Lie categories

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Spring 2023

Motivation

We begin with a physical interpretation of the notion of a category:

Mathematics	**Physics**
Objects	States
Morphisms	Processes
Composition of morphisms	Succession of processes
Invertible morphism	Reversible process

In this way, a given physical system can be viewed as a category.

Usually, the set of all states (*phase space*) of a given physical system has more structure, e.g. it is a topological space or a smooth manifold, and the same holds for the set of processes.

We are interested in categories whose set of objects and set of morphisms are smooth manifolds – categories *internal* to **Diff**.

Example from statistical physics

Consider the space

$$\Delta^n = \{(p_0, \ldots, p_n) \in [0,1]^{n+1} \mid \sum_{i=0}^n p_i = 1\}$$

of probability distributions on the finite set $\{0, \ldots, n\}$. In physics, Δ^n corresponds to the set of configurations of a statistical ensemble of particles, each of which can be in one of the "microstates" $\{0, \ldots, n\}$.

The *expected surprise* (or *entropy*) of $p := (p_i)_{i=0}^n \in \Delta^n$ is given as

$$S(p) = -\sum_{i=0}^n p_i \log p_i.$$

Second law of thermodynamics asserts that the only transitions which can occur are

$$\mathcal{C} = \{(q,p) \in \Delta^n \times \Delta^n \mid S(q) - S(p) \geq 0\} = (\Delta S)^{-1}([0,\infty))$$

where (q,p) is interpreted as the transition $p \to q$.

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Notation

A (small) category $\mathcal{C} \rightrightarrows \mathcal{X}$ consists of

- A set of objects $\mathcal{X}$,
- A set of morphisms $\mathcal{C}$.

It comes equipped with *source* and *target* maps $s, t\colon \mathcal{C} \to \mathcal{X}$

$$s(x \xrightarrow{g} y) = x, \quad t(x \xrightarrow{g} y) = y,$$

and with the *composition* and *unit* maps

$$\begin{aligned} m &\colon \mathcal{C}^{(2)} \to \mathcal{C}, & m(g, h) &= gh, \\ u &\colon \mathcal{X} \to \mathcal{C}, & u(x) &= 1_x, \end{aligned}$$

where $\mathcal{C}^{(2)} = \{(g, h) \in \mathcal{C} \times \mathcal{C} \mid s(g) = t(h)\}$ is the set of composable pairs of morphisms. (One should keep in mind the associativity axiom, etc.)

Notation

We also denote:

$$\underbrace{\mathcal{C}_x = s^{-1}(x)}_{\substack{\text{morphisms in } \mathcal{C} \\ \text{starting at } x}}, \quad \underbrace{\mathcal{C}^y = t^{-1}(y)}_{\substack{\text{morphisms in } \mathcal{C} \\ \text{ending at } y}}, \quad \underbrace{\mathcal{C}_x^y = \mathcal{C}_x \cap \mathcal{C}^y}_{Hom(x,y)}.$$

The precomposition and postcomposition by a morphism $g \in \mathcal{C}$ are expressed in this notation as the following maps:

$$L_g : \mathcal{C}^{s(g)} \to \mathcal{C}^{t(g)}, \quad h \mapsto gh,$$
$$R_g : \mathcal{C}_{t(g)} \to \mathcal{C}_{s(g)}, \quad h \mapsto hg.$$

The axioms of a Lie category

A *Lie category* is a category $\mathcal{C} \rightrightarrows \mathcal{X}$, such that:

- $\mathcal{C}$ and $\mathcal{X}$ are smooth manifolds.
- $s, t\colon \mathcal{C} \to \mathcal{X}$ are smooth submersions.
- $m\colon \mathcal{C}^{(2)} \to \mathcal{C}$ and $u\colon \mathcal{X} \to \mathcal{C}$ are smooth maps.

We require the object manifold $\mathcal{X}$ to have no boundary, but allow $\mathcal{C}$ to have one – if it does, we also impose the following condition:

- $s|_{\partial\mathcal{C}}$ and $t|_{\partial\mathcal{C}}$ are submersions.

We call this the *regular boundary* condition.

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First properties

For the moment, assume $\partial\mathcal{C} = \emptyset$.

