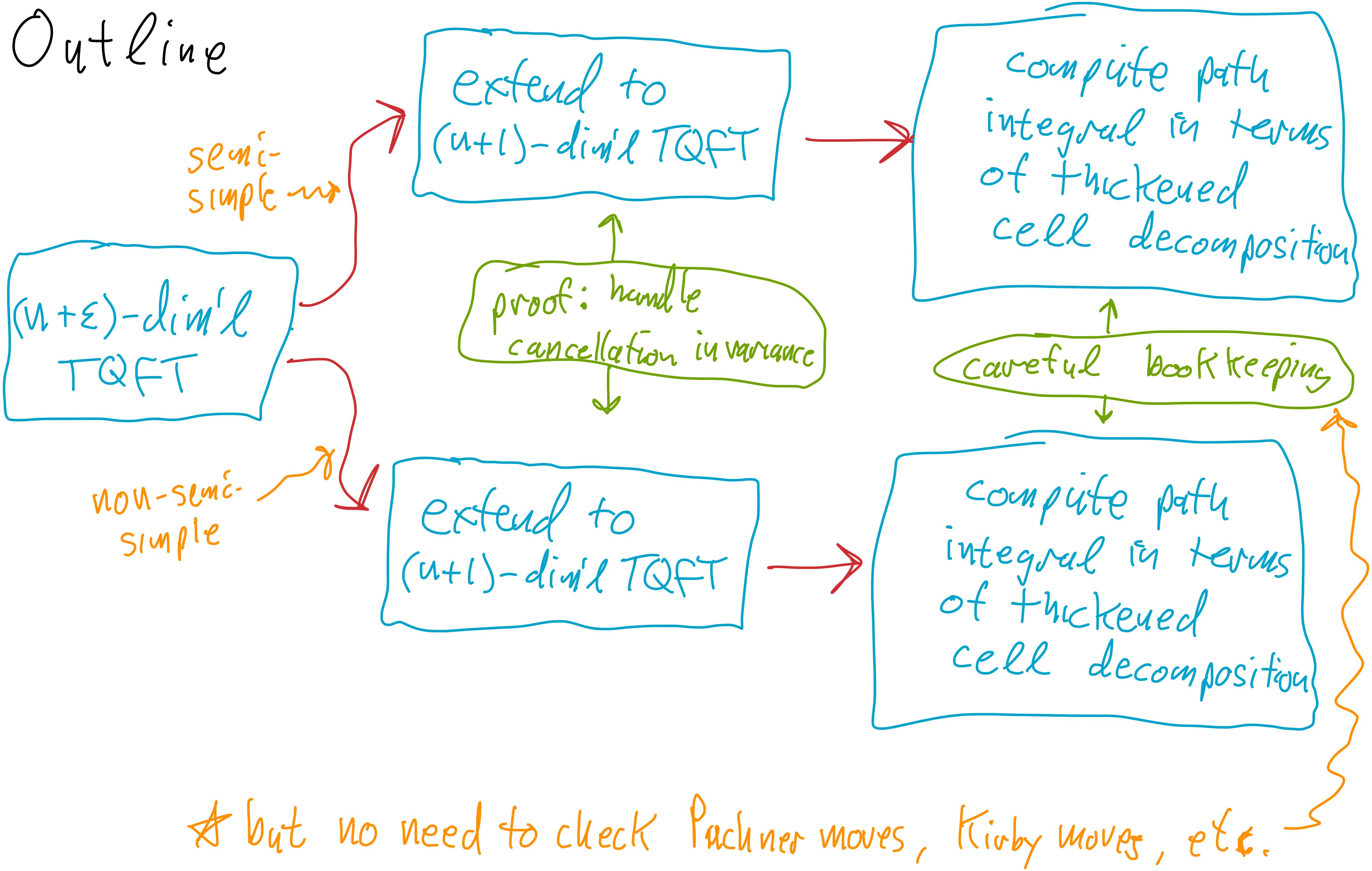
From $(n+\varepsilon)$-dimensional TQFTs to
$(n+1)$-dimensional TQFTs to
state sums

(with and without semisimplicity assumptions)

non-semisimple part is joint work with David Reutter

Outline



(n+ε)-dim'l TQFT

semi-simple

extend to (n+1)-dim'l TQFT

compute path integral in terms of thickened cell decomposition

proof: handle cancellation invariance

careful bookkeeping

non-semi-simple

extend to (n+1)-dim'l TQFT

compute path integral in terms of thickened cell decomposition

★ but no need to check Pachner moves, Kirby moves, etc.

(oriented, unoriented, spin, ...)

$n+\varepsilon$-dimensional ^ TQFTs are plentiful and easy to construct

- ingredient: functors $\mathcal{C}^k : (k\text{-balls}, \left\{\begin{matrix}\text{diffeo}\\ \text{homeo}\end{matrix}\right\}\text{-morphisms}) \longrightarrow \underline{\mathrm{Set}}$

  ($0 \leq k \leq n$; usually linearized when $k=n$)

- examples: ① $\mathcal{C}(X) := \{\text{maps } X \to T\}$ e.g. $T = BG$

  ② $\mathcal{C}(X) := \{C\text{-string diagrams on } X\}$

  C: H-pivotal n-category, $H = \mathrm{Spin}, SO, \ldots$

- require $\{\mathcal{C}^k\}$ to be compatible with gluing, restrictions to boundary

★ Extend $\mathcal{C}$ from balls to manifolds (of dimension $0 \ldots n$) via colimit construction (over poset of ball decompositions)

$\mathcal{C}(M^n)$: "skein module" (vector space)

$\mathcal{C}(Y^{n-1})$: objects of cylinder category

$\mathcal{C}(Y^{n-1} \times I)$: morphisms of cylinder category

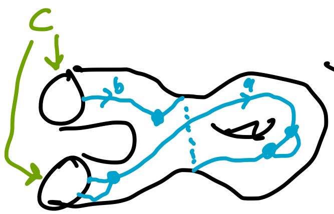
} linear 1-category

fully extended $\rightarrow$ $\vdots$

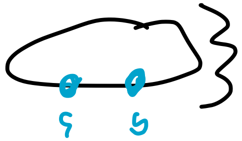
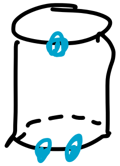
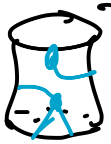
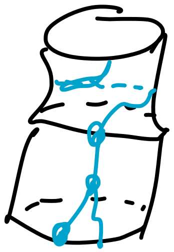
$(n-k)$-category $A(X^k)$, j-morphisms $= \mathcal{C}(X \times B^j)$

Example $n=2$

∂-condition

$A(M^2;c) = \mathbb{k}[\{$  $\}]/\sim$

$\mathcal{C}(X) = \{c\text{-string diagrams on } X\}$
$\mathcal{C}$: pivotal $\otimes$-category

$A(S^1) = $
- obj : $\{$  $\}$
- mor : $A($  $) = \mathbb{k}[\{$  $\}]/\sim$
- composition:  ← stacking cylinders

$A(M^n;c)$, $c \in \mathcal{C}(\partial M)$

Note: $\{A(M;\bullet)\}$ affords an action of the 1-cat $A(\partial M)$

"easy to construct" ⟷ never had to choose handle decomposition or triangulation; never had to check combinatorial "moves" (Pachner moves, handle slides, ...)

"plentiful" ⟷ no finiteness or semisimplicity assumptions

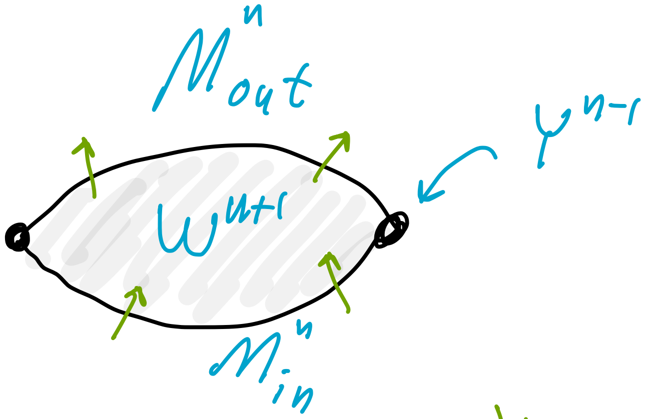
To extend from an $(n+\varepsilon)$-dim'l theory to an $(n+1)$-dim'l theory, we will have to

① resort to combinatorial descriptions of $(n+1)$-manifolds

② impose fairly stringent conditions on input n-cat

from $n+\varepsilon$ to $n+1$

- for every bordism



$\partial W = M_{out} \cup_Y \overline{M_{in}}$

$\partial M_{out} = Y = \partial M_{in}$

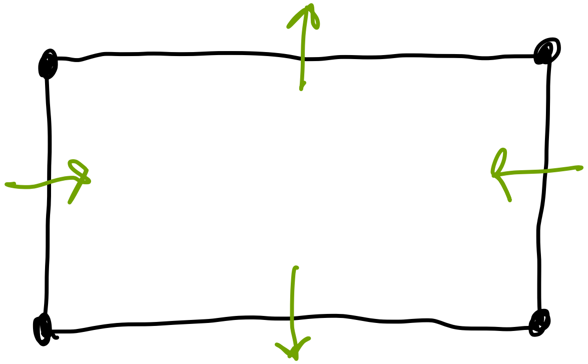
want: $Z(W^{n+1}) : A(M_{in}) \to A(M_{out})$

