & \lambda_{\imath\jmath}^{-1} \end{pmatrix}\right].$$

- ∇_χ locally gauge equivalent to the form

$$\nabla_\chi = dz\left(\partial_z + \begin{pmatrix} 0 & t_\chi \\ 1 & 0 \end{pmatrix}\right), \quad \text{where } t \text{ transforms as } \textbf{projective connection},$$

$$t_\chi(z) = \big(\varphi'(z)\big)^2\, \tilde{t}_\chi\big(\varphi(z)\big) - \frac{1}{2}\{\varphi, z\}, \qquad \{\varphi, z\} := \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2.$$

Note that $q_\chi = t_\chi - t_0 \in H^0(C, K^2)$ for t_0 fixed, giving **coordinates on** $\mathrm{Op}(C)$:

$$q_\chi = \sum_{r=1}^{d} E_r(\chi)\theta_r, \qquad \{\theta_r; r = 1, \ldots, d\} \text{ basis for } H^0(C, K^2).$$

Geometric Langlands-correspondence

A special case of the geometric Langlands-correspondence (Beilinson-Drinfeld,...),

$$\boxed{{}^{\mathrm{L}}\mathfrak{g} \text{ local systems } (\mathcal{E}, \nabla) \text{ on } C} \quad \leftrightarrow \quad \boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G(C)}$$

with G: complex group[1] with Lie algebra $\mathfrak{g}$, ${}^{\mathrm{L}}\mathfrak{g}$: Langlands dual Lie algebra of $\mathfrak{g}$, can be represented as the **correspondence** between **opers** $\chi = (\mathcal{E}_{\mathrm{op}}, \nabla_\chi)$,

$$\nabla_\chi = dz\left(\partial_z + \begin{pmatrix} 0 & t_\chi \\ 1 & 0 \end{pmatrix}\right), \qquad t_\chi = t_0 + q_\chi, \qquad q_\chi = \sum_{r=1}^{d} E_r(\chi)\theta_r,$$

and the $\mathcal{D}$-**modules** represented by the eigenvalue equations

$$\mathsf{H}_r\Psi = E_r(\chi)\Psi.$$

[1] split, reductive

Single-valuedness condition

For given pair $(\chi, \bar{\chi})$, where $\chi \in \mathrm{Op}(X)$, $\bar{\chi} \in \overline{\mathrm{Op}}(X)$, we may consider the pair of complex conjugate eigenvalue equations

$$\begin{aligned} \mathsf{H}_r \, \Psi &= E_r(\chi)\Psi, \qquad & r &= 1, \ldots, d, \\ \bar{\mathsf{H}}_r \, \Psi &= \bar{E}_r(\bar{\chi})\Psi, \qquad & d &:= 3g - 3 + n. \end{aligned} \tag{1}$$

The Hitchin Hamiltonian have singularities at **wobbly** bundles admitting nilpotent Higgs fields. $\mathrm{Bun}_G^{\mathrm{vs}}$: moduli space of stable bundles which are not wobbly.

We may then look for smooth solutions on $\mathrm{Bun}_G^{\mathrm{vs}}$ locally of the form

$$\Psi(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{k,l} C_{kl} \psi_k(\mathbf{x}) \bar{\psi}_l(\bar{\mathbf{x}}), \qquad \begin{aligned} \mathsf{H}_r \, \Psi &= E_r(\chi)\Psi, \\ \bar{\mathsf{H}}_r \, \Psi &= \bar{E}_r(\bar{\chi})\Psi, \end{aligned} \qquad r = 1, \ldots, d,$$

($\mathbf{x} = (x_1, \ldots, x_d)$: local coordinates on $\mathrm{Bun}_G^{\mathrm{vs}}$) which are **single-valued**.

This requires that the nontrivial monodromies that the local sections $\psi_r(\mathbf{x})$ and $\bar{\psi}_s(\bar{\mathbf{x}})$ will generically have (e.g. around wobbly loci) cancel each other.