- $\mathcal{C}_x$, C^x are smooth manifolds, for any $x \in \mathcal{X}$ (by implicit map theorem).
- $\mathcal{C}^{(2)} = (s \times t)^{-1}(\Delta_{\mathcal{X}})$ is a smooth manifold (by transversality theorem), so the requirement of m being smooth is sensible.
- The latter implies that $L_g : \mathcal{C}^{s(g)} \to \mathcal{C}^{t(g)}$ and $R_g : \mathcal{C}_{t(g)} \to \mathcal{C}_{s(g)}$ are smooth maps. This means we obtain a covariant functor $\mathcal{C} \to$ **Diff**, given by $x \mapsto \mathcal{C}^x$, $g \mapsto L_g$, and a contravariant one: $x \mapsto \mathcal{C}_x$, $g \mapsto R_g$.
- $u : \mathcal{X} \to \mathcal{C}$ is a section of $s : \mathcal{C} \to \mathcal{X}$, so an injective immersion and a homeomorphism onto its image, with continuous inverse $s|_{u(\mathcal{X})}$.

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First properties

In the case $\partial\mathcal{C} \neq \emptyset$, the regular boundary condition ensures that all these properties still hold, and moreover that:

- $\partial(\mathcal{C}_x) = \mathcal{C}_x \cap \partial\mathcal{C}$ and $\partial(\mathcal{C}^x) = \mathcal{C}^x \cap \partial\mathcal{C}$.
- $\mathcal{C}^{(2)} \subset \mathcal{C} \times \mathcal{C}$ is a smooth manifold with corners (luckily for us, transversality theorems for manifolds with corners were developed in the 90's). Can you guess what its corners are?

Example 1: Lie monoids

A *Lie monoid* is a Lie category with $\mathcal{X} = \{*\}$. Equivalently, it is a monoid M together with a structure of a smooth manifold, possibly with a boundary, so that $m\colon M \times M \to M$ is smooth.

Specific examples:

- Lie groups.
- $[0, \infty)$ or $\mathbb{H}^n$ for addition; $[0, \infty)$ for multiplication.
- Matrices $\mathbb{F}^{n \times n}$ for multiplication; more generally, endomorphisms $\mathrm{End}(V)$ of a finite-dim. vector space V.
- Finite-dimensional unital algebras (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{H}$).
- $\det^{-1}(0, 1] \subset \mathbb{R}^{n \times n}$, $\overline{\mathbb{D}} \subset \mathbb{C}$, $\cup_{k=0}^{\infty} \frac{1}{2^k} S^1 \subset \mathbb{C}$, $\overline{\mathbb{B}^4} \subset \mathbb{H}, \ldots$

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Example 2: A triviality producing a slight non-triviality

If X is a smooth manifold without boundary and M a Lie monoid,

$$\mathcal{C} = X \times M \times X \rightrightarrows X$$

is a *trivial Lie category*, with composition defined as

$$(z, g, y)(y, h, x) = (z, gh, x).$$

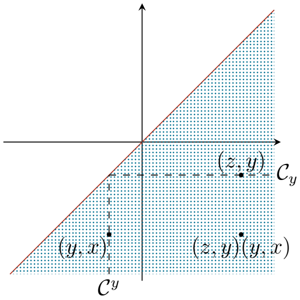
For $M = \{e\}$ the trivial group, we get the *pair groupoid*.

Example 2: A triviality producing a slight non-triviality

Now consider the pair groupoid over $\mathbb{R}$ and "throw half the arrows away", i.e. consider

$$\mathcal{C} = \{(y, x) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\} \rightrightarrows \mathbb{R}.$$

This is just the order category on $\mathbb{R}$! Easy to visualize:



Example 3: Endomorphism category

Let $\pi\colon E \to X$ be a vector bundle with fibre V. The set

$$\mathrm{End}(E) = \{\xi\colon E_x \to E_y \mid x, y \in X \text{ and } \xi \text{ is linear}\}$$

is a category over X, with obvious structure maps.

Bonus: $\mathrm{End}(E)$ is enriched over **Vect**, i.e. the Hom-sets are vector spaces and composition is bilinear.

Example 3: Endomorphism category

Moral: Lie categories describe smooth families of endomorphisms of an abstract structure, parametrized by the base manifold.

Example 4: Bundles of Lie monoids

A *bundle of Lie monoids* is a Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$ with $s = t =: p$.