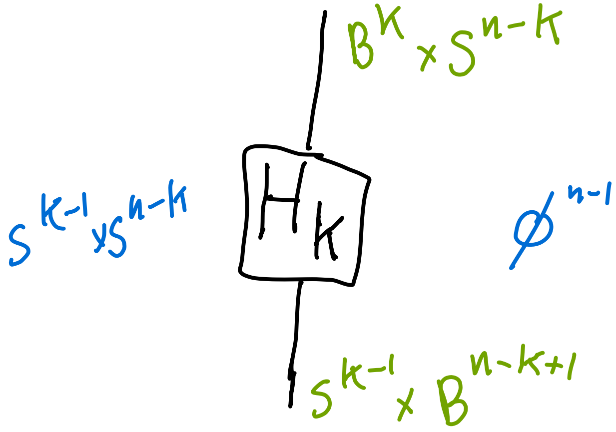
A(Y)-module map

compatible with gluing of (n+1)-manifolds and homeomorphisms of n-manifolds

k-handle bordism

$$H_k = B^k \times B^{n+1-k} : S^{k-1} \times B^{n+1-k} \to B^k \times S^{n-k}$$





Std. topological fact:

$$\left[(n+1)\text{-manifolds}\right] \cong \left[\text{handle decompositions}\right] / \sim$$

① distant reordering

② isotopy of attachment

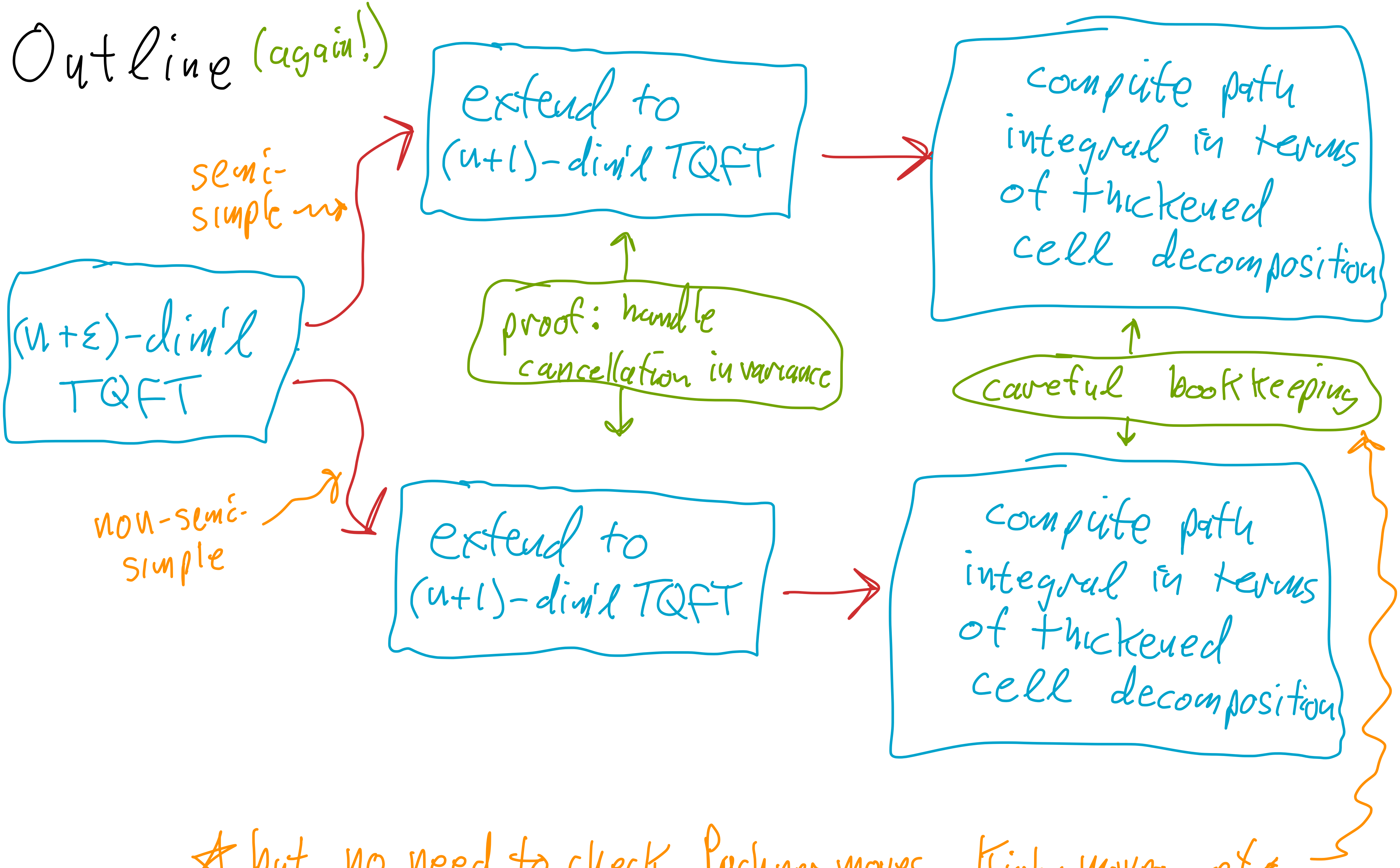
③ handle cancellations

Strategy:

Define $Z(H_0)$, $Z(H_1)$, … … $Z(H_{n+1})$. Then verify ③.

(① and ② come for free.)

Outline (again!)



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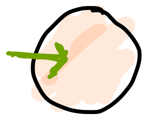
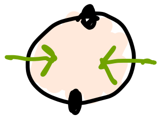
Warm-up: $n=1$ case

- Choose $Z(D^2): A(S^1) \to k$ $\quad H_0$ (0-handle)



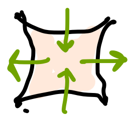
$\rightsquigarrow$ Pairing $P_0 : A(B^1) \otimes A(B^1) \to \mathbb{k}$

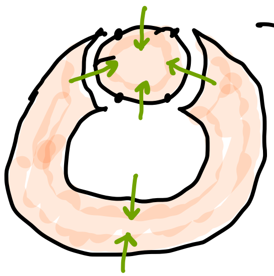
$$x \otimes y \longmapsto Z(B^2)(x \cup y)$$



$\rightsquigarrow$ If P_0 is non-degenerate, $\exists$ copairing

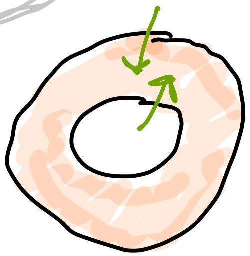
$$Q_0 : {}_1A(B^1)_2 \otimes {}_3A(B^1)_4 \to {}_1A(B^1)_4 \otimes {}_3A(B^1)_2$$

$\rightsquigarrow$ Define $Z(H_1) = Z\Big($  $\Big) = Q_0$

$\rightsquigarrow$ Compute $Z(S^1 \times I) = Z\Big[$  $\Big] = Z(H_0) \circ Z(H_1)$

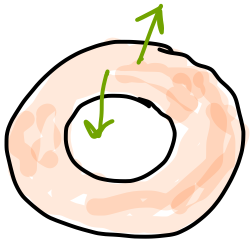
$= Z(H_0) \circ Q_0$

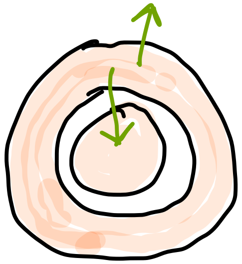
$\rightsquigarrow$ Pairing $P_1 = Z(S^1 \times I) : A(S^1) \otimes A(S^1) \to \mathbb{K}$



$\rightsquigarrow$ If P_1 is non-degenerate, $\exists$ copairing

$$Q_1 : \mathbb{K} \to A(S^1) \otimes A(S^1)$$



$\rightsquigarrow$ compute $Z(H_2) = Z\Big[$  $\Big] = Z(H_0) \circ Q_1$

Conclusion: If extension from $1+\varepsilon$ to $1+1$ exists, then it is completely determined by choice of $Z(H_0): A(S^1) \to \mathbb{K}$. A necessary condition for $Z(H_0)$ to extend is that the pairings P_0 and P_1 are non-degenerate.

Thm (W, Reutter). Let $A(\cdot)$ be an $n+\varepsilon$-dimensional TQFT as above. Choose $Z(B^{n+1}) = Z(H_0) : A(S^n) \to \mathbb{K}$.

Then $Z(\cdots)$ extends to a full $n+1$-dim'l TQFT if and only if the inductively defined pairings

$$P_K : A(S^K \times B^{n-K}) \otimes A(S^K \times B^{n-K}) \to \mathbb{K} \quad , \; 0 \leq K \leq n-1$$

are non-degenerate.

Remark 1: If $P_0, P_1, \ldots, P_m$ are non-degenerate, then can define $Z(\cdots)$ on $n+1$-dim'l handlebodies, all handles of index $\leq m+1$, invariant under handle cancellations of index $\leq m+1$.

