Exhaustive Symbolic Regression

(or how to find the best function for your data)

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IST Lisbon

Overview

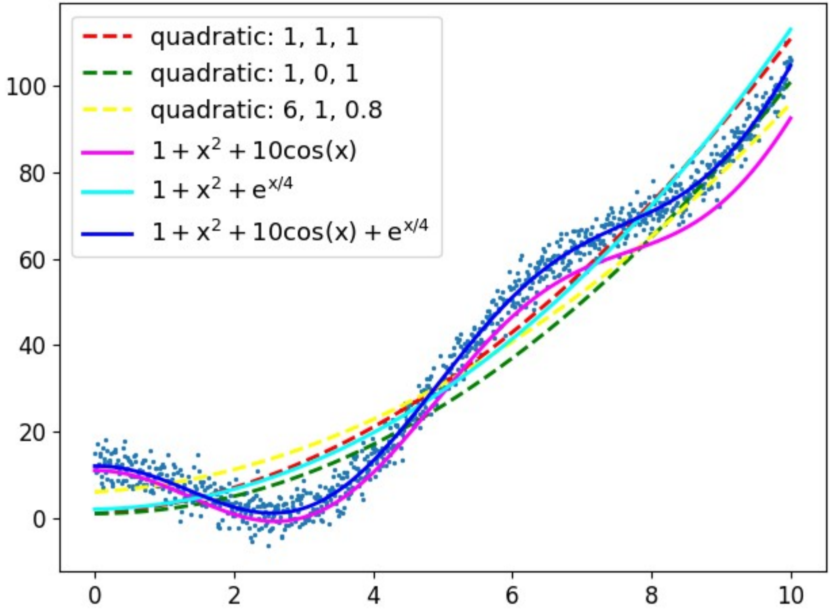
- Symbolic regression (SR)
- Exhaustive Symbolic Regression (ESR)
- Minimum description length as model selection principle
- Applications to date:
  - Cosmic expansion rate
  - Galaxy dynamics (radial acceleration relation)
- Upgrades in the works

Symbolic Regression overview

- Discover *functions* describing a dataset rather than parameters of predefined function

Numerical regression: $y = 6 + 1x + 0.8x^2$

Symbolic regression: $y = 1 + x^2 + 10\cos(x)$



Symbolic Regression overview

- Discover *functions* describing a dataset rather than parameters of predefined function
- Difficulties:
  - Larger search space makes convergence harder
  - Optimisation methods of numerical regression not applicable
- Advantages:
  - Much more general (reduces confirmation bias)
  - Easy to prevent overfitting
  - Highly interpretable

Numerical regression: $y = 6 + 1x + 0.8x^2$

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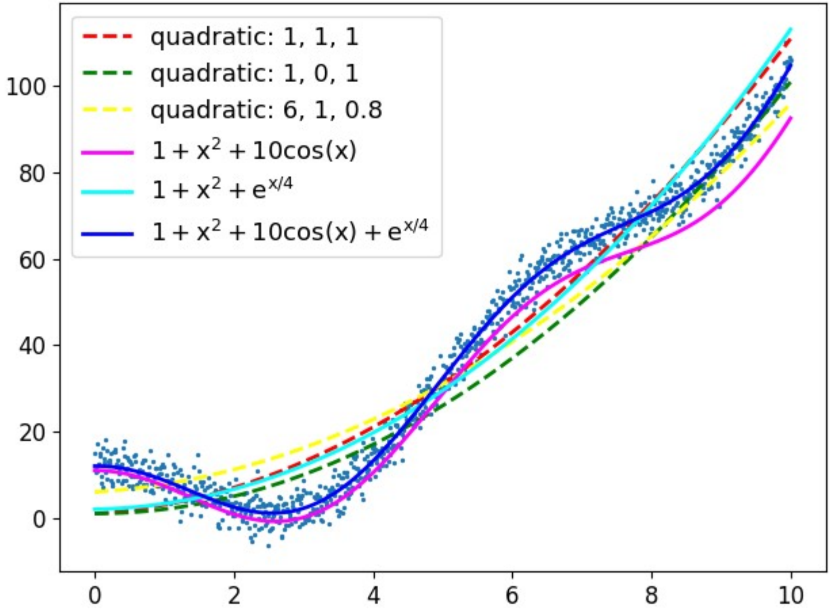
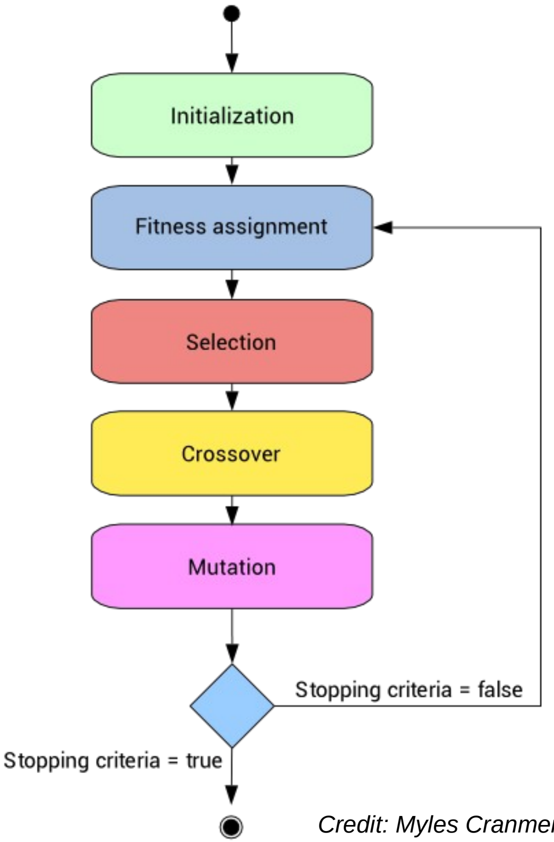
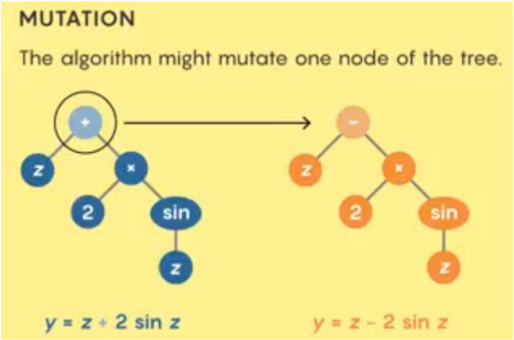


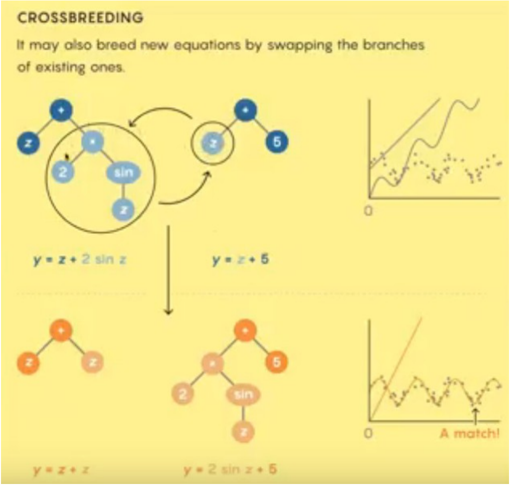
Traditional Symbolic Regression

I. Generating functions

Genetic Algorithm (e.g. *PySR*, *DataModeler*)







Credit: Myles Cranmer

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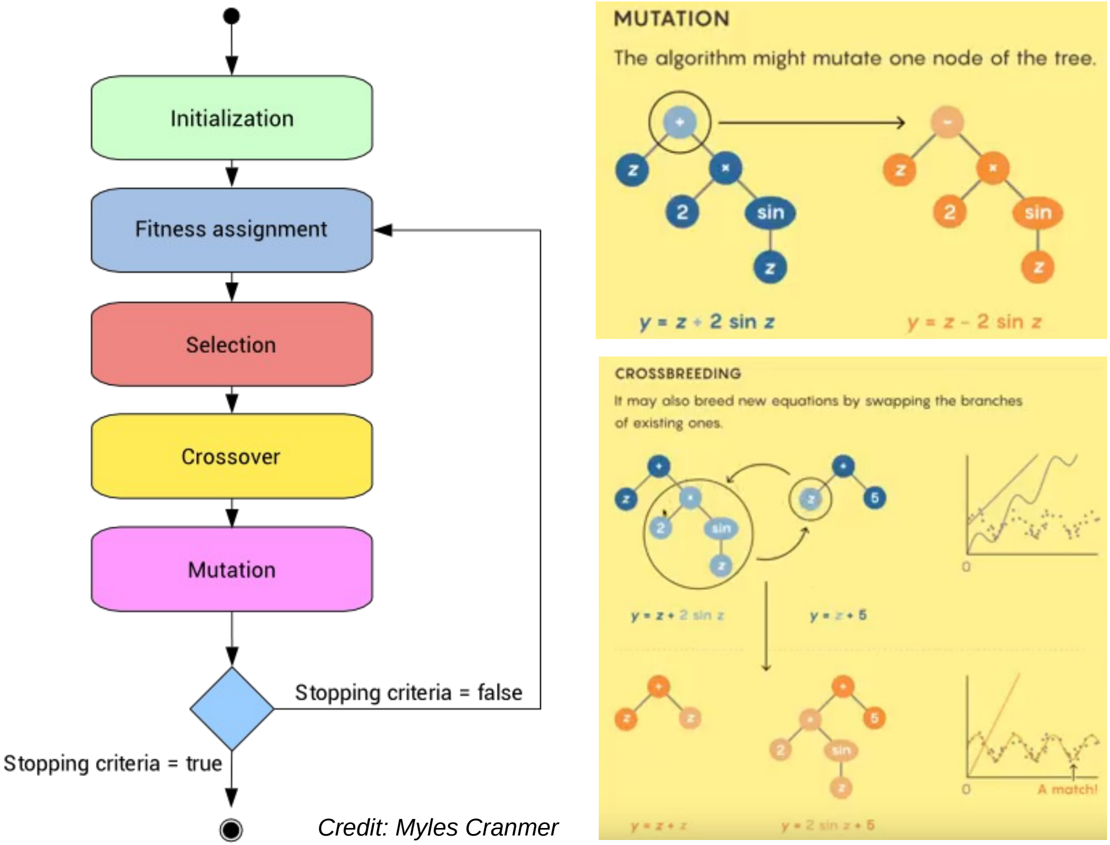

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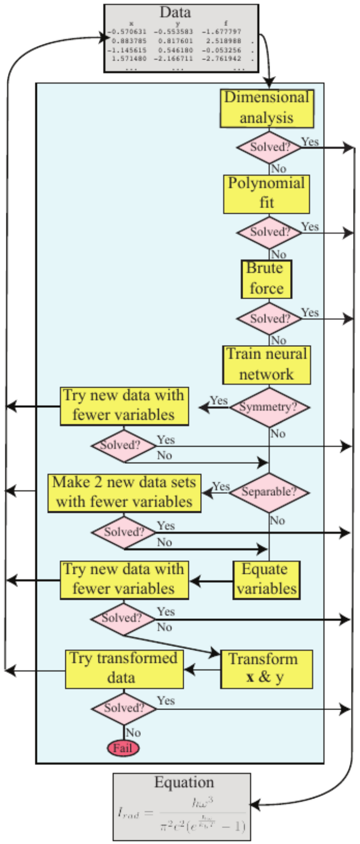
"Physics inspired" (e.g. *AI Feynman*)



Better for exact data with symmetries, worse with noise

Unlike GA, doesn't produce entire population of possible equations

Udrescu & Tegmark 2020

Traditional Symbolic Regression

II. Assessing functions

- Problem: Can always get 0 error with some (very complex) function
- Solution: *two* objectives, accuracy and simplicity
- The best equations are the ones that cannot be made more accurate without also being made more complex