Separation of variables – making GL explicit

Theorem: There exist[2] explicit integral transformations of the form

$$\Psi(\mathbf{x},\bar{\mathbf{x}}) = \int d\mathbf{u}d\bar{\mathbf{u}}\; K(\mathbf{x},\bar{\mathbf{x}}\,|\mathbf{u},\bar{\mathbf{u}})\Phi(\mathbf{u},\bar{\mathbf{u}}), \tag{2}$$

$\mathbf{u} = (u_1,\dots,u_D)$, $\bar{\mathbf{u}} = (\bar{u}_1,\dots,\bar{u}_D)$, such that (1) is equivalent to

$$(\partial_{u_r}^2 - t(u_r))\phi(u_r,\bar{u}_r) = 0, \quad (\bar{\partial}_{\bar{u}_r}^2 - \bar{t}(\bar{u}_r))\phi(u_r,\bar{u}_r) = 0, \quad \forall r.$$

The proof generalises a paradigm from the theory of quantum integrable models called Separation of Variables (SOV; Sklyanin). It follows that

$$\Phi(\mathbf{u},\bar{\mathbf{u}}) = \prod_{r=1}^{D}\phi_r(u_r,\bar{u}_r), \qquad \begin{aligned}&(\partial_{u_r}^2 - t(u_r))\phi_r(u_r,\bar{u}_r) = 0,\\ &(\bar{\partial}_{\bar{u}_r}^2 - \bar{t}(\bar{u}_r))\phi_r(u_r,\bar{u}_r) = 0.\end{aligned} \tag{3}$$

The transformation (2) is invertible. Single-valuedness of the kernel K implies that single-valuedness of Φ and Ψ are equivalent. Single-valued solutions to (3) are unique up to normalisation, so $\phi_r(u,\bar{u}) = \phi(u,\bar{u})$ for all r.

[2] Sklyanin, Frenkel '95, Enriquez-Feigin-Roubtsov, Enriquez-Roubtsov, Felder-Schorr, Ribault-J.T., Frenkel-Gukov-J.T.,...., Dinh-J.T.

Single-valuedness

The separation of variables maps single-valued solutions of pairs of complex-conjugate Hitchin eigenvalue equations to

$$(\partial_u^2 - t(u))\phi(u,\bar{u}) = 0, \qquad (\bar{\partial}_{\bar{u}}^2 - \bar{t}(\bar{u}))\phi(u,\bar{u}) = 0.$$

It is possible to show that smooth solutions can be represented in the form

$$\phi(u,\bar{u}) = \chi(u)\cdot C\cdot \chi^\dagger(\bar{u}), \qquad \chi(u) = (\chi_1(u),\chi_2(u)), \qquad (\partial_u^2 - t(u))\chi(u) = 0.$$

The matrix C can be brought to diagonal form by a change of basis. It can be shown that up to changes of C induced by changes of basis the only choice of C which gives single-valued solutions $\phi(u,\bar{u})$ is $C = \mathrm{diag}(1,-1)$. This is single-valued if the monodromy is in $\mathrm{SU}(1,1)$, which is conjugate to $\mathrm{SL}(2,\mathbb{R})$.

Remark: May allow solutions defined up to a sign, corresponding to solutions $\phi(u,\bar{u})$ defined on a cover of C.

Real opers I

A **real** oper is an oper with real **monodromy**. Opers are in one-to-one correspondence to **projective structures** on C. Indeed, one may use the ratios

$$A(z)=\frac{\chi_1(z)}{\chi_2(z)},\quad \chi_i(z):\ \text{ linearly independent solutions of }\ (\partial_z^2-t(z))\chi_i(z)=0,$$

to define new local coordinates $w=A(z)$ on C. Features:

- Oper represented by $\tilde{t}(w)=0$,
- Changes of coordinates $w'(w)$: Möbius transformations

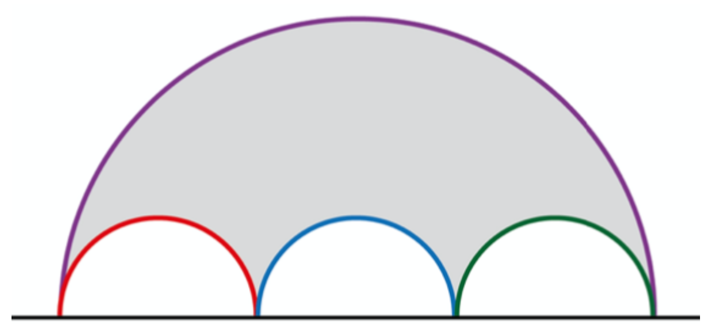
An atlas $\{U_\imath;\imath\in\mathcal{I}\}$ for C with transition functions represented by Möbius transformations defines a **projective structure**, allowing one to represent C as a cover of a fundamental domain in $\mathbb{C}$.