Remark 2: Proof depends only 2-functor

A: ($n-1$-manifolds, n-manifolds, homeomorphisms) $\longrightarrow$ target 2-cat

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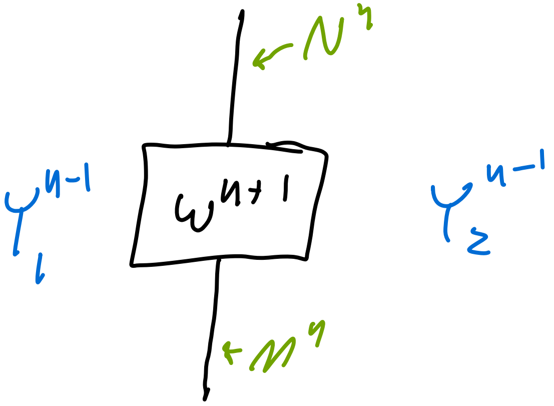
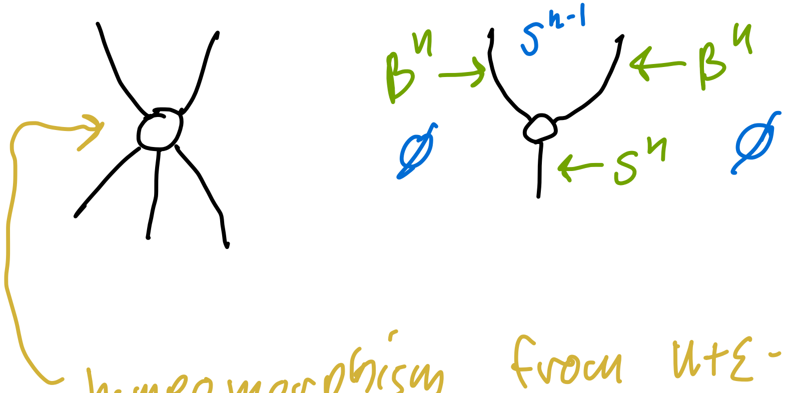
Earlier results:

- W. 2006 – semisimple, positive def. case
- Lurie 2009 – ∞-cat case, framed manifolds, non-pivotal n-cats

Proof:

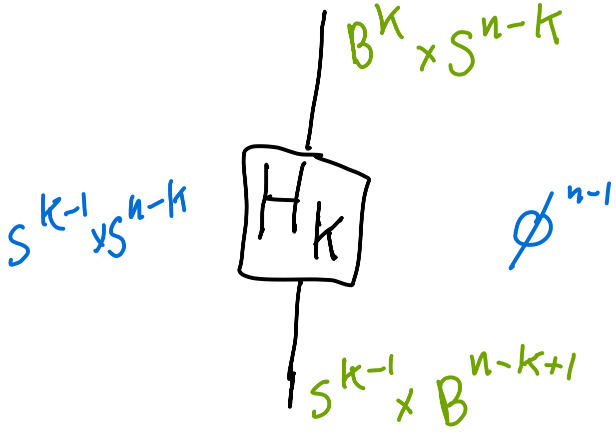


[All diagrams are in target 2-cat, but most labels are in source bordism 2-cat.]



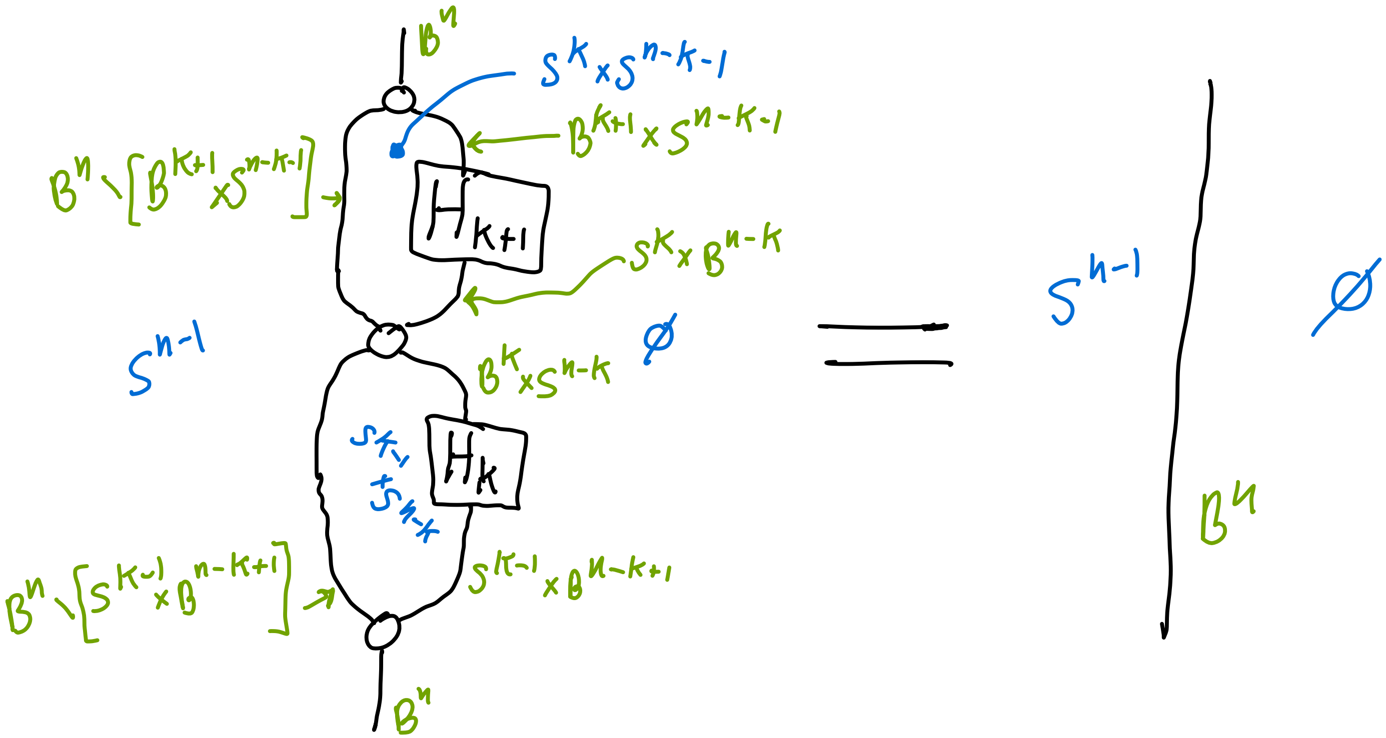
homeomorphism from $n+\varepsilon$-dim'l TQFT

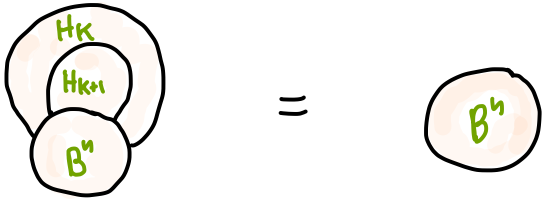
We want to define $Z(H_K)$



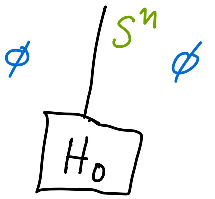
Satisfying handle cancellation:

A





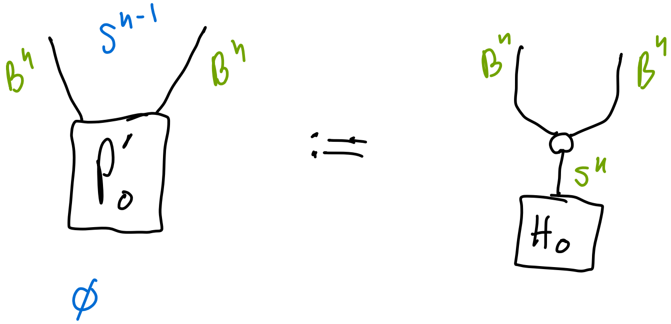
Assume $\exists\ Z(H_0)$



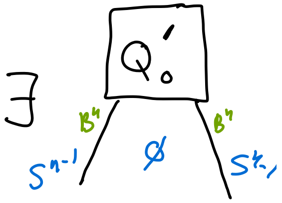
$H_0 = B^{n+1}$

such that the induced pairings P_0'

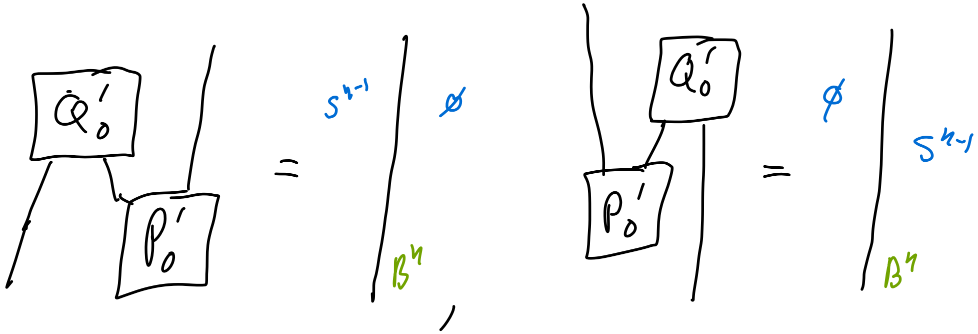
B



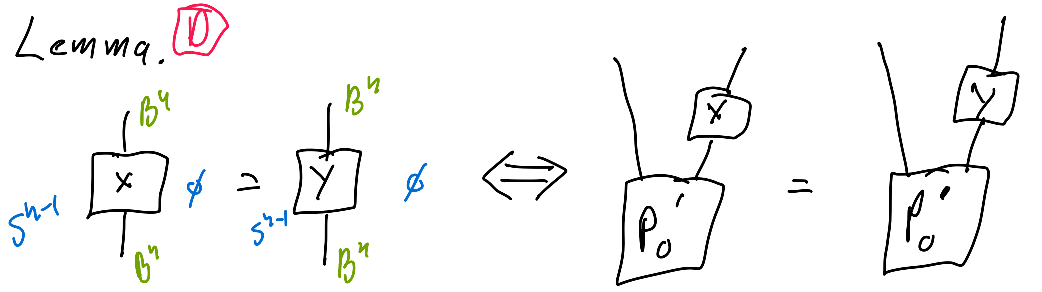
is non-degenerate, in the sense that $\exists$



C



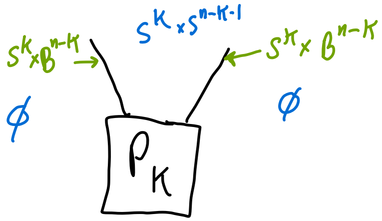
Lemma. ⓓ

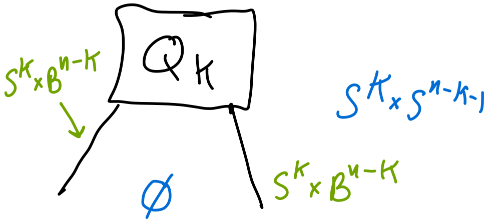


Pf. Easy.