Concrete examples: (again let $E \to X$ be a vector bundle)

- Endomorphism bundle $E^* \otimes E \xrightarrow{p} X$, a subcategory of $\mathrm{End}(E)$.
- Exterior bundle

$$\Lambda(E) = \bigoplus_{i=0}^{\operatorname{rank} E} \Lambda^k(E),$$

  where composition of $\alpha \in \Lambda^k(E_x)$ and $\beta \in \Lambda^\ell(E_x)$ is just $\alpha \wedge \beta \in \Lambda^{k+\ell}(E_x)$, and the units are $1_x = 1 \in \mathbb{F} = \Lambda^0(E_x)$.

Note that bundles of Lie monoids are not necessarily locally trivial.

Bundles of Lie monoids enriched over **Vect** would rightfully be called *bundles of unital associative algebras*.

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Example 5: Lie groupoids

A *Lie groupoid* is a Lie category with all morphisms invertible. These structures enjoy a lot of attention from geometers.

Well-known examples include:

- Bundles of Lie groups (e.g. any vector bundle for addition).
- The fundamental groupoid of a smooth manifold, i.e. homotopy classes of paths, relative to endpoints.
- Monodromy groupoid of a foliation on a manifold. It consists of homotopy classes of paths, relative to endpoints, contained within the leaves.
- The gauge groupoid of a principal bundle $\pi\colon P \xrightarrow{G} M$. It consists of G-equivariant maps between π-fibres.
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Reversibility

In any category $\mathcal{C}$, its *core*

$$\mathcal{G}(\mathcal{C}) = \{g \in \mathcal{C} \mid g \text{ is invertible}\}$$

forms a subcategory. Natural question: if $\mathcal{C}$ is a Lie category, under what conditions is $\mathcal{G}(\mathcal{C})$ a Lie groupoid? When is it open in $\mathcal{C}$?

In general, $\mathcal{G}(\mathcal{C})$ is neither: an easy counterexample is provided by the "disjoint union" of pair groupoid and order category on $\mathbb{R}$.

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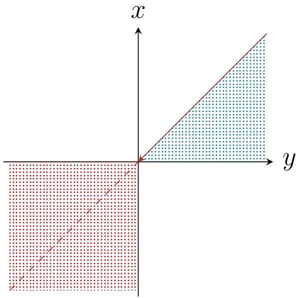

Units dictate the invertibles

To provide sufficient conditions for the questions about openness and smoothness of $\mathcal{G}(\mathcal{C})$, we observe the following.

Lemma

For any Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$, there holds:

$$u(\mathcal{X}) \subset \operatorname{Int}\mathcal{C} \textit{ implies } \mathcal{G}(\mathcal{C}) \subset \operatorname{Int}\mathcal{C},$$
$$u(\mathcal{X}) \subset \partial\mathcal{C} \textit{ implies } \mathcal{G}(\mathcal{C}) \subset \partial\mathcal{C}.$$

Lie monoid case: all invertible elements are either in the interior, or in the boundary.

Sufficient conditions for $\mathcal{G}(\mathcal{C})$ to be a Lie groupoid

Motivated by the last result, we say that a Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$ has a *normal boundary*, if either of the following holds:

- $u(\mathcal{X}) \subset \operatorname{Int} \mathcal{C}$.
- $\partial\mathcal{C} \subset \mathcal{C}$ is a wide subcategory, i.e. a subcategory with $u(\mathcal{X}) \subset \partial\mathcal{C}$.

Theorem

If $\mathcal{C} \rightrightarrows \mathcal{X}$ is a Lie category with a normal boundary, then $\mathcal{G}(\mathcal{C})$ is an embedded Lie subcategory of $\mathcal{C}$. More precisely:

- *If $u(\mathcal{X}) \subset \operatorname{Int} \mathcal{C}$, then $\mathcal{G}(\mathcal{C})$ is open in $\operatorname{Int} \mathcal{C}$.*
- *If $u(\mathcal{X}) \subset \partial\mathcal{C}$, then $\mathcal{G}(\mathcal{C})$ is open in $\partial\mathcal{C}$.*

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First consequence: $\mathcal{G}$ is a functor

If $F\colon \mathcal{C} \to \mathcal{D}$ is a smooth functor between Lie categories with normal boundaries, functoriality implies $F(\mathcal{G}(\mathcal{C})) \subset \mathcal{G}(\mathcal{D})$, so we can define $\mathcal{G}(F) = F|_{\mathcal{G}(\mathcal{C})}$.