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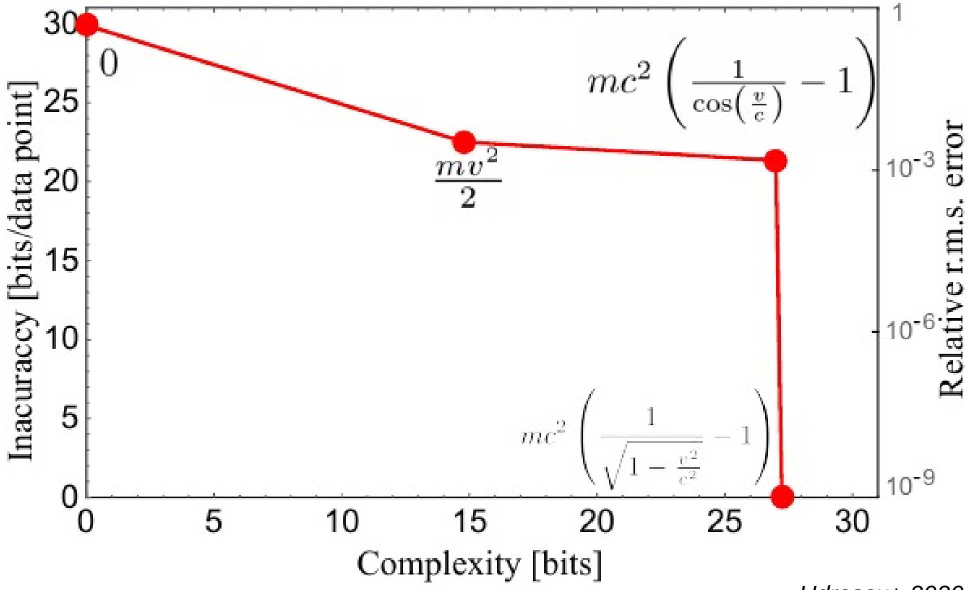

Udrescu+ 2020

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II. Assessing functions

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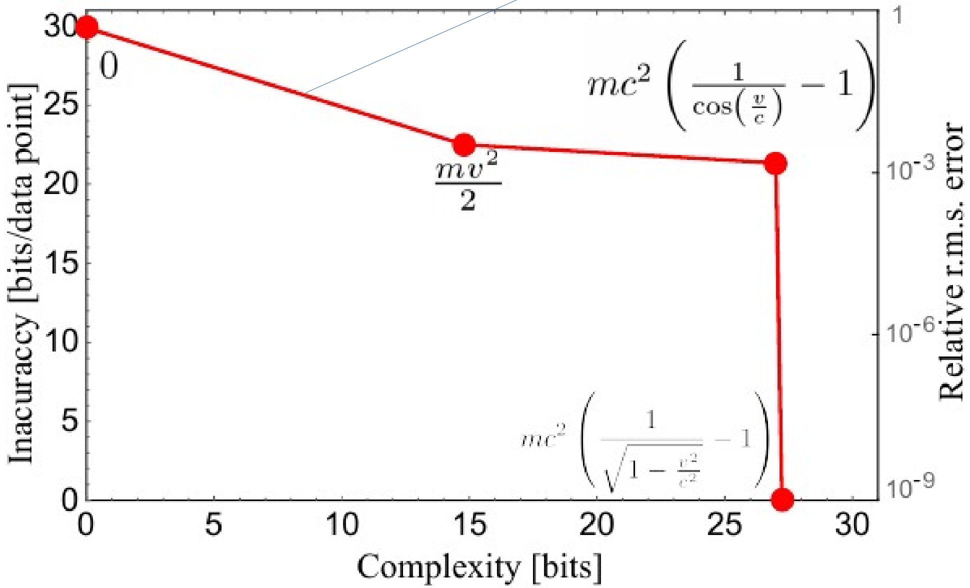
- The best equations are the ones that cannot be made more accurate without also being made more complex ("Pareto-optimal")



Udrescu+ 2020

Exhaustive Symbolic Regression

Designed to overcome two problems:

- Stochastic method may fail to find any given function

- Typical accuracy definitions fail to account for data uncertainties, and complexity definition is largely arbitrary. The two are incommensurable.

Exhaustive Symbolic Regression

Designed to overcome two problems:

- Stochastic method may fail to find any given function

  → Search exhaustively, complexity-by-complexity

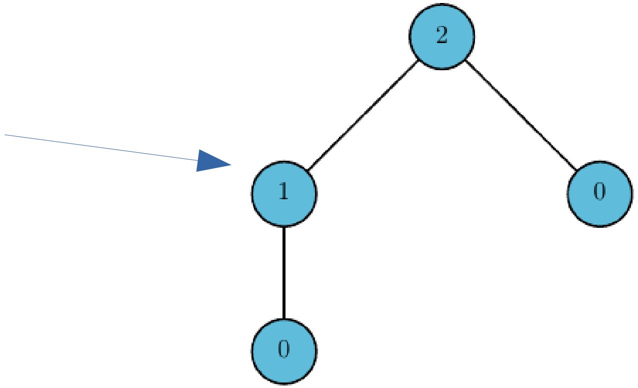
- Typical accuracy definitions fail to account for data uncertainties, and complexity definition is largely arbitrary. The two are incommensurable.

  → Use *Minimum Description Length (MDL) principle*

Exhaustive Symbolic Regression

I. Function generation & optimisation

1) Generate all possible trees with given complexity = #nodes, with placeholder operators labelled by arity (number of arguments to operator)



2) Decorate with all operator permutations

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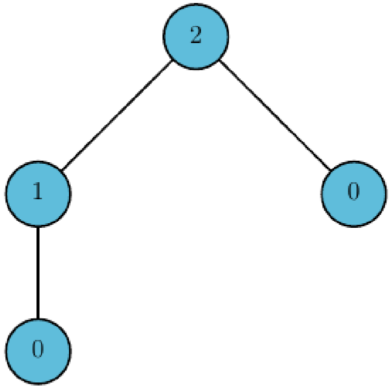
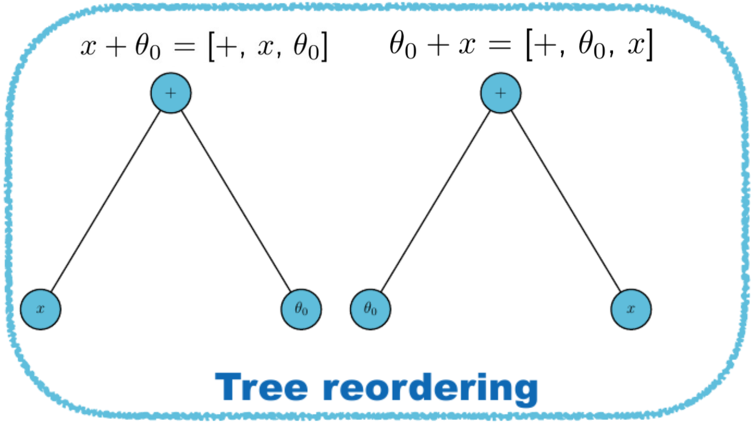


2) Decorate with all operator permutations

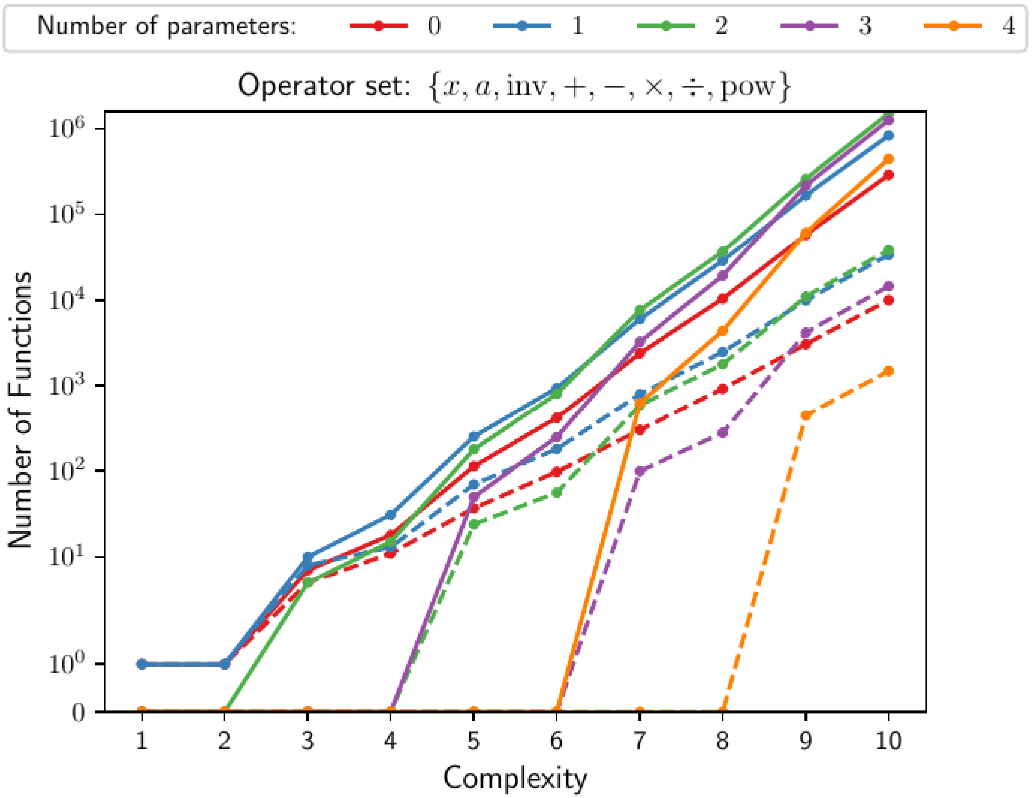
3) Simplify and remove duplicates (*tree reordering*, *parameter permutations*, *simplifications*, *reparametrisation invariance*, *parameter combinations*)