Define $P_0 = P_0' \amalg P_0'$, $Q_0 = Q_0' \amalg Q_0'$

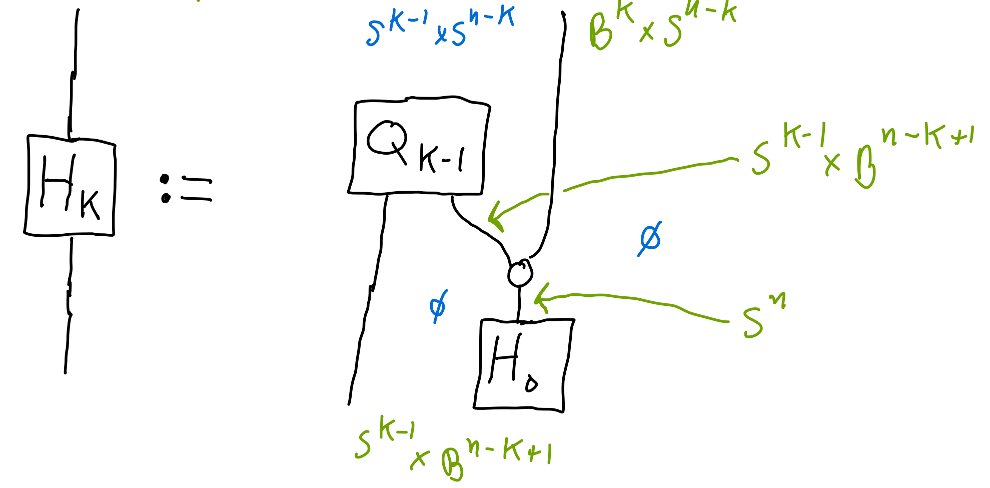
Inductive assumptions:

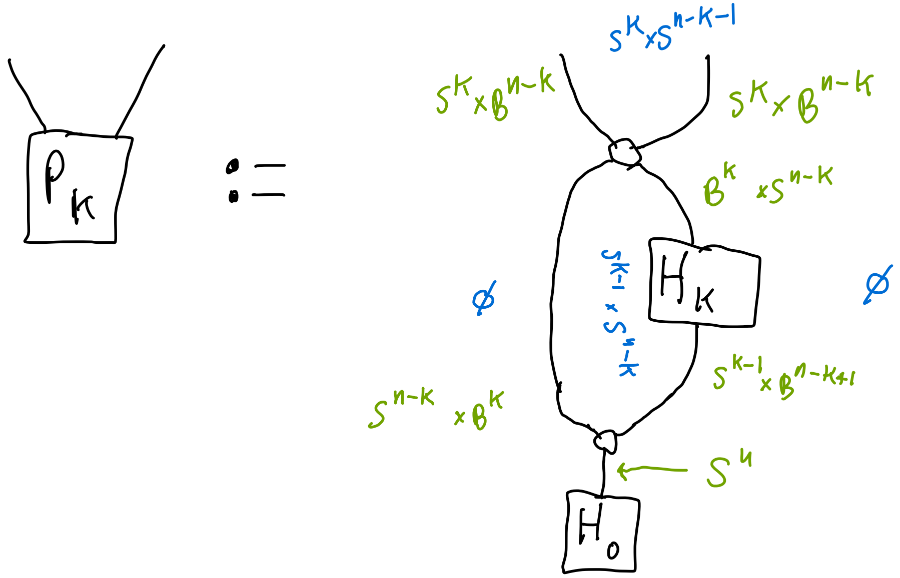




satisfying two zig-zags

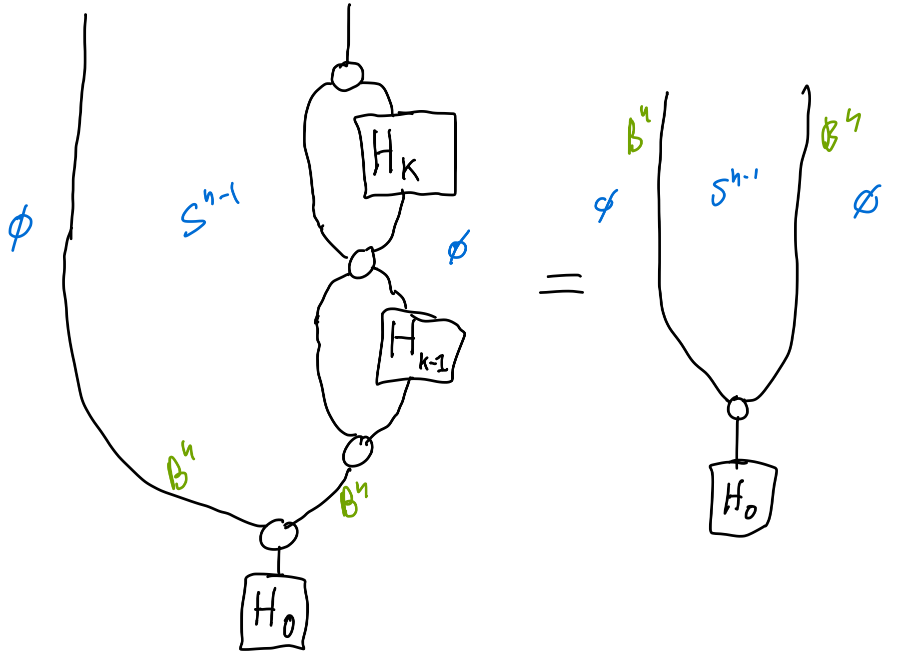
Inductive steps....