We thus obtain a functor

$$\mathcal{G}\colon \mathbf{LieCat}_{\partial} \to \mathbf{LieGrpd}$$

from the category of Lie categories with normal boundary, to the category of Lie groupoids (without boundary).

Infinitesimal counterparts of Lie categories

A *Lie algebroid* over a smooth manifold $\mathcal{X}$ is a vector bundle $A \to \mathcal{X}$, together with:

- A Lie bracket on its sections:

$$[\cdot, \cdot] \colon \Gamma^\infty(A) \times \Gamma^\infty(A) \to \Gamma^\infty(A),$$

- A morphism of vector bundles

$$\rho \colon A \to T\mathcal{X},$$

  called the *anchor* of A.

The following form of Leibniz rule is also required to hold:

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta,$$

for all $\alpha, \beta \in \Gamma^\infty(A)$ and $f \in C^\infty(\mathcal{X})$.

Infinitesimal counterparts of Lie categories

Generalizing the construction of Lie algebras of Lie groups:

A *left-invariant* vector field on a Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$ is a vector field $X \in \mathfrak{X}(\mathcal{C})$ such that:

- X is tangent to t-fibres, i.e. $X_g \in \ker \mathrm{d}t_g$ for all $g \in \mathcal{C}$.
- $\mathrm{d}(L_g)_h(X_h) = X_{gh}$ for all $(g, h) \in \mathcal{C}^{(2)}$.

Lemma

The vector space $\mathfrak{X}^L(\mathcal{C})$ is closed under the Lie bracket, and canonically isomorphic to the vector space $\Gamma^\infty(A^L(\mathcal{C}))$ of sections of the vector bundle $A^L(\mathcal{C}) = u^*(\ker \mathrm{d}t)$ over $\mathcal{X}$.

The isomorphism is given by restricting to the units, $\mathrm{ev}(X) = X|_{u(\mathcal{X})}$, so all information of a left-invariant vector field is contained at the units $u(\mathcal{X})$.

Infinitesimal counterparts of Lie categories

The *left Lie algebroid* of a Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$ is the vector bundle $A^L(\mathcal{C}) \to \mathcal{X}$, endowed with the Lie bracket $[\cdot, \cdot]$ on its sections as induced by the isomorphism ev, together with the anchor map $\rho^L \colon A^L(\mathcal{C}) \to T\mathcal{X}$,

$$\rho^L = \mathrm{d}s|_{A^L(\mathcal{C})}.$$

We similarly define the *right Lie algebroid* $A^R(\mathcal{C}) \to \mathcal{X}$. On a Lie groupoid, the left and right algebroid are isomorphic – the isomorphism is induced by inversion.

If we are given a morphism $F \colon \mathcal{C} \to \mathcal{D}$ over $\mathrm{id}_{\mathcal{X}}$, it induces a morphism of left (resp. right) Lie algebroids, given by

$$\Gamma^\infty(A^L(\mathcal{C})) \ni \alpha \mapsto \mathrm{d}F \circ \alpha \in \Gamma^\infty(A^L(\mathcal{D})).$$

Upshot: A^L and A^R are functors **LieCat** $\to$ **LieAlgd**.

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Are the Lie algebroids of $\mathcal{C}$ and $\mathcal{G}(\mathcal{C})$ the same?

Proposition

Let $\mathcal{C} \rightrightarrows \mathcal{X}$ be a Lie category. If the units of $\mathcal{C}$ are contained in the interior of $\mathcal{C}$, i.e. $u(\mathcal{X}) \subset \operatorname{Int} \mathcal{C}$, then the left and right Lie algebroids of $\mathcal{C}$ are isomorphic to the Lie algebroid of its core $\mathcal{G}(\mathcal{C})$.

On the other hand, if $\partial\mathcal{C} \subset \mathcal{C}$ is a wide subcategory, then the Lie algebroid $A(\mathcal{G}(\mathcal{C}))$ of the core will always fail to be isomorphic to the two Lie algebroids of $\mathcal{C}$, since the rank of the vector bundle $A(\mathcal{G}(\mathcal{C}))$ is one less than the rank of $A^L(\mathcal{C})$ and $A^R(\mathcal{C})$.