4) Calculate maximum-likelihood parameter values

5) Repeat for other desired complexities

Simplifications make an exhaustive search feasible



168 million

Naïve estimate: $\sum_{j=1}^{n} j^k = H_n^{(-k)}$

5.2 million

Total number of decorated trees

134,234

Number of unique equations

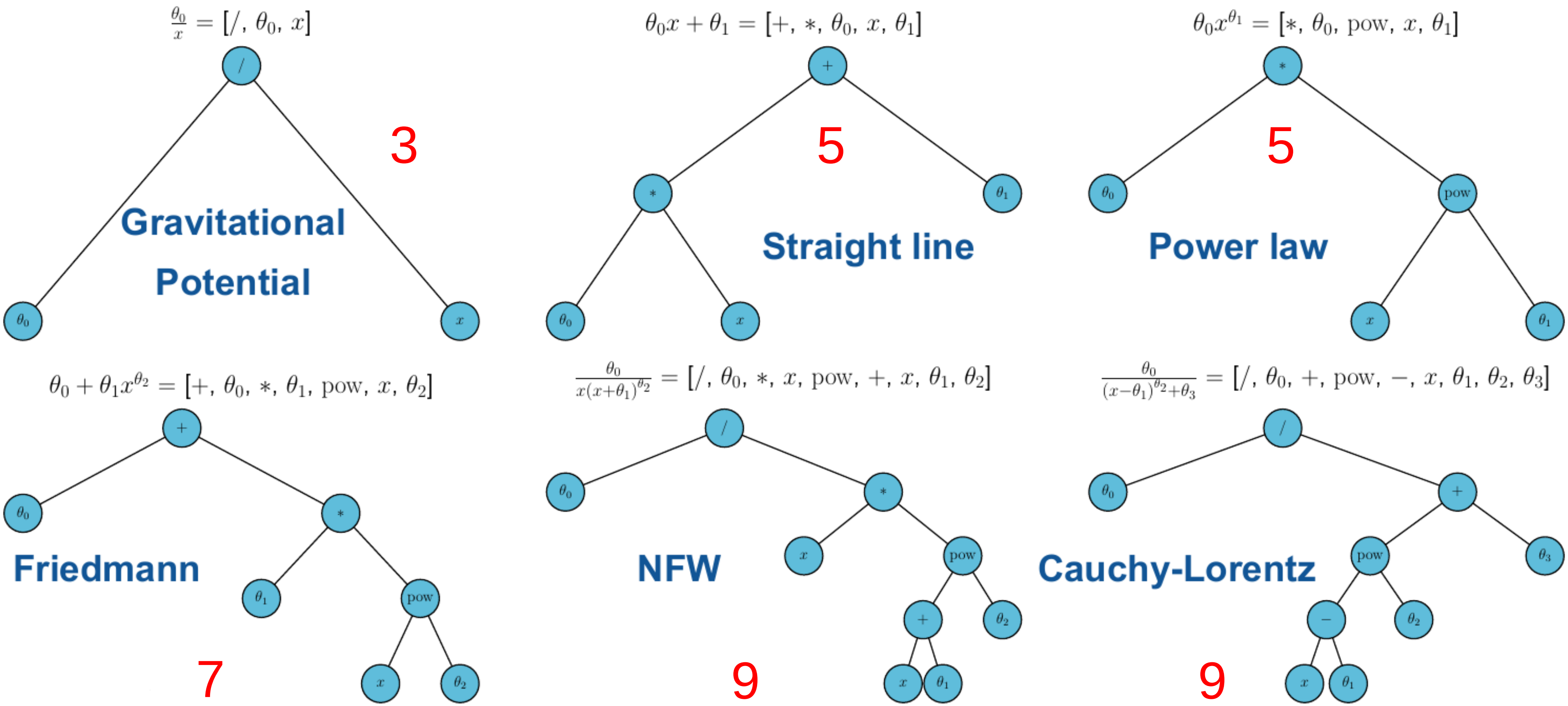
119,861

Number of equations containing at least one parameter

1400

Times fewer equations to consider than our naïve guess

Many physics functions have complexity < 10



Exhaustive Symbolic Regression

II. Model selection principle: *minimum description length*

$$L(D) = L(H) + L(D \mid H)$$

Description length

Hypothesis

Residuals

- Purpose of functional fit is *data compression*
- Most information-efficient function has minimum *L(D)*

Exhaustive Symbolic Regression

II. Model selection principle: *minimum description length*

$$L(D) = L(H) + L(D\,|\,H)$$

Description length ($L(D)$) — Hypothesis ($L(H)$) — Residuals ($L(D\,|\,H)$)

$$L(D) = -\log(\mathscr{L}(\hat{\theta})) + k\log(n) + \sum_i^p \log(|\hat{\theta}_i|/\Delta_i) + p\log(2) + \sum_j \log(c_j)$$

Shannon-Fano coding ($-\log(\mathscr{L}(\hat{\theta}))$)

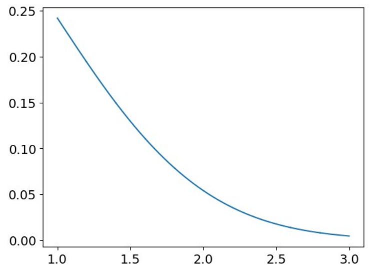
k nodes with n basis functions needs $\log\left(n^k\right)$ nats ($k\log(n)$)

Send p parameters, each with precision Δ_i ($\sum_i^p \log(|\hat{\theta}_i|/\Delta_i) + p\log(2)$)

Integer constants ($\sum_j \log(c_j)$)

- Purpose of functional fit is *data compression*
- Most information-efficient function has minimum *L(D)*
- Both accuracy and complexity expressed in nats → can be combined
- Accounts for both functional and parametric complexity. Accuracy is likelihood.

Test case 0: Benchmarking



- *feynman_I_6_2a* dataset from the *SRBench 2022 Competition*

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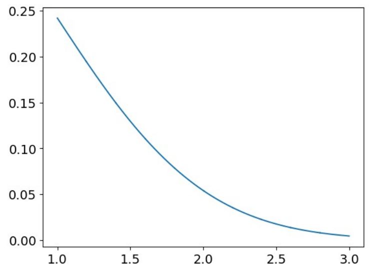

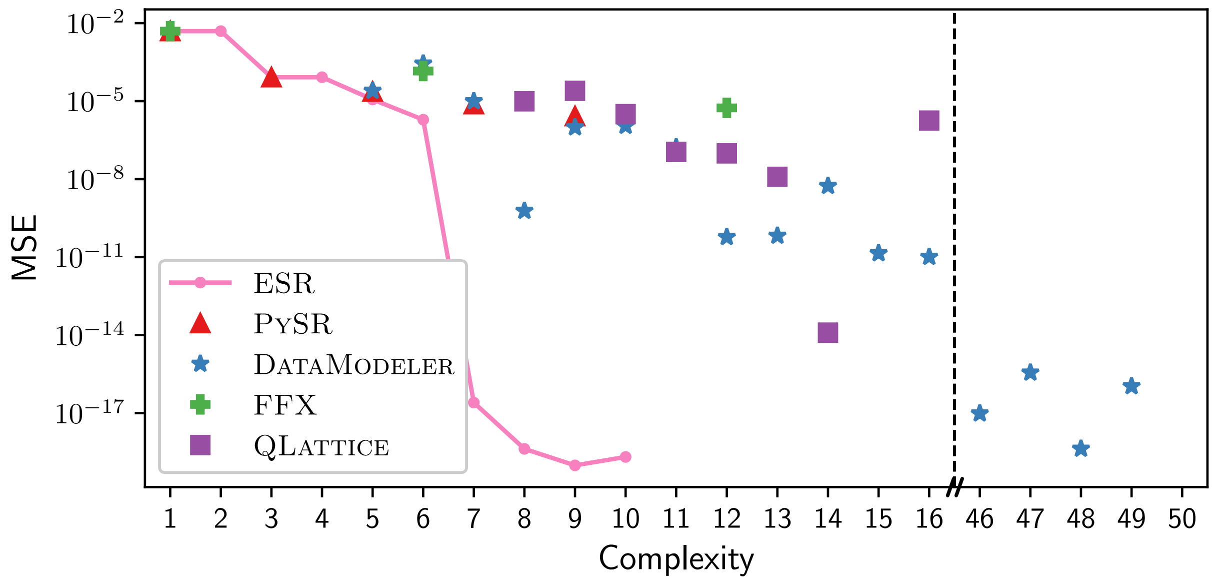

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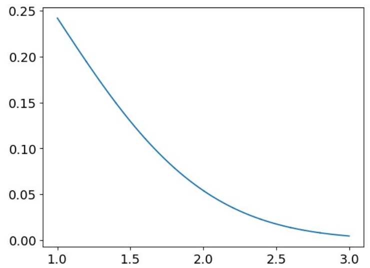

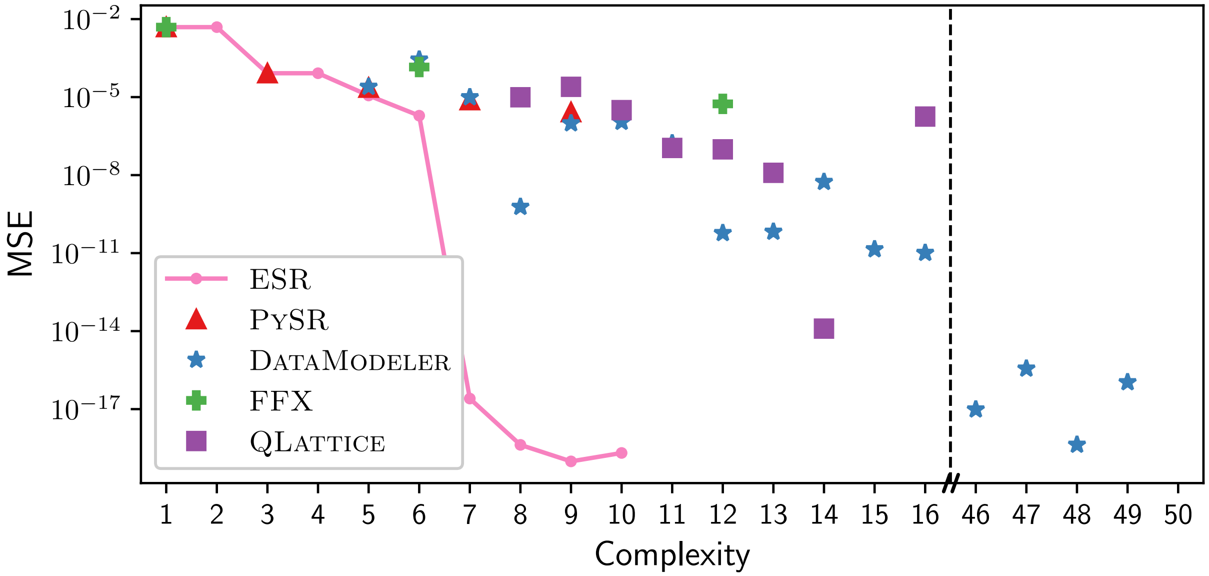

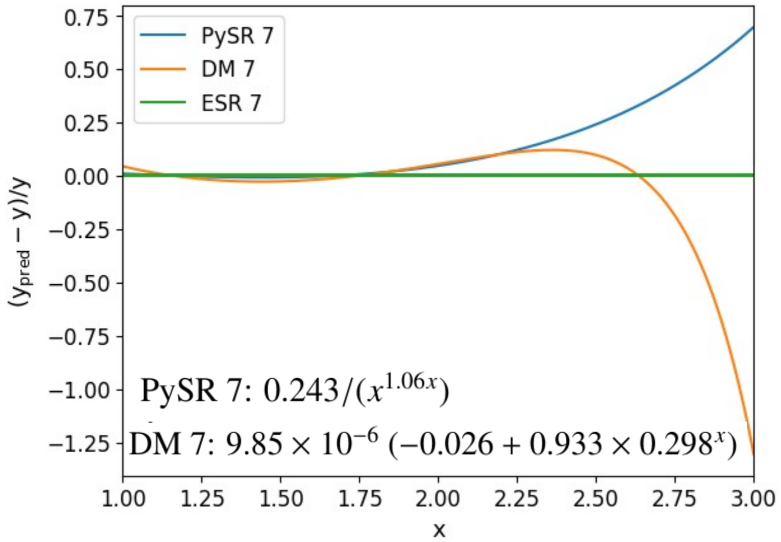

$$y = \theta_1 \theta_0^{x^2}$$
$$\theta_0 = 0.6065$$
$$\theta_1 = 0.3989$$

- *feynman_I_6_2a* dataset from the *SRBench 2022 Competition*
- Not only does ESR get by far the lowest error...