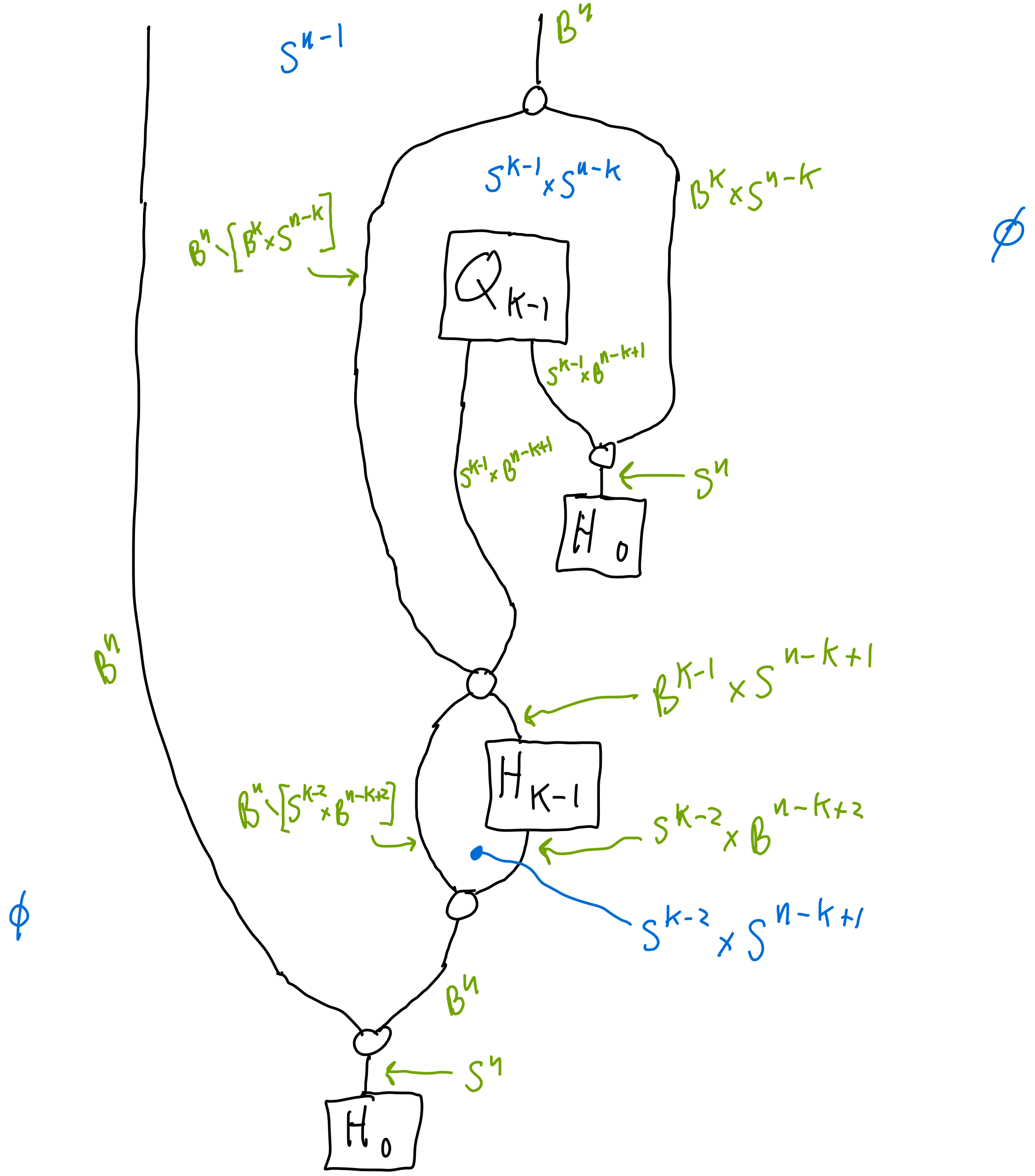


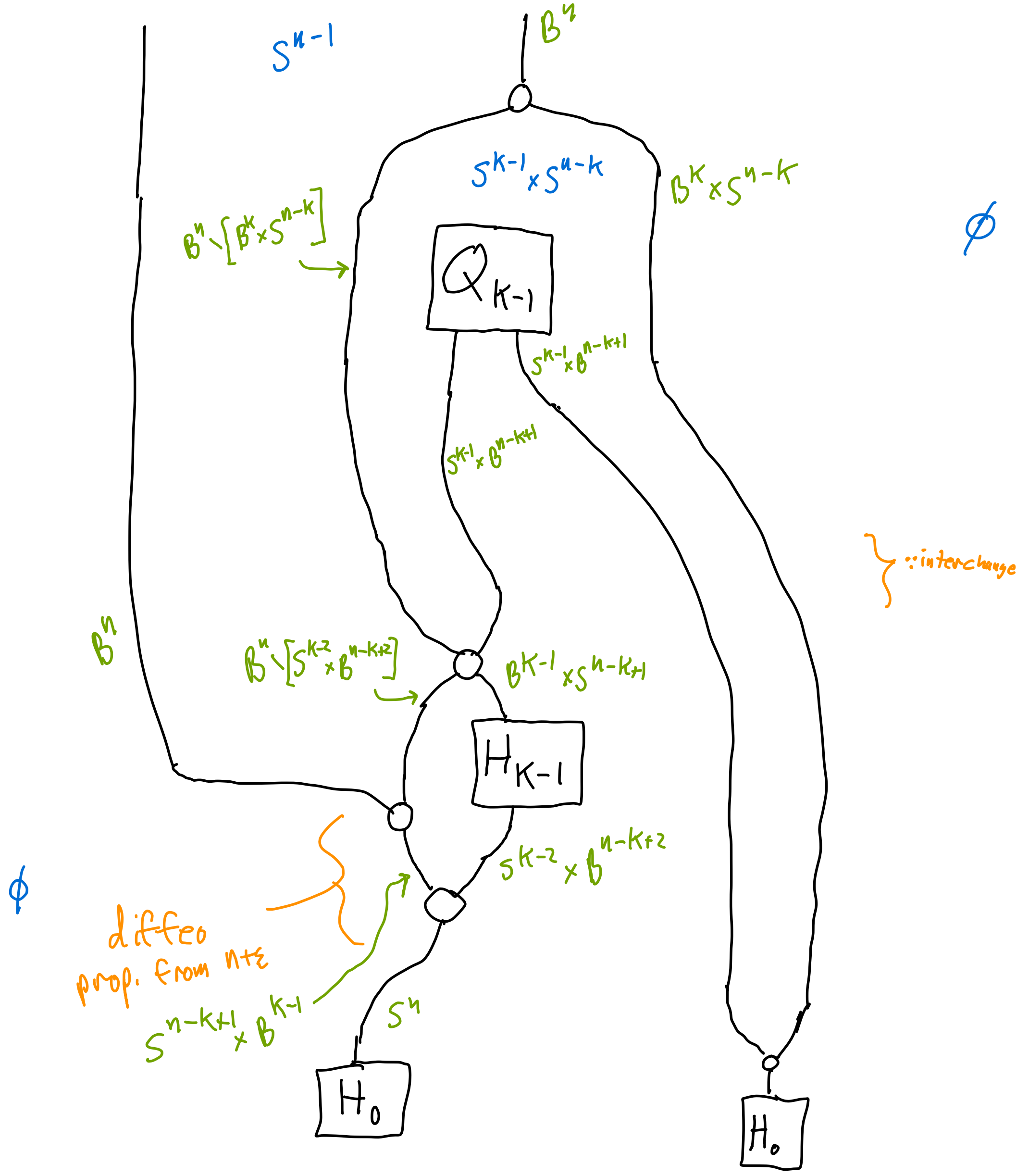
:= copairing of P_K, if it exists

handle cancellation. By Ⓑ + Ⓓ, Ⓐ follows from

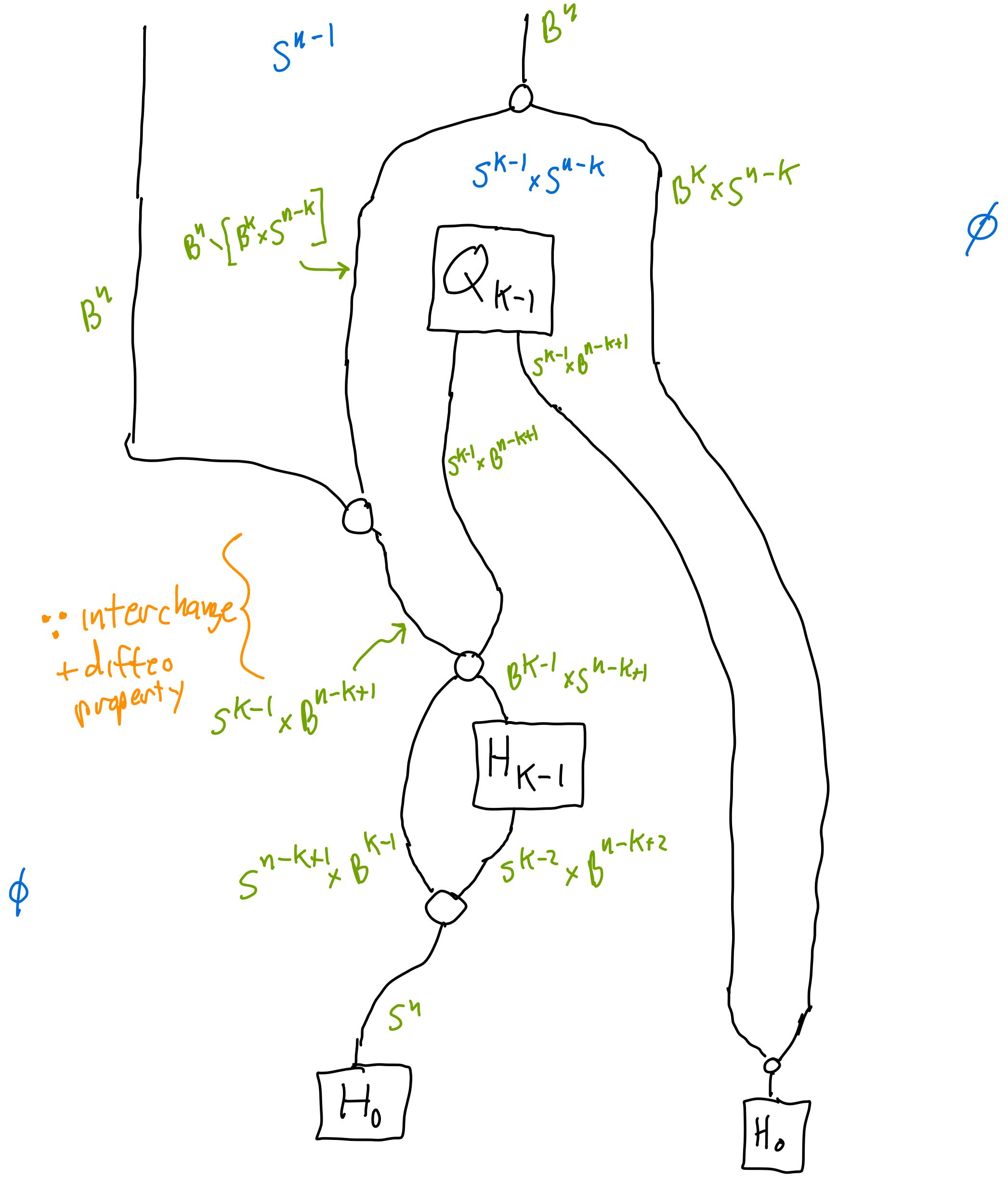


expand H_K


S^{n-1}
B^n
$S^{k-1} \times S^{n-k}$
$B^k \times S^{n-k}$
ϕ
$B^n \setminus [B^k \times S^{n-k}]$
Q_{k-1}
$S^{k-1} \times B^{n-k+1}$
$S^{k-1} \times B^{n-k+1}$
S^n
H_0
B^n
$B^{k-1} \times S^{n-k+1}$
H_{k-1}
$B^n \setminus [S^{k-2} \times B^{n-k+2}]$
$S^{k-2} \times B^{n-k+2}$
ϕ
$S^{k-2} \times S^{n-k+1}$
B^n
S^n
H_0