Example:

Monodromy Fuchsian

$\rightsquigarrow$ uniformisation

e.g. $C=C_{1,1}$:

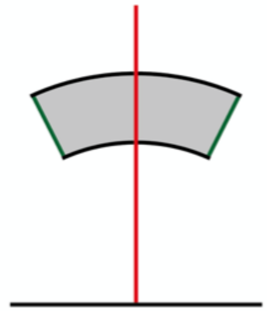


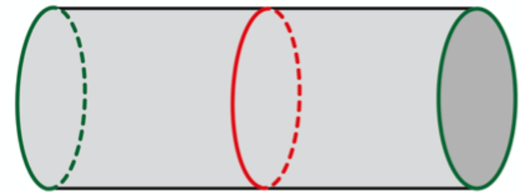
Real opers II – Grafting (Surgery on projective structures)

Grafting:

Cut figure on bottom

left open along γ,

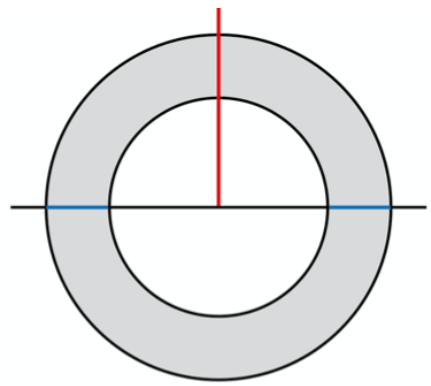


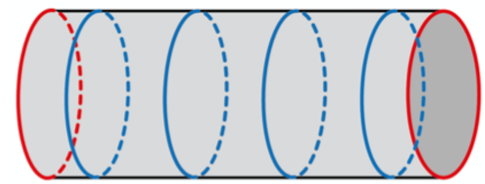


insert multiple copies

of the annulus,

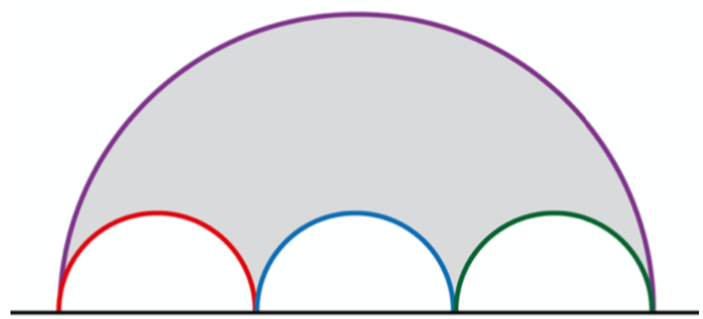
reglue along red curves.

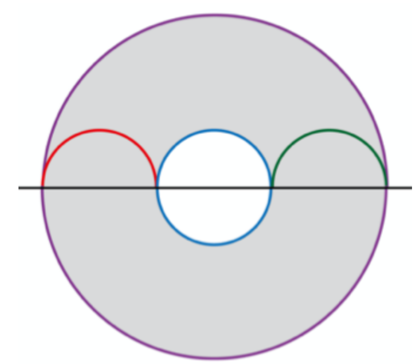




Result depicted in the

figure on bottom right.





Real opers III – Goldman's classification

All real opers can be constructed in this way:

Integer-measured laminations Λ: Homotopy classes of finite collections of simple non-intersecting noncontractible closed curves on S with integral weights, such that

- The weight of a curve is non-negative unless the curve is peripheral.
- A lamination containing a curve of weight zero is equivalent to the lamination with this curve removed.
- A lamination containing two homotopic curves of weights k and $l \sim$ a lamination with one of the two curves removed and weight $k+l$ assigned to the other.

The set of all such laminations on C is denoted as $\mathcal{ML}_C(\mathbb{Z})$. **Half**-integer measured laminations $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$ can be defined in an analogous way.

Theorem (Goldman '87): Real projective structures are in one-to-one correspondence to half-integer measured laminations $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$.

In other words: **All** real projective structures can be obtained from the uniformising projective structure by grafting along a $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$.