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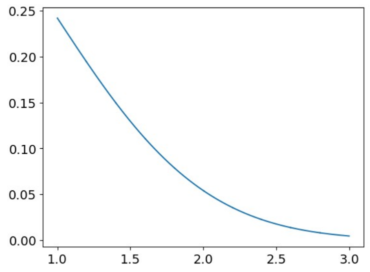
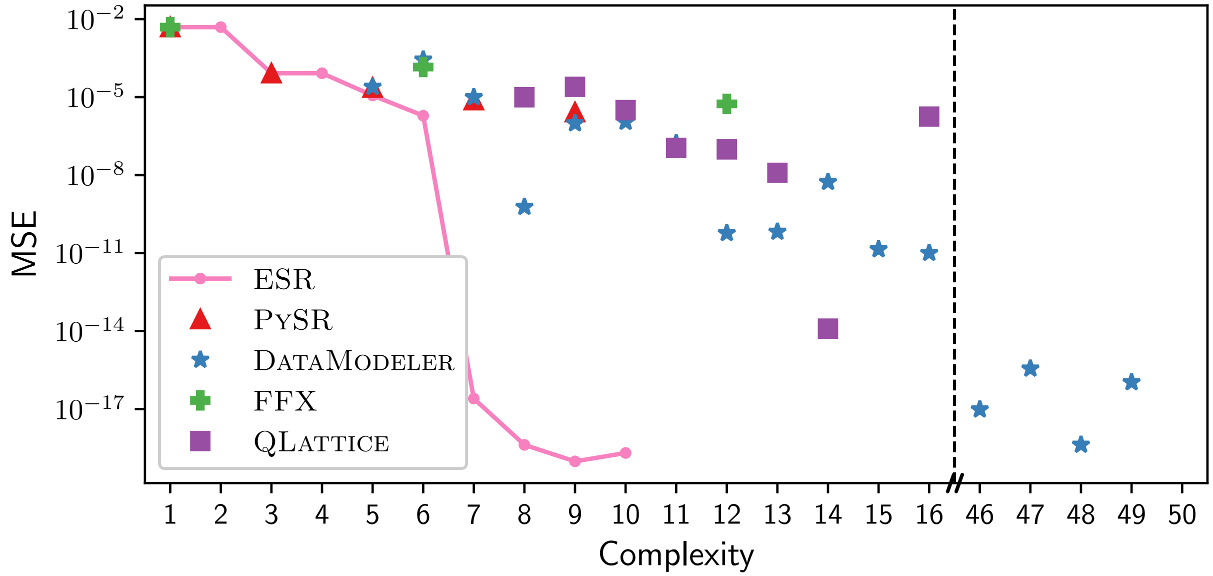
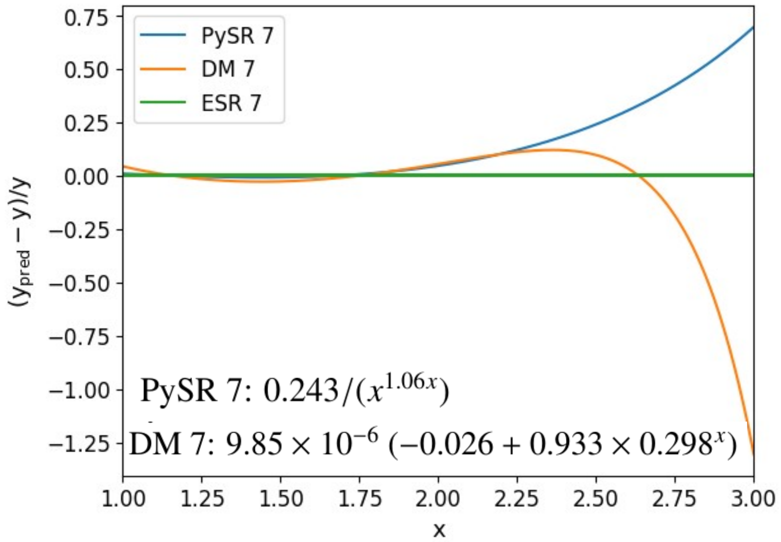






- *feynman_I_6_2a* dataset from the *SRBench 2022 Competition*
- Not only does ESR get by far the lowest error... it discovers the standard normal!

$$y = \theta_1 \theta_0^{x^2}$$
$$\theta_0 = 1/\sqrt{e}$$
$$\theta_1 = 1/\sqrt{2\pi}$$

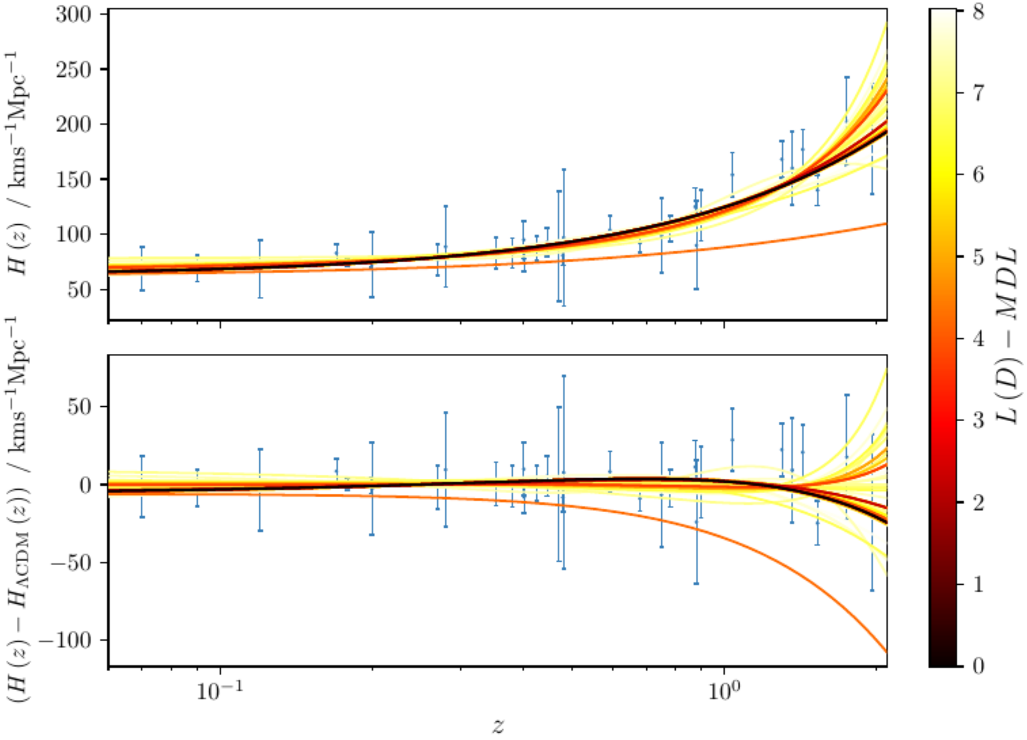
Test case 1: The law of cosmic expansion

- Can we determine the functional form of cosmic expansion without assuming ΛCDM?
- How good is the Friedmann equation relative to other functions?

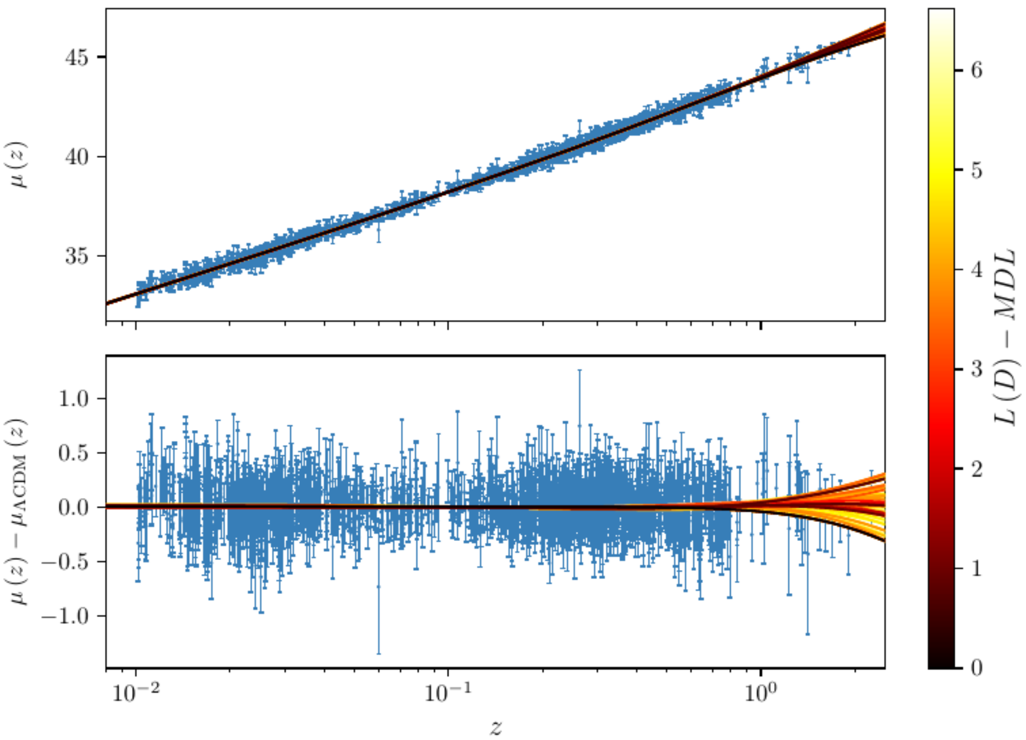
$$H(z)^2_{\Lambda\mathrm{CDM}} = \theta_0 + \theta_1\,(1+z)^3 \qquad H(z)^2_{\Lambda\mathrm{fluid}} = \theta_0 + \theta_1\,(1+z)^{\theta_2}$$

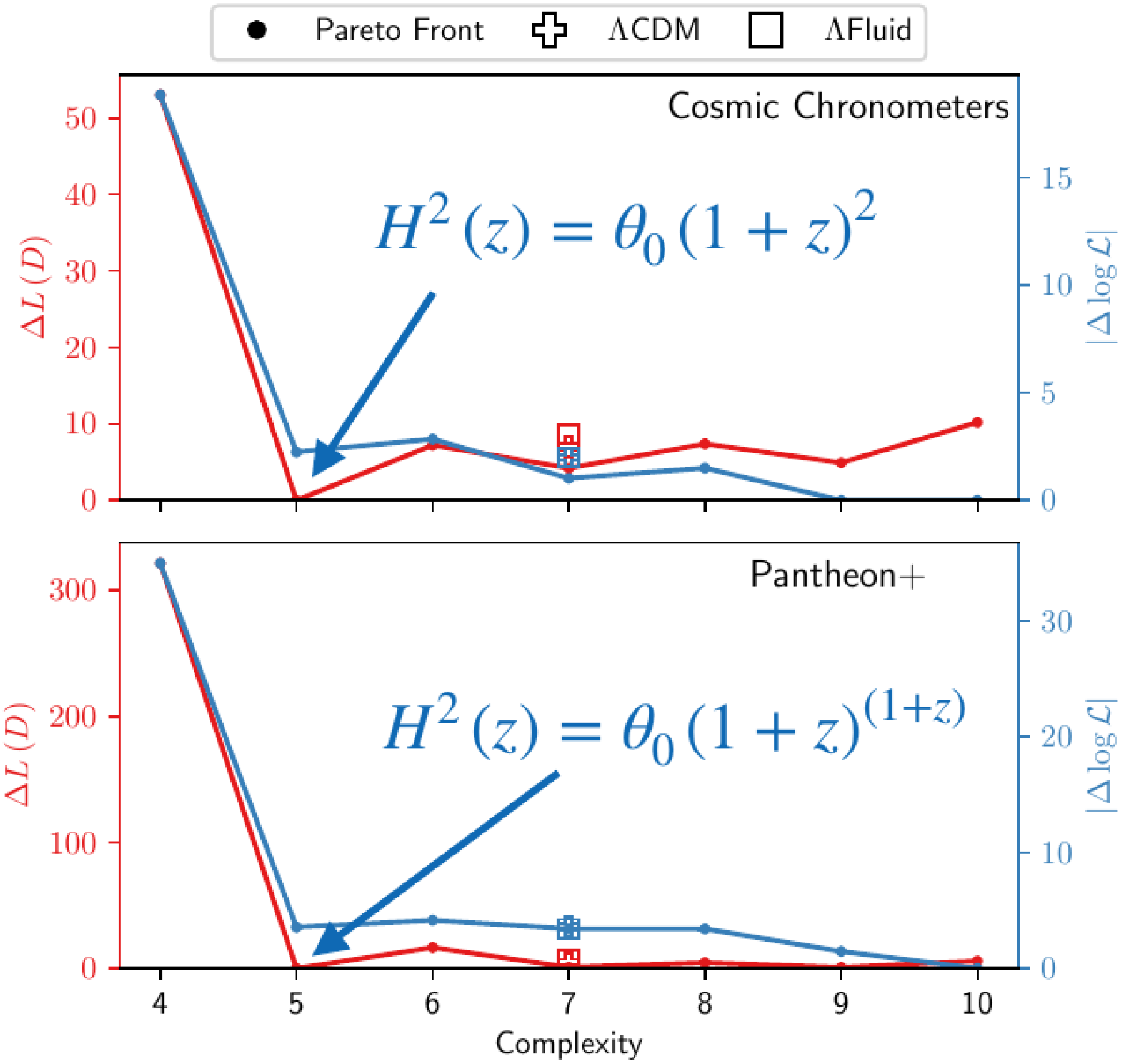
- Data:
  - Cosmic chronometers (32 data points) (Moresco et al 2022)
  - Type 1a Supernovae (1590 data points) (Pantheon+, Scolnic et al 2021)
- Basis operators: $\{x \equiv 1+z, \theta, \mathrm{inv}, +, -, \times, \div, \mathrm{pow}\}$