S^{n-1}
B^n
$S^{k-1} \times S^{n-k}$
$B^k \times S^{n-k}$
$\emptyset$
$B^n \setminus [B^k \times S^{n-k}]$
Q_{k-1}
$S^{k-1} \times B^{n-k+1}$
$S^{k-1} \times B^{n-k+1}$
}: interchange
B^n
$B^n \setminus [S^{k-2} \times B^{n-k+2}]$
$B^{k-1} \times S^{n-k+1}$
H_{k-1}
$S^{k-2} \times B^{n-k+2}$
$\emptyset$
diffeo
prop. from n+ε
$S^{n-k+1} \times B^{k-1}$
S^n
H_0
H_0


S^{n-1}
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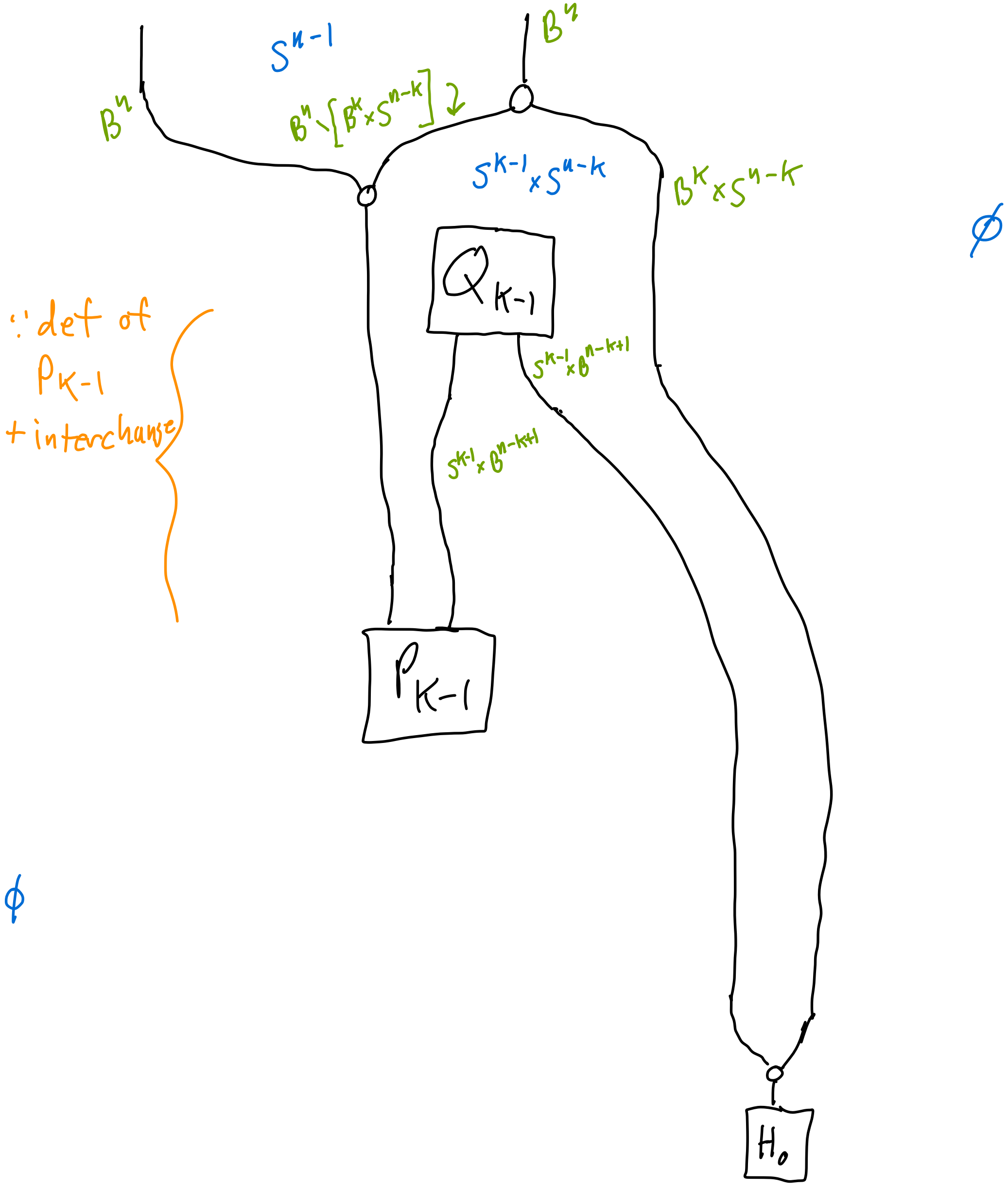
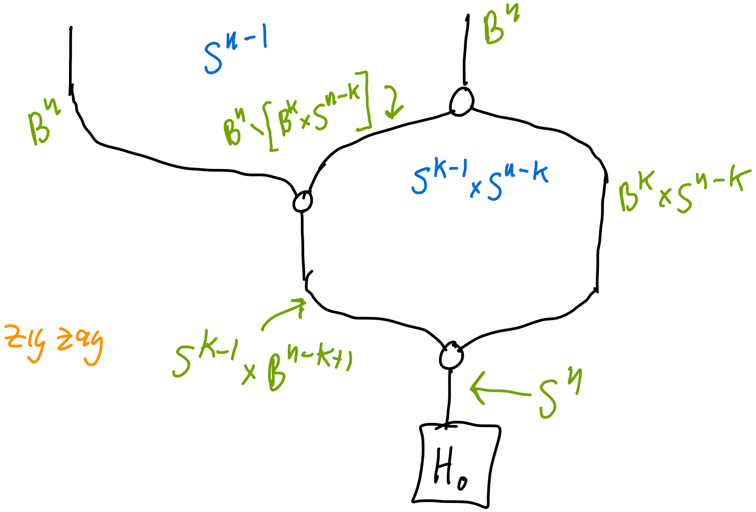
$B^k \times S^{n-k}$
$\emptyset$
Q_{k-1}
$S^{k-1} \times B^{n-k+1}$
$S^{k-1} \times B^{n-k+1}$
∵ interchange + diffeo property
$S^{k-1} \times B^{n-k+1}$
$B^{k-1} \times S^{n-k+1}$
H_{k-1}
$S^{n-k+1} \times B^{k-1}$
$S^{k-2} \times B^{n-k+2}$
$\emptyset$
S^n
H_0
H_0

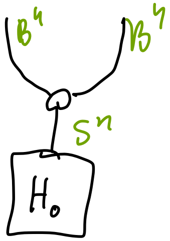


∵ def of P_{K-1} + interchange



∵ P-Q zig zag

=



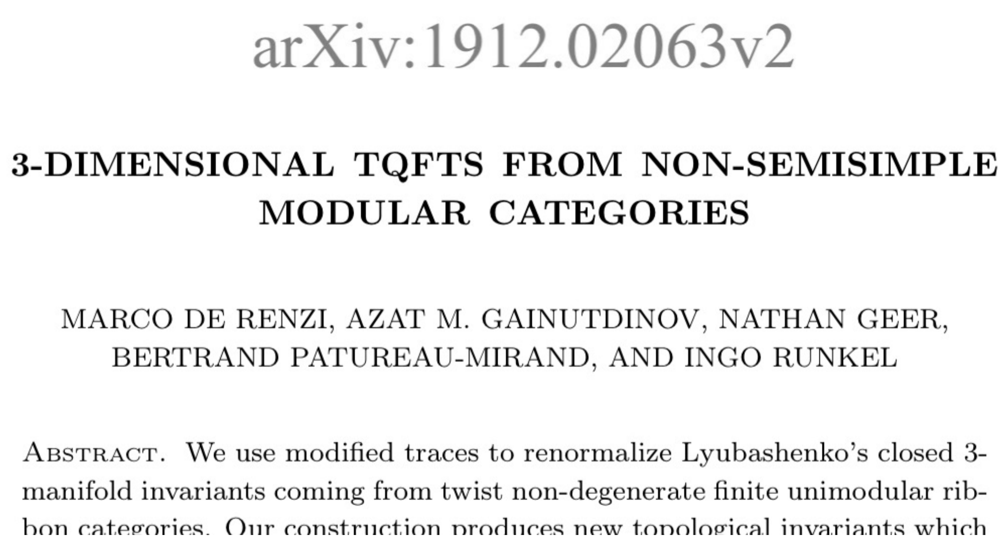
by diffeo property

QED

Mysterious (to me) non-semisimple TQFT constructions:

1990s: Lyubashenko, Kuperberg, Hennings, ...