The work of Etingof, Frenkel and Kazhdan

The work[3] of Etingof, Frenkel and Kazhdan (EFK) deepens the story considerable by introducing harmonic analysis aspects.

Up to now: Solution to eigenvalue problem restricted to **single-valuedness** only.

EFK introduce a smooth algebraic moduli space $\mathrm{Bun}_G^{\mathrm{rs}}(C)$ by considering the stack of bundles $\mathrm{Bun}_G^{\circ}(C)$ with automorphisms in the center of G, and forgetting automorphisms. On $\mathrm{Bun}_G^{\mathrm{rs}}(C)$ introduce

- Line bundle of half-densities $\Omega_{\mathrm{Bun}}^{1/2} := |K_{\mathrm{Bun}}^{1/2}|$, where $|\mathcal{L}| = \mathcal{L} \otimes \bar{\mathcal{L}}$,
- $\mathcal{S}_G$ space of smooth compactly supported sections of $\Omega_{\mathrm{Bun}}^{1/2}$, and
- define a Hermitian form $\langle .,. \rangle$ by

$$\langle v, w \rangle := \int_{\mathrm{Bun}_G^{\mathrm{rs}}} v\, \bar{w}, \qquad v, w \in \mathcal{S}_G.$$

- **Hilbert space** $\mathcal{H}_G$: Completion of $\mathcal{S}_G$ with respect to $\langle .,. \rangle$.

[3] arXiv:1908.09677, arXiv:2103.01509, arXiv:2106.05243

The work of Etingof, Frenkel and Kazhdan II

EFK formulate a set of conjectures and prove them in some cases, leading to the following picture:

- The eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ generated by single-valued solutions to the pair of eigenvalue equations with eigenvalues $(\chi, \bar{\chi})$ are contained in $\mathcal{H}_G$, at most one-dimensional, and non-vanishing only if $\bar{\chi}$ is complex conjugate to χ.
- The Hilbert spaces $\mathcal{H}_G$ admit an orthogonal decomposition into the spaces $\mathcal{H}_{\chi,\bar{\chi}}$ (completeness)

EFK furthermore introduce a family of integral operators called Hecke operators, roughly of the form

$$(\mathbf{H}_\lambda f)(\mathcal{E}) := \int_{Z_\lambda(\mathcal{E},P')} q_1^*(f),$$

- $Z_\lambda(\mathcal{E}, P')$ – space of all possible λ-Hecke modifications of bundle $\mathcal{E}$ at point P', isomorphic to closure $\overline{\mathrm{Gr}}_\lambda$ of orbit $\mathrm{Gr}_\lambda = G[\![t]\!] \cdot \lambda(t) \,/\, G[\![t]\!]$.
- $q_1^*(f)$ pull-back of f under correspondence between bundles defined by Hecke modifications.

The work of Etingof, Frenkel and Kazhdan III

EFK conjecture and prove in some cases that

- the Hecke operators extend to a family of commuting compact normal operators on $\mathcal{H}_G$,
- $\mathcal{H}_G$ decomposes into eigenspaces of the Hecke operators,
- the eigenspaces coincide with the eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ of the Hitchin Hamiltonians,
- and the eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ are non-trivial only if χ is a real oper.

It seems quite remarkable that the conditions of single-valuedness and $\bar{\chi}$ being the complex conjugate of χ turn out to be equivalent to single-valuedness and square-integrability in this case.

The work of EFK uses many results from previous work on geometric Langlands as groundwork or input for the functional-analytic extension studied in their work.

Expect further fruitful interplay between algebro-geometric and analytic aspects.

Analytic Langlands correspondence: A brief summary

Analytic Langlands correspondence:

$$\boxed{\text{Real opers}} \quad \leftrightarrow \quad \boxed{\left\{\begin{array}{c}\text{single-valued}\\ L^2\text{-normalisable}\end{array}\right\} \text{Hitchin eigen-}\mathcal{D}\text{-modules}}$$

Intepretation from the perspective of integrable models:

Analytic Langlands	**Quantum Integrable models**
$\mathcal{D}$-modules	Local IM: Commuting Hamiltonians
Hecke operators	Non-local IM: Baxter Q-operators
Uniformising oper	Ground state
Grafting	**Creation operators**