Cosmic Chronometers



Pantheon+ Supernovae





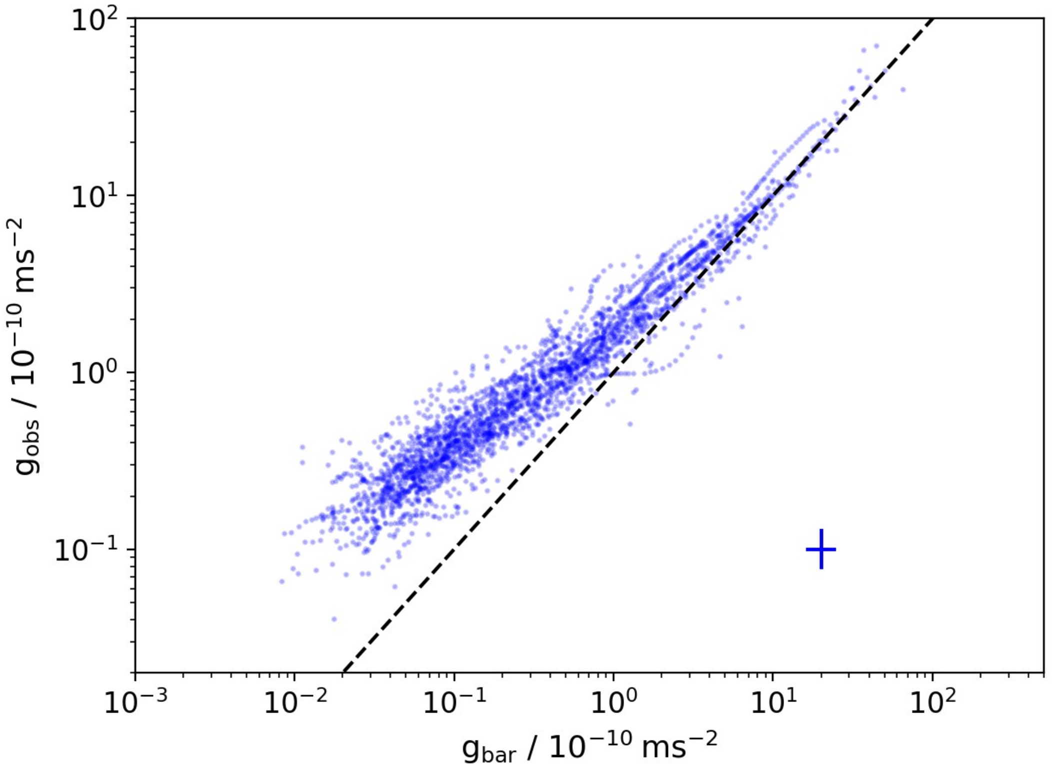
- ΛCDM ranked 39th for cosmic chronometers and 37th for SNe

- Best functions approximate ΛCDM at low z, but are simpler

- ~200 functions (up to complexity 10) more accurate than ΛCDM for Pantheon+

Test case 2: The radial acceleration relation

- Relates acceleration sourced by baryons (g_{bar}) to total acceleration as measured by rotation velocity (g_{obs})
- 2,696 points from 153 late-type galaxies (SPARC sample)
- Regularity and low scatter hard to understand in ΛCDM



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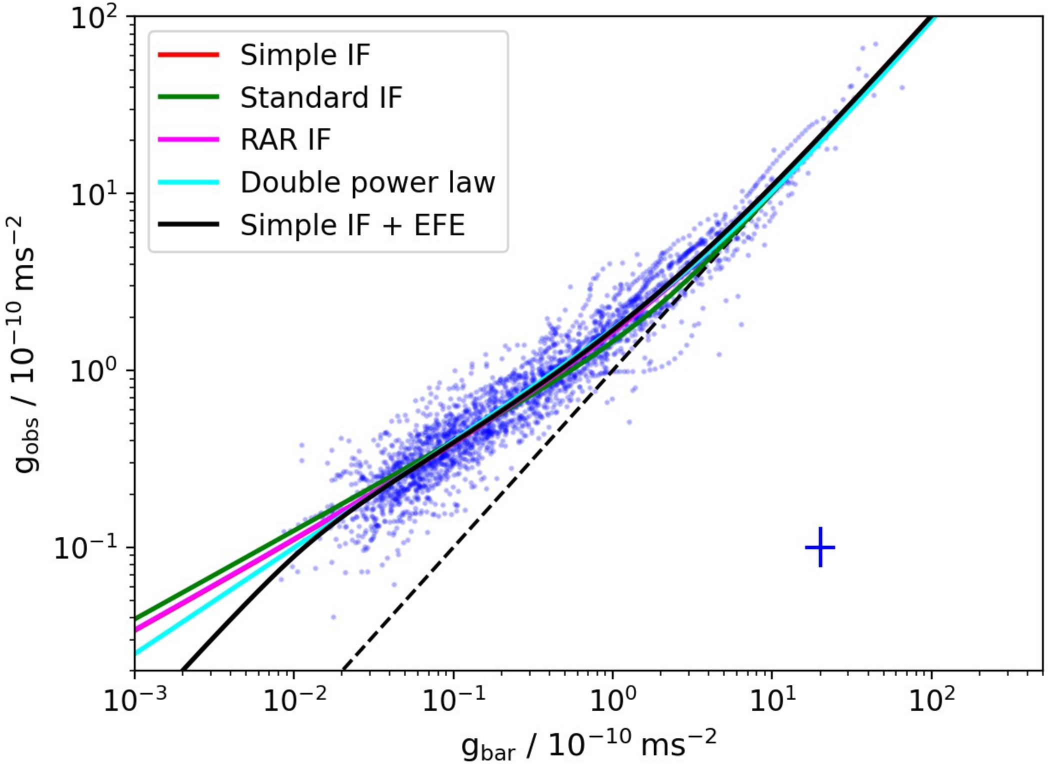
<u>MOND Interpolating Functions (IFs)</u>

Simple — $g_{obs} = g_{bar}/2 + \sqrt{g_{bar}^2/4 + g_{bar}g_0}$

Standard — $g_{obs} = \frac{1}{\sqrt{2}}\sqrt{g_{bar}^2 + \sqrt{g_{bar}^2\left(g_{bar}^2 + 4g_0^2\right)}}$

RAR — $g_{obs} = g_{bar}/(1 - \exp(-\sqrt{g_{bar}/g_0}))$



1) Are the MOND IFs optimal descriptions of the RAR?

2) Do optimal solutions satisfy the MOND limits (and hence may be considered new IFs)?

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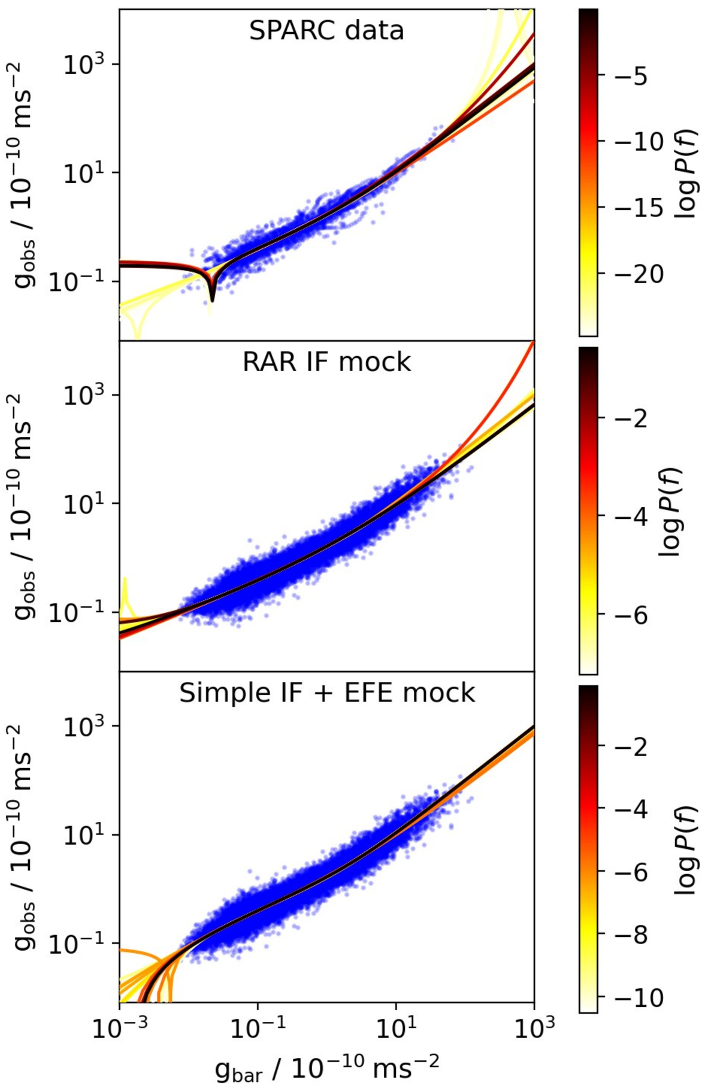

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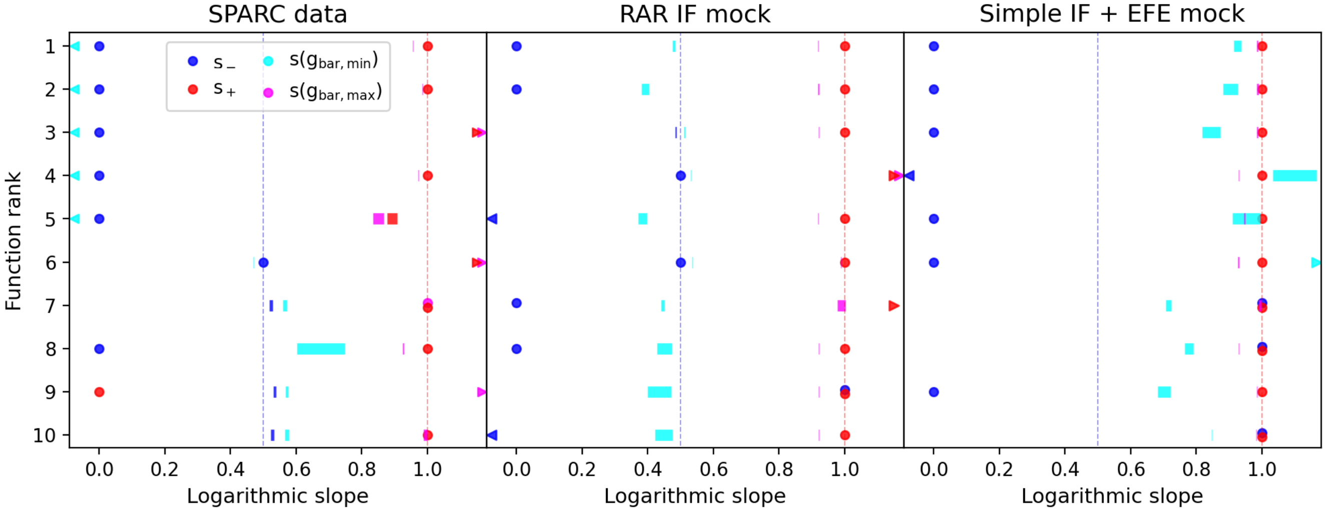