RT-ish:



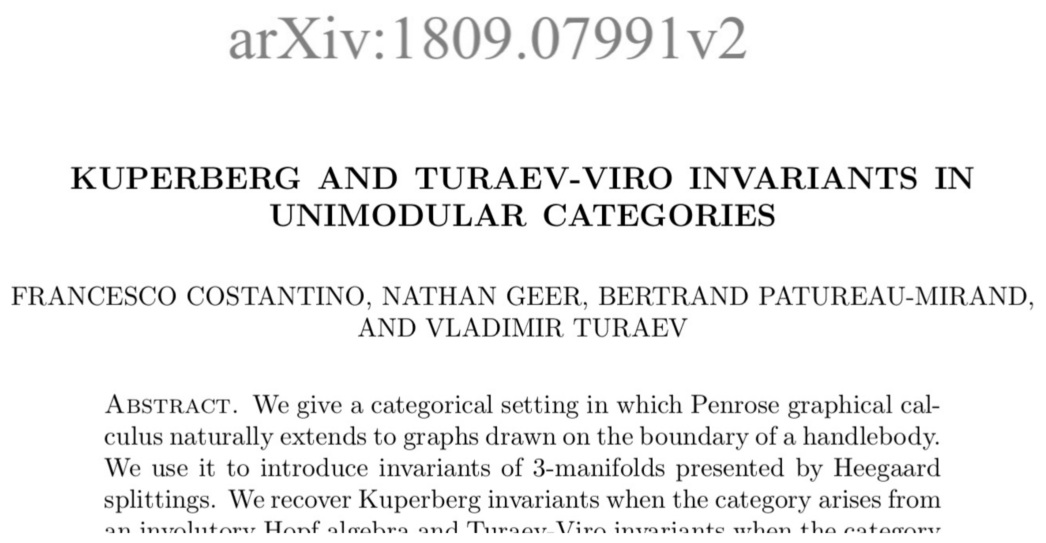
arXiv:1912.02063v2

3-DIMENSIONAL TQFTS FROM NON-SEMISIMPLE MODULAR CATEGORIES

MARCO DE RENZI, AZAT M. GAINUTDINOV, NATHAN GEER,
BERTRAND PATUREAU-MIRAND, AND INGO RUNKEL

Abstract. We use modified traces to renormalize Lyubashenko's closed 3-manifold invariants coming from twist non-degenerate finite unimodular ribbon categories. Our construction produces new topological invariants which

TV-ish:



arXiv:1809.07991v2

KUPERBERG AND TURAEV-VIRO INVARIANTS IN UNIMODULAR CATEGORIES

FRANCESCO COSTANTINO, NATHAN GEER, BERTRAND PATUREAU-MIRAND,
AND VLADIMIR TURAEV

Abstract. We give a categorical setting in which Penrose graphical calculus naturally extends to graphs drawn on the boundary of a handlebody. We use it to introduce invariants of 3-manifolds presented by Heegaard splittings. We recover Kuperberg invariants when the category arises from an involutory Hopf algebra and Turaev-Viro invariants when the category

$\rightsquigarrow$ consider non-$\otimes$-unital $\otimes$-categories.

$\rightsquigarrow$ non-$\otimes$-unital skein theory [D. Jordan]

(this is still work in progress)

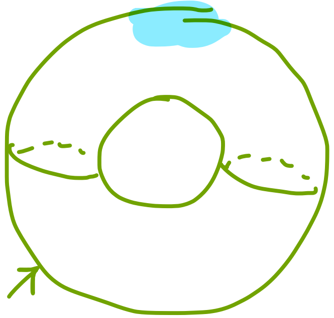
$n=3$, $\mathcal{P}$ = non-$\otimes$-unital ribbon category (e.g. projective ideal in....)

$$A(M^3) := \mathbb{K}\big[\{\mathcal{P}\text{-ribbon-graphs in } M\}\big] \Big/ \langle \text{local relations with non-empty } \partial\text{-condition} \rangle$$

Then

$A(S^3)^* \longleftrightarrow$ space of "modified traces"

$Z(H_0) \in A(S^3)^* \longleftrightarrow$ choice of modified trace pairing P_0 non-degenerate ⟷ modified trace non-degenerate

$A\Big[$  $\Big]$ ⟷ coend of

$S^1\times D^2$

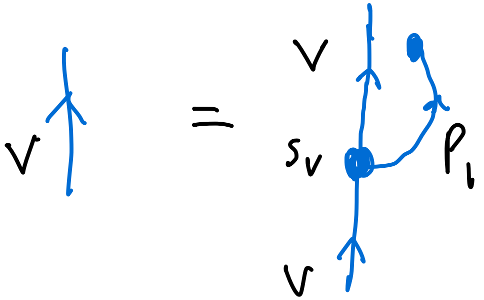
$\circlearrowleft$

$\mathrm{Rep}(P)$

$X\otimes Y \mapsto \mathrm{mor}_P(X^*\otimes Y, -)$

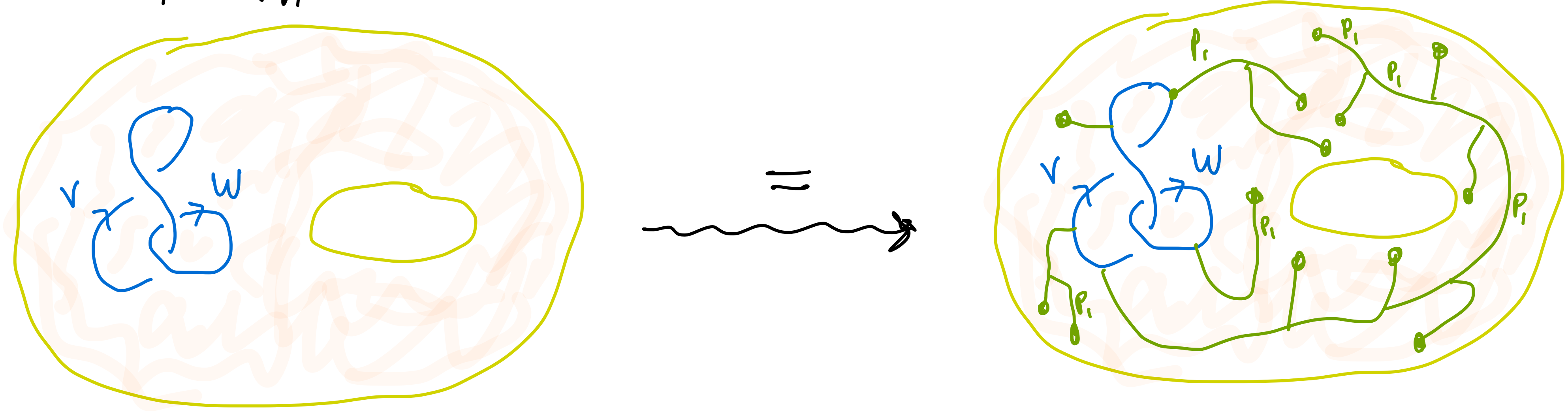
$Z(H_2): A(S^1\times D^2)\to A(D^2\times S^1) \rightsquigarrow$ integral for this coend

Note: In Geer et al examples, P_1 (projective cover of $\otimes$-unit) has the following "weak $\otimes$-unit" property:



$\forall$ objects $V \in \mathcal{P}$

So can fill M^3 with "tendrils":



Non-semisimple Crane-Yetter 3+1-dim'l TQFT

$\mathcal{P}$ as above. W^4 oriented 4-manifold. G: $\mathcal{P}$-ribbon graph in ∂W. Goal: evaluate $Z(W)(G) \in \Bbbk$.

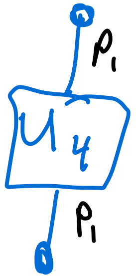
Choose handle decomposition $W_0 \subset W_1 \subset W_2 \subset W_3 \subset W_4 = W$

(W_i = 0-handles $\cup \cdots \cup$ i-handles)

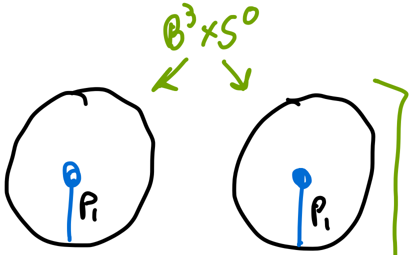
Recall from above

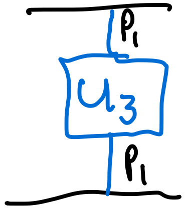
$$Z(H_i): A(B^i \times S^{3-i}) \to A(S^{i-1} \times B^{4-i})$$