However, we conjecture that in this case $A^L(\mathcal{C}) \cong A^R(\mathcal{C})$.

The ranks of a morphism

The *left* and *right* *rank* of a morphism $g \in \mathcal{C}$ are given by

$$\operatorname{rank}^L_{\mathcal{C}}(g) = \operatorname{rank} \mathrm{d}(L_g)_{1_{s(g)}}, \quad \operatorname{rank}^R_{\mathcal{C}}(g) = \operatorname{rank} \mathrm{d}(R_g)_{1_{t(g)}}.$$

Some nomenclature:

- $g \in \mathcal{C}$ has *full* left rank, if $\operatorname{rank}^L_{\mathcal{C}}(g) = \dim \mathcal{C} - \dim \mathcal{X}$.
- $g \in \mathcal{C}$ has *constant* left rank, if L_g has constant rank.

Clearly, any invertible morphism has full and constant rank.

This is a generalization of rank from linear algebra – given a matrix A from the Lie monoid $\mathbb{R}^{n\times n}$, there holds:

$$\operatorname{rank}_{\mathbb{R}^{n\times n}}(A) = n \operatorname{rank}(A).$$

Some nontrivial properties of ranks

- The subset $\{g \in \mathcal{C} \mid \operatorname{rank}^L_{\mathcal{C}}(g) \text{ is full}\} \subset \mathcal{C}$ is open in $\mathcal{C}$.
- Left rank of g is constant on invertibles: $\operatorname{rank}^L_{\mathcal{C}}(g) = \operatorname{rank} \mathrm{d}(L_g)_h$ holds for any $h \in \mathcal{G}(\mathcal{C})^{s(g)}$. Again, "units dictate invertibles."
- Left translations define an integrable singular distribution on $\mathcal{C}$:

$$D = \coprod_{g \in \mathcal{C}} \operatorname{Im} \mathrm{d}(L_g)_{1_{s(g)}} \subset \ker \mathrm{d}t \subset T\mathcal{C}.$$

The integral manifold of D through $g \in \mathcal{C}$ is $L_g(\mathcal{G}(\mathcal{C})^{s(g)})$, whose dimension equals $\operatorname{rank}^L_{\mathcal{C}}(g)$.

Extensions of Lie categories to groupoids

We have seen examples of Lie categories arising from Lie groupoids, by restricting to a submanifold, e.g. the order category or $[0, \infty)$ for addition. We want to understand them better.

An *extension* of a Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$ is a Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{X}$ together with smooth injective immersive functor $F \colon \mathcal{C} \to \mathcal{G}$ over the identity $\mathrm{id}_{\mathcal{X}}$. In other words, $\mathcal{G}$ is a Lie groupoid such that $\mathcal{C}$ is its wide Lie subcategory.

Properties of extendable Lie categories

A Lie category $\mathcal{C} \rightrightarrows \mathcal{X}$ that is extendable to a Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{X}$ enjoys the following properties:

- Cancellativity of morphisms: all left and right translations are injective.
- All morphisms have full and constant ranks.
- If $\dim \mathcal{G} = \dim \mathcal{C}$, then $A^L(\mathcal{C}) \cong A(\mathcal{G}) \cong A^R(\mathcal{C})$.

Thank you for your attention.

Further research:

- Ongoing: constructing an isomorphism between left an right Lie algebroid when $\partial\mathcal{C} \subset \mathcal{C}$ is a wide subcategory.
- Open questions: Are Lie monoids parallelizable? Are Hom-sets $\mathcal{C}_x^y$ submanifolds? (We know this holds e.g. for extendable categories.)
- Existence of a class of Lie algebroids that integrate to Lie categories, but not groupoids.
- Infinite dimensional Lie categories – e.g. infinitely many microstates in a canonical ensemble, or the tensor bundle $\oplus_{k=0}^{\infty} \otimes^k E$.
- Multiplicative structures, e.g. multiplicative symplectic forms and multiplicative Ehresmann connections on Lie categories.
- Haar systems, smooth sieves and smooth Grothendieck sites, Morita equivalences, ...