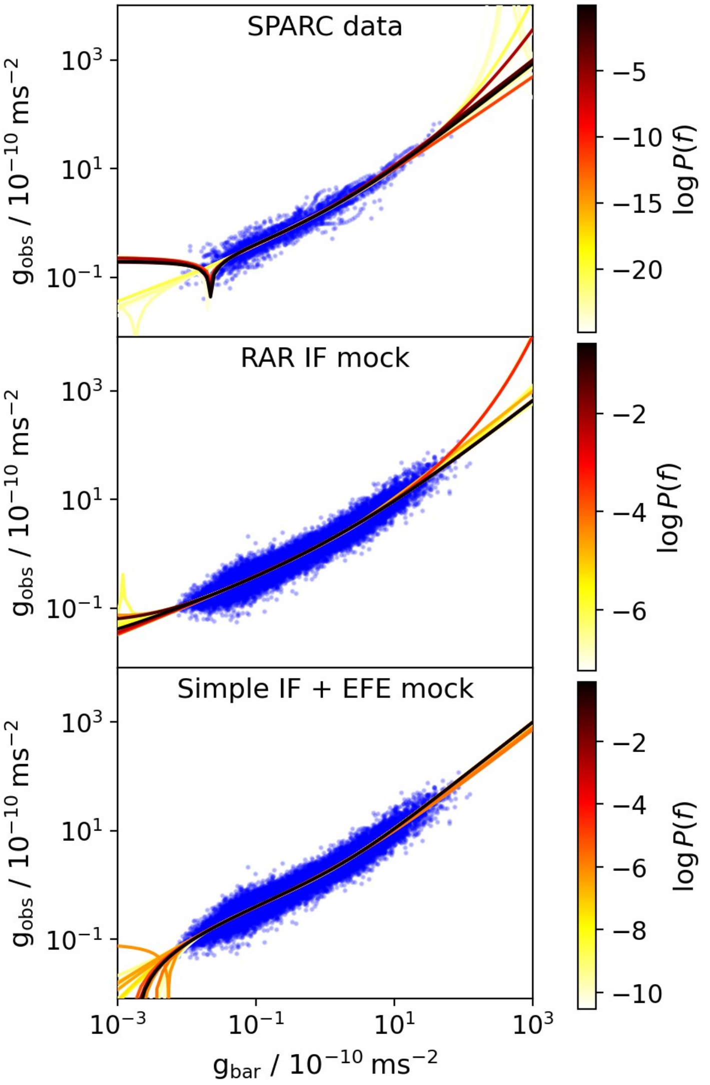

- Newtonian limit often found; deep-MOND limit rarely
- Can’t recover MOND behaviour even from MOND mocks!
→ Uncertainties and dynamic range of data insufficient

Upgrades I

- In a fully Bayesian formulation, compare the evidence:

$$P(f_i|D) = \frac{1}{P(D)} \int P(D|f_i, \boldsymbol{\theta}_i)\, P(\boldsymbol{\theta}_i|f_i)\, P(f_i)\, \mathrm{d}\boldsymbol{\theta}_i \qquad \log P(f_i|D) = -\log P(f_i) - \log \mathcal{Z}(D|f_i)$$

$$\log \mathcal{Z}(D|f_i) \simeq \log H(D, f_i, \hat{\boldsymbol{\theta}}_i) + \frac{p}{2}\log 2\pi - \frac{1}{2}\log\left|\det \hat{\boldsymbol{I}}^H\right| \simeq \frac{p}{2}\log 2\pi - BIC/2 \quad \left(\hat{I}^H_{\alpha\beta} = -\partial_\alpha \partial_\beta \log H(D, f_i, \boldsymbol{\theta}_i)|_{\hat{\theta}_i}\right)$$

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To overcome prior-dependence, use the *Fractional Bayes Factor*:

$$B_b = \frac{q_1(b)}{q_2(b)}, \quad q_i(b) = \frac{\int P\left(D|f_i, \boldsymbol{\theta}_i\right) P\left(\boldsymbol{\theta}_i|f_i\right) \mathrm{d}\boldsymbol{\theta}_i}{\int P\left(D|f_i, \boldsymbol{\theta}_i\right)^b P\left(\boldsymbol{\theta}_i|f_i\right) \mathrm{d}\boldsymbol{\theta}_i}$$

Upgrades II

- A prior that captures physicists' expectations for operator combinations

    Currently klog(n), but want sin(x0)+sin(x1) to be *a priori* more likely than sin(sin(x0+x1))...

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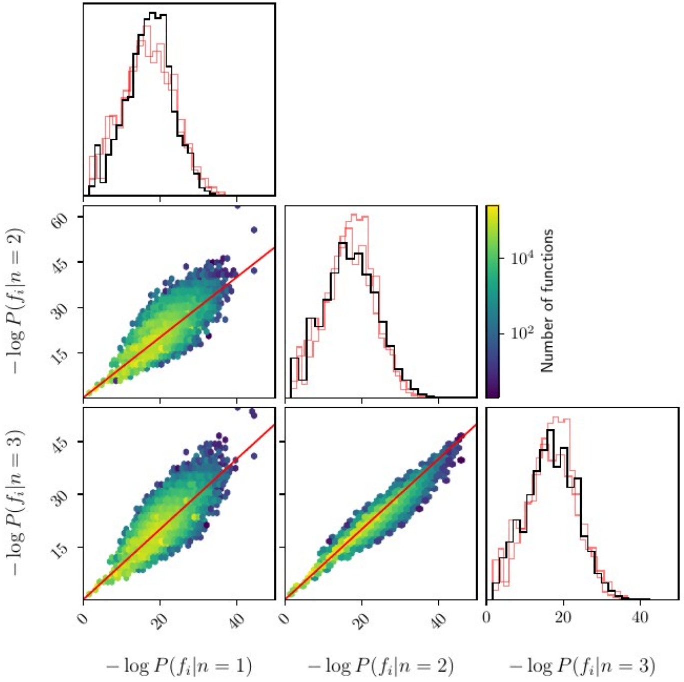
  “Katz back-off model” determines probability of next operator given *n* preceding operators based on a training set of equations (from Feynman’s *Lectures on Physics*)



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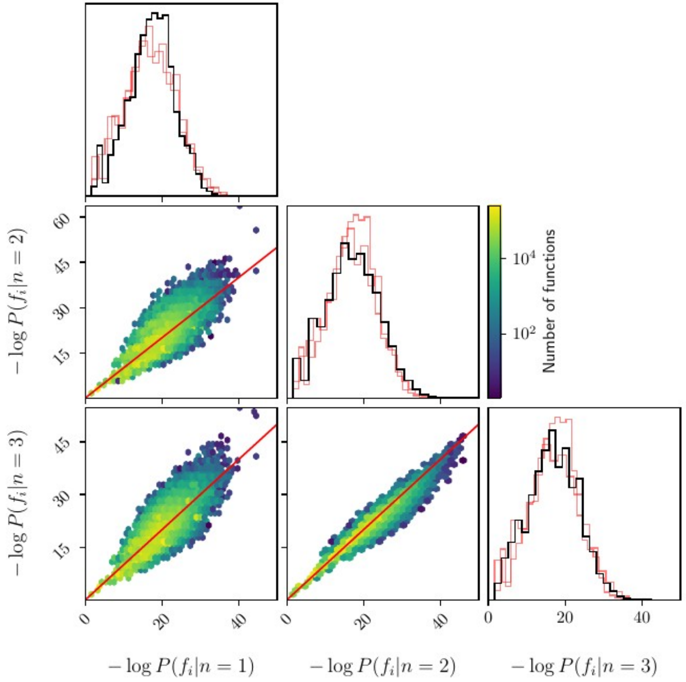

- Reaching higher complexity:

  Starting with the best ESR functions, evaluate all "unit edits" to the function to follow a low-description-length path out to higher complexity

Conclusions

- *Exhaustive Symbolic Regression*: Guaranteed to find the best simple function for any dataset
- *Minimum description length* affords principled combination of accuracy and simplicity for model comparison
- Cosmic chronometers and supernovae don't uniquely favour ΛCDM
- The radial acceleration relation doesn't uniquely favour MOND
- Improvements and many more applications — including yours!

https://github.com/DeaglanBartlett/ESR https://zenodo.org/record/7339113

Extra Slides

Precision of constants - tradeoff between accuracy and information needed

True $\hat{\theta}_i$ uniformly distributed within $\pm\Delta_i/2$ of transmitted value - may not transmit true MLP

Taylor expand log-likelihood about true value:

$$-\log(\mathscr{L}(\hat{\theta}+\mathbf{d})) \approx -\log(\mathscr{L}(\hat{\theta})) + \frac{1}{2}\mathbf{d}^{\mathrm{T}}\,\mathrm{I}\,\mathbf{d} \qquad \mathrm{I}_{ij} = \left.\frac{\mathrm{d}^2(-\log\mathcal{L})}{\mathrm{d}\theta_i\,\mathrm{d}\theta_j}\right|_{\hat{\theta}}$$

Giving expected contribution to description length

$$L(\mathbf{\Delta}) = \frac{1}{2}\sum_{ij}\langle \mathrm{I}_{ij} d_i d_j\rangle - \sum_i \log(\Delta_i)$$

Minimise this:

$$L(\Delta_i) = \frac{1}{24}\mathrm{I}_{ii}\Delta_i^2 - \log(\Delta_i) \quad \Longrightarrow \quad \Delta_i = \left(\frac{12}{\mathrm{I}_{ii}}\right)^{1/2}$$

The Description Length of a Function

$$L(D) = -\log(\mathscr{L}(\hat{\theta})) + k\log(n) - \frac{p}{2}\log(3) + \sum_i^p \left(\frac{1}{2}\log(\mathrm{I}_{ii}) + \log(|\hat{\theta}_i|)\right)$$

$$\frac{1}{2}\left(p\log(N) - 2\log(\mathscr{L})\right) = \frac{1}{2}\mathrm{BIC}$$

(for large number of data points, N)

- Description length looks like BIC plus corrections due to structural complexity (prior on model)
- For large N, equivalent to minimising the BIC (an approximation to the evidence)

$$P(H) = \exp(-L(D)) / \sum(\exp(-L(D)))$$

Cosmic Chronometers - Standard Clocks

$$H(z) = -\frac{1}{1+z}\frac{\mathrm{d}z}{\mathrm{d}t} \approx -\frac{1}{1+z}\frac{\delta z}{\delta t}$$

Image credit: AAS NOVA

Jimenez & Loeb 2001 (arXiv:astro-ph/0106145)
Moresco 2015 (arXiv:1503.01116)
Moresco et al . 2016 (arXiv:1601.01701)
Ratsimbazafy et al. 2017 (arXiv:1702.00418)
Stern et al. 2010 (arXiv:0907.3149)
Simon et al. 2004 (arXiv:astro-ph/0412269)
Borghi et al. 2021 (arXiv:2110.04304)
Zhang et al. 2012 (arXiv:1207.4541)
Moresco et al. 2012 (arXiv:1201.3609)
Moresco et al. 2022 (arXiv:2201.07241)

- Passively evolving stellar population are standard clocks - measure δt
- Directly measure δz
- (Cosmological) model-independent measurement of $H(z)$
- We use sample of 32 CC $H(z)$ measurements

Type Ia Supernovae - Standard Candles

$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H(z')}$$

$$\mu(z) = 5\log_{10}\left(\frac{d_L(z)}{10\,\mathrm{pc}}\right)$$

$$\mu = m_B + \alpha x_1 - \beta c - M_0$$

Amplitude (m_B)

Stretch (αx_1)

Colour (βc)

Rest-frame magnitude (M_0)

Pantheon+ sample with SH0ES Cepheid calibration

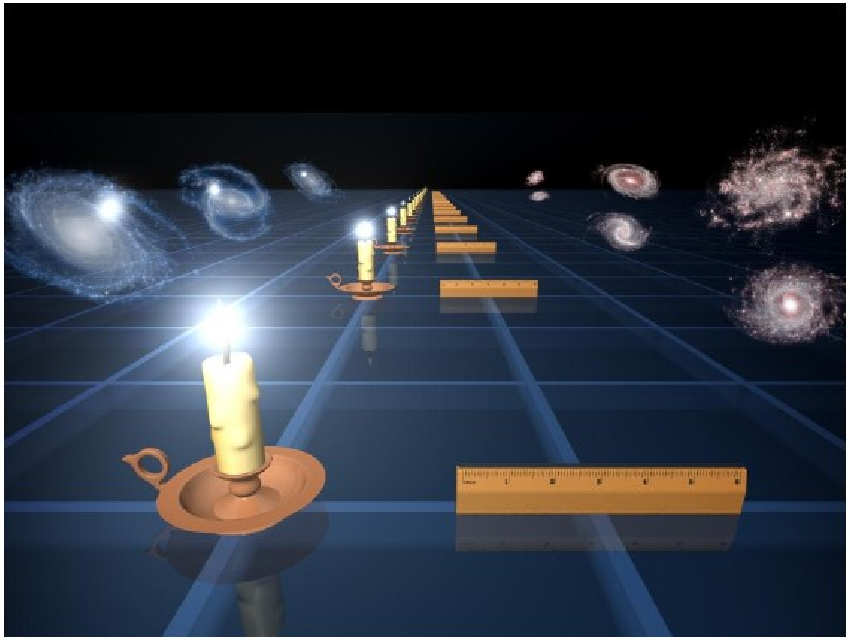

Image credit: NASA

Scolnic et al. 2021 (arXiv:2112.03863)
Riess et al. 2022 (arXiv:2112.04510)

CCs

Rank	$y(x)$ / $\mathrm{km^2 s^{-2} Mpc^{-2}}$	Complexity	Parameters			Codelength			
			θ_0	θ_1	θ_2	Residuals[1]	Function[2]	Parameter[3]	Total
1	$\theta_0 x^2$	5	3883.44	-	-	8.36	5.49	2.53	16.39
2	$\lvert\theta_0\rvert^{x^{\theta_1}}$	5	3982.43	0.22	-	7.97	5.49	5.24	18.70
3	$\theta_0 \lvert\theta_1\rvert^{-x}$	5	1414.43	0.31	-	7.57	6.93	5.58	20.08
4	$\theta_0 x^{\theta_1}$	5	3834.51	2.03	-	8.35	6.93	5.08	20.36
5	$x^2(\theta_0 + x)$	7	3881.85	-	-	8.36	9.70	2.53	20.60
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
39	$\theta_0 + \theta_1 x^3$	9	3164.02	1481.71	-	7.28	12.48	3.76	23.51
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
84	$\theta_0 + \theta_1 x^{\theta_2}$	7	3322.96	1374.97	3.08	7.27	11.27	6.52	25.06

[1] $-\log \mathcal{L}(\hat{\boldsymbol{\theta}})$ [2] $k\log(n) + \sum_j \log(c_j)$ [3] $-\frac{p}{2}\log(3) + \sum_i^p (\frac{1}{2}\log(I_{ii}) + \log(|\hat{\theta}_i|))$

SNe

Rank	$y(x)$ / $\mathrm{km^2 s^{-2} Mpc^{-2}}$	Complexity	Parameters			Codelength			
			θ_0	θ_1	θ_2	Residuals[1]	Function[2]	Parameter[3]	Total
1	$\theta_0 x^x$	5	5345.02	-	-	706.18	6.93	5.11	718.22
2	$\lvert\theta_0\rvert^{x^{\theta_1}}$	9	5280.11	0.16	-	705.11	5.49	8.41	719.01
3	$\theta_0 \lvert\theta_1\rvert^{-x}$	5	1694.95	0.32	-	701.79	6.93	10.33	719.05
4	$\theta_0 x^{x^{\theta_1}}$	7	5378.69	0.78	-	702.45	9.70	6.98	719.13
5	$\lvert\theta_0\rvert^{\lvert\theta_1\rvert^x}$	5	1898.47	1.14	-	701.88	5.49	12.64	720.02
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	$\theta_0 + \theta_1 x^3$	9	3591.09	1773.63	-	701.85	12.48	8.81	723.13
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
96	$\theta_0 + \theta_1 x^{\theta_2}$	7	3280.83	2069.32	2.73	701.64	11.27	12.19	725.10

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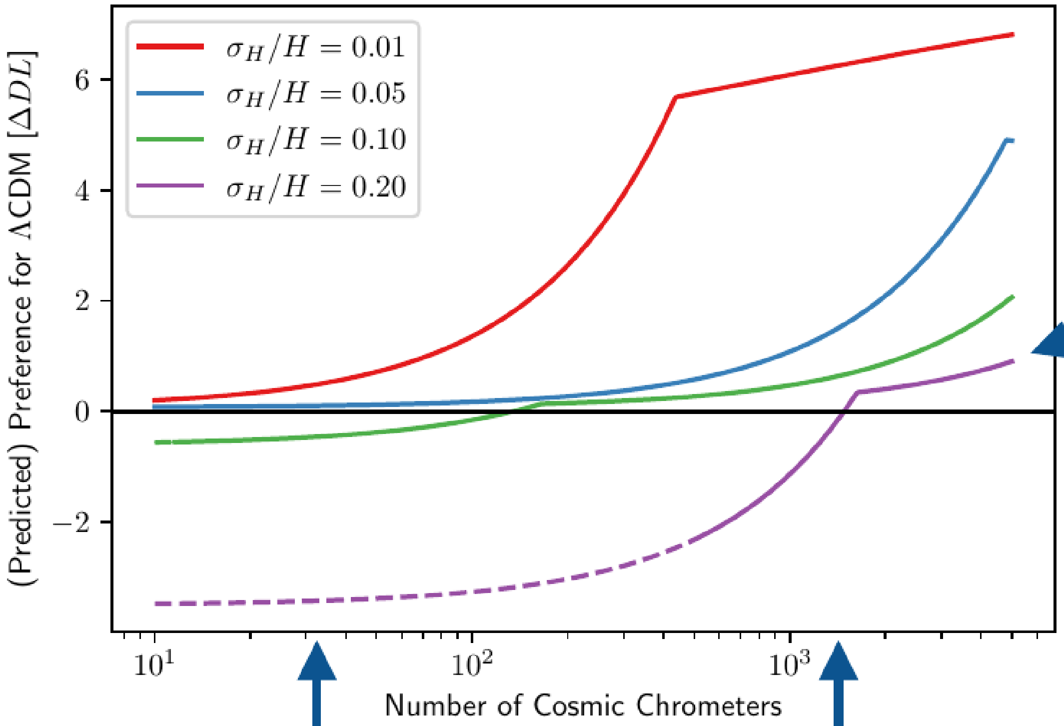
Should we have seen ΛCDM? No.

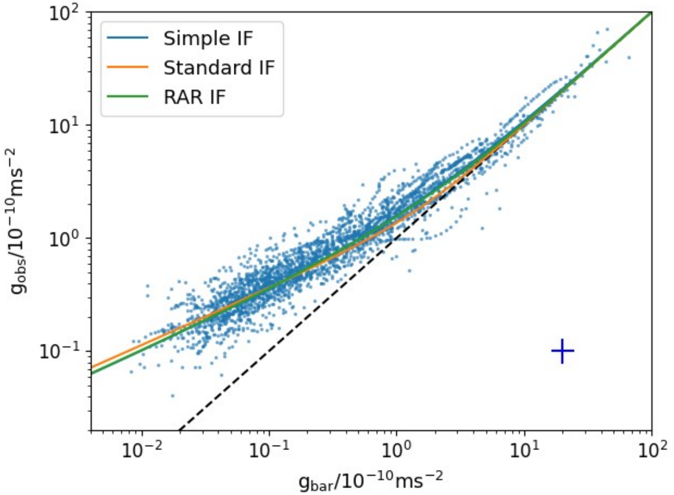
Mock cosmic chronometer data assuming ΛCDM (Planck18)



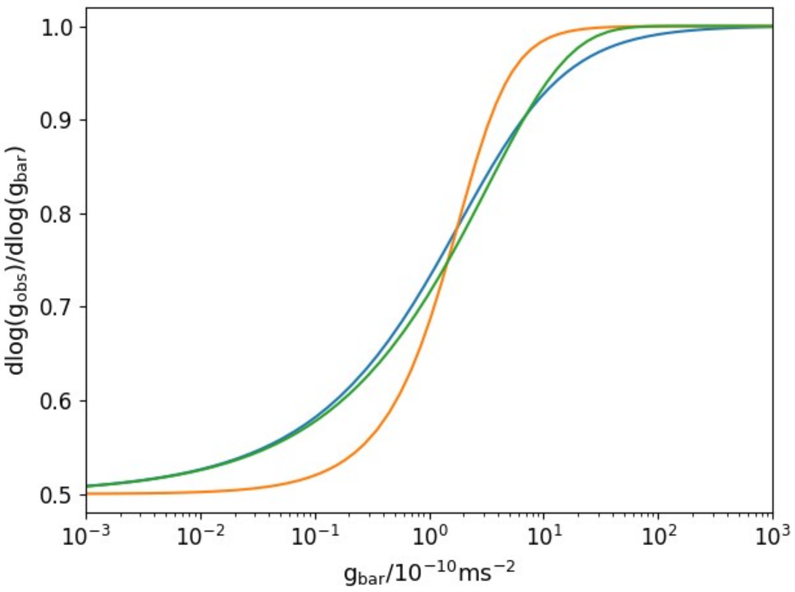
Current observational uncertainties

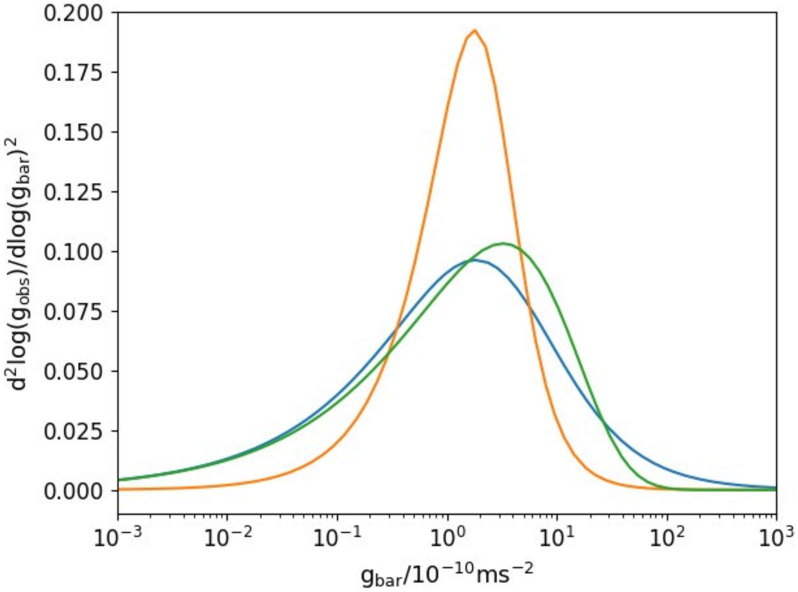
We used this many

Would need this many



First logarithmic derivative





Second logarithmic derivative

SPARC data

Rank	Function	Comp.	$P(f)$	Parameters				Description length			
				θ_0	θ_1	θ_2	θ_3	Resid.[1]	Func.[2]	Param.[3]	Total
1	$\theta_0\left(\lvert\theta_1 + x\rvert^{\theta_2} + x\right)$	9	9.3×10^{-1}	0.84	-0.02	0.38	—	-1279.1	14.5	14.0	-1250.6
2	$\lvert\lvert\theta_1\rvert^{x} + \theta_0\rvert^{\theta_2} + x$	9	6.4×10^{-2}	-0.99	0.64	0.36	—	-1279.9	12.5	19.6	-1247.9
3	$\lvert\theta_0\rvert^{\lvert\theta_1 - x\rvert^{\theta_2} - \theta_3}$	9	2.0×10^{-3}	-1.4×10^{2}	0.02	0.14	0.89	-1276.4	12.5	19.5	-1244.4
4	$\lvert\theta_0(\theta_1 + x)\rvert^{\theta_2} + x$	9	1.4×10^{-4}	0.35	-0.02	0.34	—	-1268.9	14.5	12.7	-1241.7
5	$\left\lvert\theta_0 - \lvert\theta_1 - x\rvert^{\theta_2}\right\rvert^{\theta_3}$	9	1.0×10^{-5}	-0.30	0.02	0.42	2.14	-1271.1	12.5	19.5	-1239.1
6	$\sqrt{x}\exp\left(\frac{\lvert\theta_0 + x\rvert^{\theta_1}}{2}\right)$	9	1.5×10^{-9}	-0.02	0.36	—	—	-1257.9	17.5	10.0	-1230.3
7	$\left(\frac{\lvert\theta_0\rvert^{x}}{x}\right)^{\theta_1} + x$	9	2.4×10^{-10}	1.87	-0.52	—	—	-1250.6	14.5	7.6	-1228.5
8	$\sqrt{\lvert\theta_0 + x\rvert} + \theta_1 x$	8	1.8×10^{-10}	-1.8×10^{-3}	0.72	—	—	-1245.6	12.9	4.5	-1228.2
9	$\left\lvert\theta_0 + \frac{1}{\sqrt[4]{x}}\right\rvert^{\theta_1}$	8	9.6×10^{-11}	-0.22	-2.14	—	—	-1251.1	14.3	9.2	-1227.6
10	$\left(\sqrt{x} + \frac{1}{x}\right)^{\theta_0} + x$	9	8.2×10^{-11}	-0.53	—	—	—	-1248.3	16.1	4.8	-1227.4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	$x/(\exp(\theta_0) - \lvert\theta_1\rvert^{\sqrt{x}})$	9	2.2×10^{-11}	0.03	0.44	—	—	-1250.9	17.5	7.3	-1226.1
—	Double power law	11	9.7×10^{-16}	4.65	3.96	1.03	0.60	-1252.3	17.7	18.5	-1216.1
—	Simple IF	10	5.5×10^{-25}	1.11	—	—	—	-1217.3	18.6	3.9	-1194.8
—	RAR IF	9	6.7×10^{-26}	1.13	—	—	—	-1212.8	16.1	3.9	-1192.7
—	Simple IF + EFE	59	5.0×10^{-69}	1.16	6.8×10^{-3}	—	—	-1238.9	139.9	5.6	-1093.4
—	Standard IF	14	9×10^{-150}	1.54	—	—	—	-939.5	27.9	4.1	-907.5

[1] $-\log\mathcal{L}(\hat{\boldsymbol{\theta}})$ [2] $k\log(n) + \sum_j \log(c_j)$ [3] $-\frac{p}{2}\log(3) + \sum_i^p \left(\log(I_{ii})^{1/2} + \log(\lvert\hat{\theta}_i\rvert)\right)$

RAR IF mock

Rank	Function	Comp.	$P(f)$	Parameters				Description length			
				θ_0	θ_1	θ_2	θ_3	Resid.[1]	Func.[2]	Param.[3]	Total
1	$\theta_0 + \theta_1 x + \sqrt{x}$	8	5.6×10^{-1}	9.1×10^{-3}	0.63	—	—	-2045.2	12.9	4.9	-2027.4
2	$\sqrt{\lvert\theta_0 + x\rvert} + \theta_1 x$	8	2.8×10^{-1}	3.0×10^{-3}	0.64	—	—	-2044.4	12.9	4.8	-2026.7
3	$\theta_0 x + x^{\theta_1}$	7	8.2×10^{-2}	0.64	0.49	—	—	-2045.2	11.3	8.5	-2025.5
4	$\sqrt{x}\exp\left(\frac{x^{\theta_0}}{2}\right)$	7	3.5×10^{-2}	0.36	—	—	—	-2040.7	12.5	3.5	-2024.7
5	$(\theta_0 + x)\left(\theta_1 + \frac{1}{\sqrt{x}}\right)$	9	1.1×10^{-2}	1.3×10^{-3}	0.64	—	—	-2044.5	16.1	4.8	-2023.5
6	$\frac{1}{\sqrt{\lvert\theta_0 + \frac{1}{x}\rvert}} + x$	8	8.8×10^{-3}	1.74	—	—	—	-2038.5	12.9	2.3	-2023.3
7	$(x\lvert\theta_0\rvert)^{(x\lvert\theta_1\rvert)^{\theta_2}}$	9	3.1×10^{-3}	-2.09	-1.4×10^{-4}	0.04	—	-2045.3	12.5	10.6	-2022.2
8	$\theta_0 x + \lvert\theta_1 + x\rvert^{\theta_2}$	9	2.4×10^{-3}	0.64	1.4×10^{-3}	0.49	—	-2045.4	14.5	8.9	-2022.0
9	$x\left(\lvert\theta_0 - x\rvert^{\theta_1} - \theta_2\right)$	9	2.3×10^{-3}	1.2×10^{-3}	-0.51	-0.64	—	-2045.3	14.5	8.9	-2021.9
10	$(\theta_0 - x)\left(\theta_1 - x^{\theta_2}\right)$	9	2.2×10^{-3}	-6.5×10^{-4}	-0.64	-0.51	—	-2045.4	14.5	9.0	-2021.9
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
27	$x/(\exp(\theta_0) - \exp(-\sqrt{x}))$	9	3.2×10^{-4}	-0.01	—	—	—	-2039.3	17.5	1.9	-2020.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
41	$x/(\exp(\theta_0) - \lvert\theta_1\rvert^{\sqrt{x}})$	9	1.1×10^{-4}	-5.0×10^{-3}	0.38	—	—	-2042.1	17.5	5.7	-2018.9
—	RAR IF	9	1.0×10^{-3}	1.14	—	—	—	-2041.1	16.1	3.9	-2021.1
—	Double power law	11	3.4×10^{-8}	1.25	1.47	0.90	0.54	-2047.2	17.7	18.7	-2010.8
—	Simple IF	10	2.8×10^{-11}	1.12	—	—	—	-2026.2	18.6	3.9	-2003.7
—	Standard IF	14	2.9×10^{-55}	1.54	—	—	—	-1934.4	27.9	4.1	-1902.4
—	Simple IF + EFE	59	5.9×10^{-64}	1.12	0	—	—	-2026.2	139.9	3.9	-1882.4

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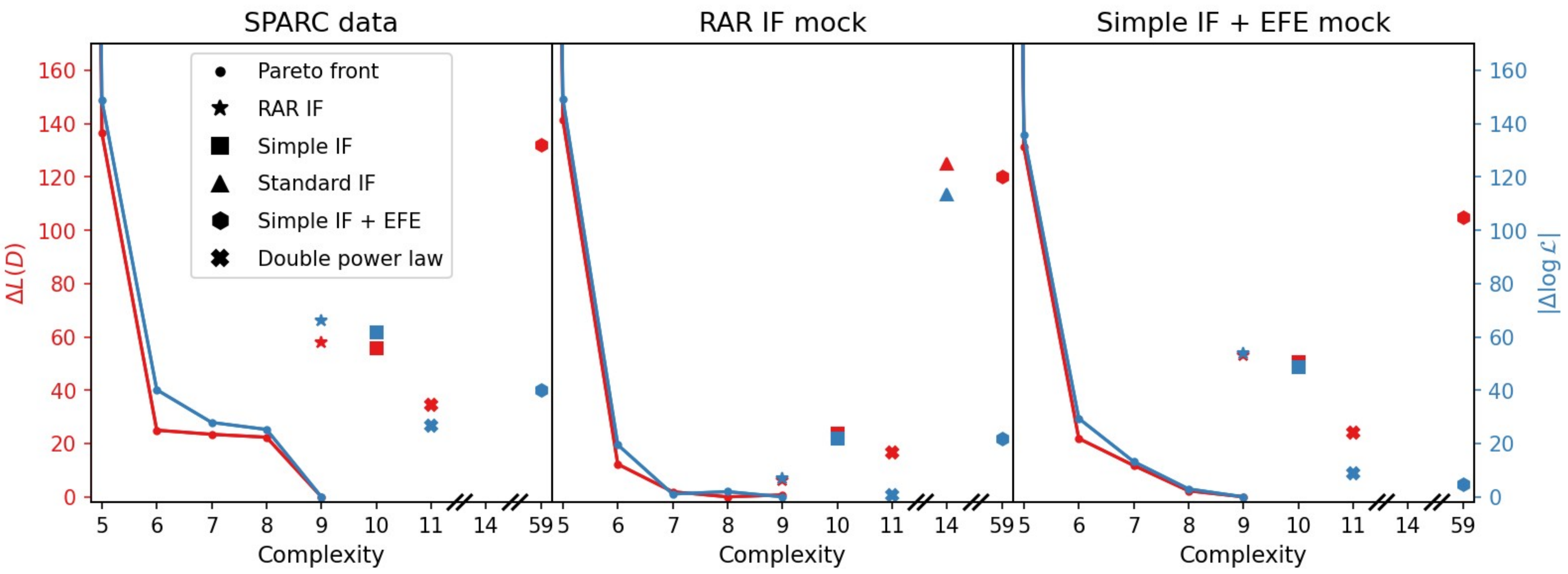
Simple IF + EFE mock

Rank	Function	Comp.	$P(f)$	Parameters				Description length			
				θ_0	θ_1	θ_2	θ_3	Resid.[1]	Func.[2]	Param.[3]	Total
1	$\theta_0 + \sqrt{x^2 + 2x}$	9	8.9×10^{-1}	-0.06	—	—	—	-2017.7	14.5	3.1	-2000.0
2	$\theta_0 + \sqrt{x\|\theta_1 + x\|}$	8	9.3×10^{-2}	-0.06	1.97	—	—	-2017.9	12.9	7.3	-1997.8
3	$-\|\theta_0\|^{\sqrt{x}} + \theta_1 + x$	8	5.6×10^{-3}	0.26	0.95	—	—	-2017.9	12.9	10.1	-1995.0
4	$(\theta_0 - x)\left(\theta_1 - x^{\theta_2}\right)$	9	3.3×10^{-3}	3.1×10^{-3}	-0.71	-0.53	—	-2019.7	14.5	10.7	-1994.4
5	$x^{\theta_0} - \theta_1(\theta_2 - x)$	9	2.4×10^{-3}	0.39	0.79	0.12	—	-2020.9	14.5	12.3	-1994.1
6	$\|\theta_0 - x\|^{\theta_1} - \theta_2 x$	9	2.0×10^{-3}	5.5×10^{-3}	0.48	-0.71	—	-2019.1	14.5	10.6	-1994.0
7	$x\|\theta_0\|^{-\|\theta_1\|^{x^{\theta_2}}}$	9	1.7×10^{-3}	0.04	-0.16	0.33	—	-2018.1	12.5	11.9	-1993.8
8	$x\left(\theta_0 + \|\theta_1 + x\|^{\theta_2}\right)$	9	1.5×10^{-3}	0.71	0.01	-0.53	—	-2018.7	14.5	10.6	-1993.7
9	$\|\theta_0\|^{\|\theta_1\|^{x^{\theta_2}}} + x$	9	6.5×10^{-4}	7.0×10^{-6}	0.03	0.17	—	-2016.7	12.5	11.4	-1992.8
10	$\exp\left(\theta_0 - \frac{1}{\sqrt[4]{x}}\right) + x$	9	5.5×10^{-4}	0.57	—	—	—	-2014.0	17.5	3.9	-1992.6
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
21	$x/(\exp(\theta_0) - \|\theta_1\|^{\sqrt{x}})$	9	1.8×10^{-5}	0.03	0.44	—	—	-2014.2	17.5	7.4	-1989.3
—	Double power law	11	3.4×10^{-11}	3.53	3.31	0.98	0.60	-2012.3	17.7	18.6	-1976.0
—	Simple IF	10	1.2×10^{-22}	1.11	—	—	—	-1972.1	18.6	3.9	-1949.6
—	RAR IF	9	7.0×10^{-24}	1.13	—	—	—	-1966.9	16.1	3.9	-1946.8
—	Simple IF + EFE	59	3.8×10^{-57}	1.19	8.6×10^{-3}	—	—	-2016.0	139.9	5.9	-1870.2
—	Standard IF	14	2×10^{-141}	1.54	—	—	—	-1708.3	27.9	4.1	-1676.3

[1] $-\log\mathcal{L}(\hat{\boldsymbol{\theta}})$ [2] $k\log(n) + \sum_j \log(c_j)$ [3] $-\frac{p}{2}\log(3) + \sum_i^p(\log(I_{ii})^{1/2} + \log(|\hat{\theta}_i|))$



$$\theta_0\left(|\theta_1 + x|^{\theta_2} + x\right)$$

$$\theta_0 + \theta_1 x + \sqrt{x}$$

$$\theta_0 + \sqrt{x^2 + 2x}$$

ESR readily Pareto-dominates all literature fits