Define

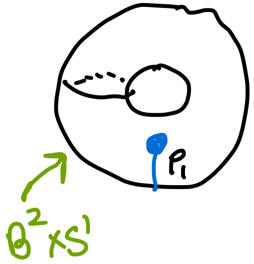
$Z(H_4)(\emptyset) =$ 

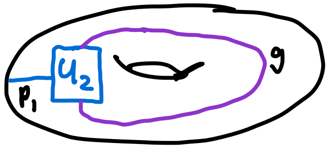
$\in A(S^3)$

$Z(H_3)\Big[$ 

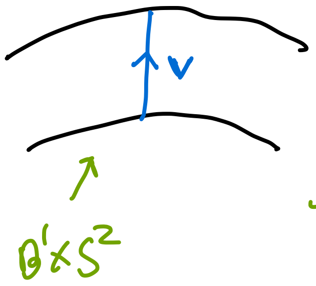
$\Big] =$ 

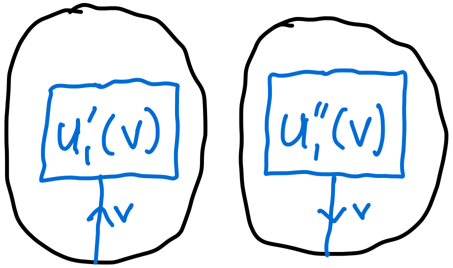
$\in A(S^2 \times B^1)$

$Z(H_2)\Big[$ 

$\Big] =$ 

$\in A(S^1 \times B^2)$

$Z(H_1)\Big[$ 

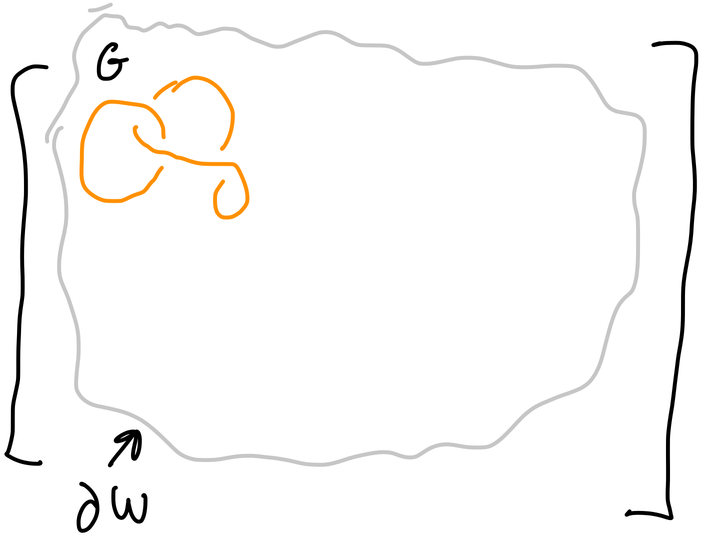
$\Big] =$ 

$\in A(S^0 \times B^3)$

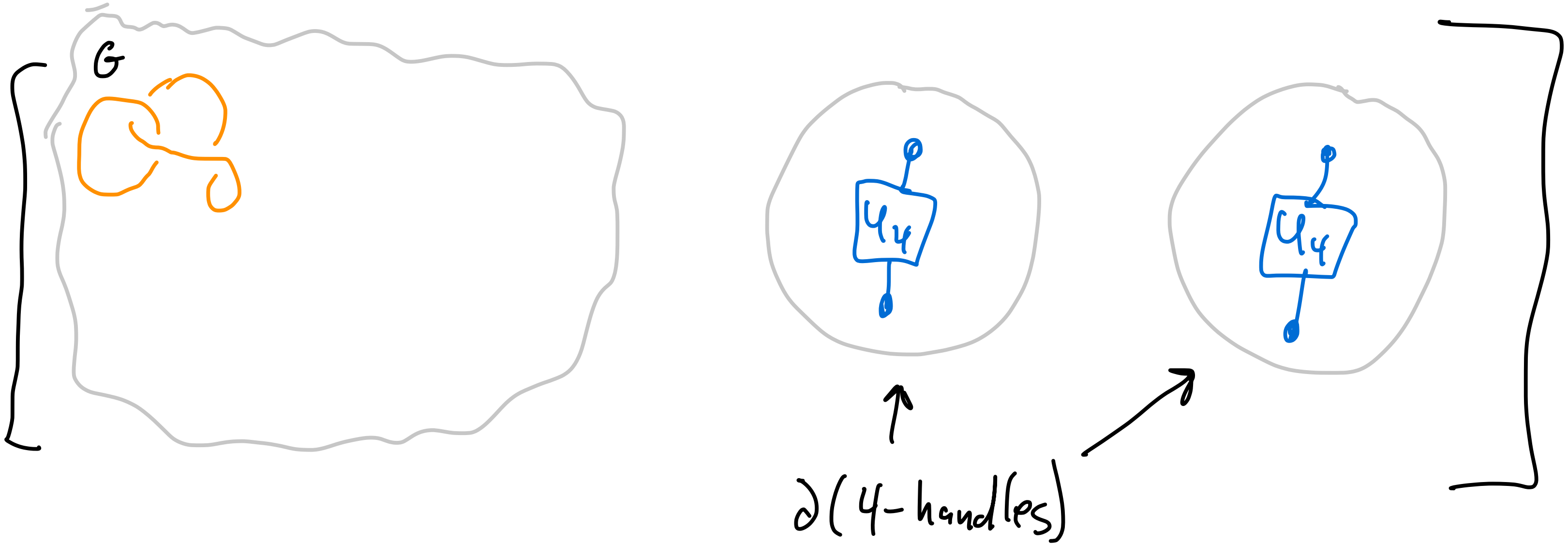
$Z(H_0) = \mathrm{mtr} : A(S^3) \to \Bbbk$

Then....

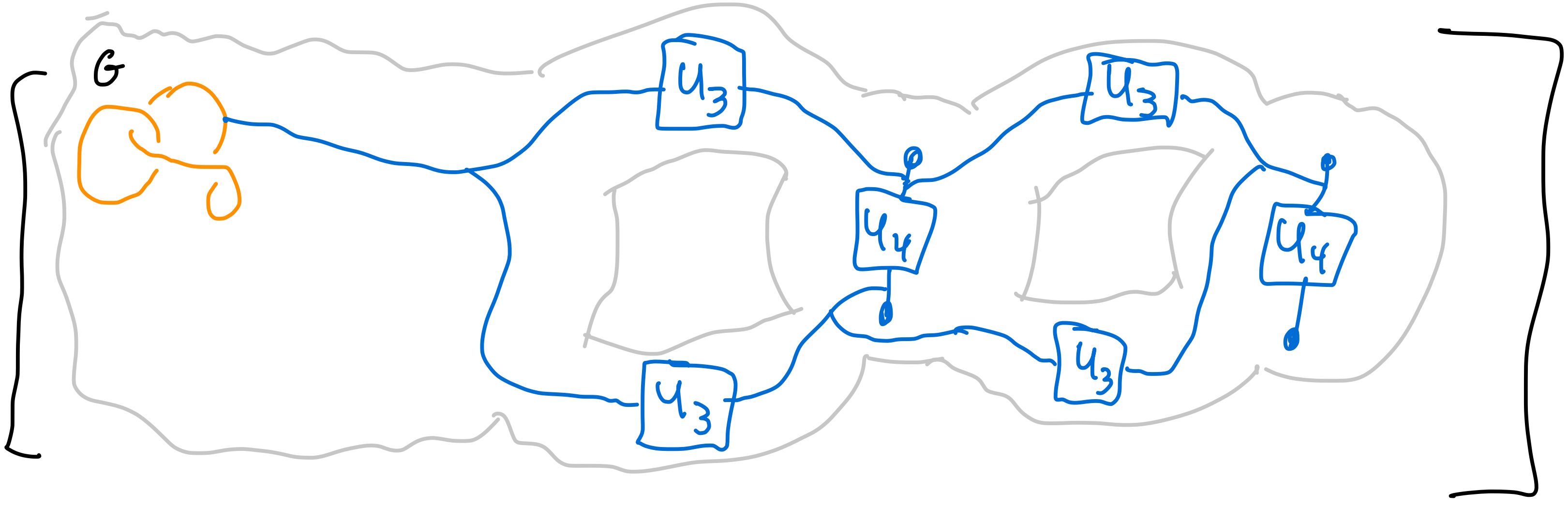
$Z(W_4)$



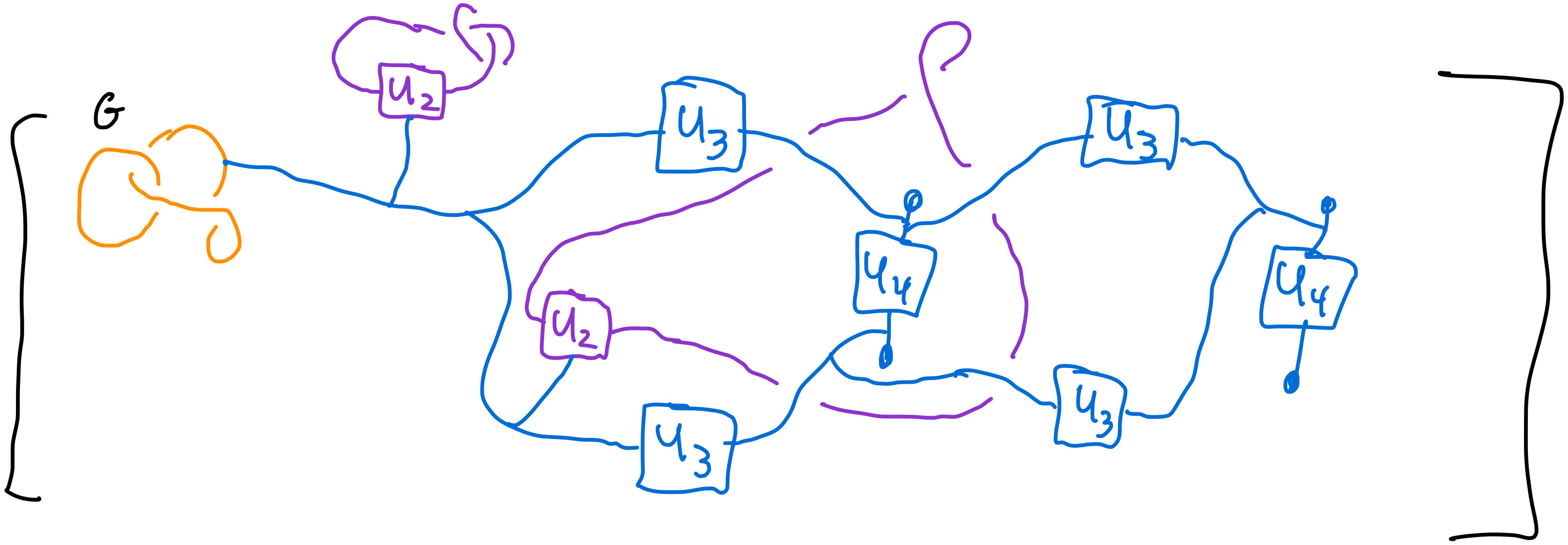
$= Z(W_3)$



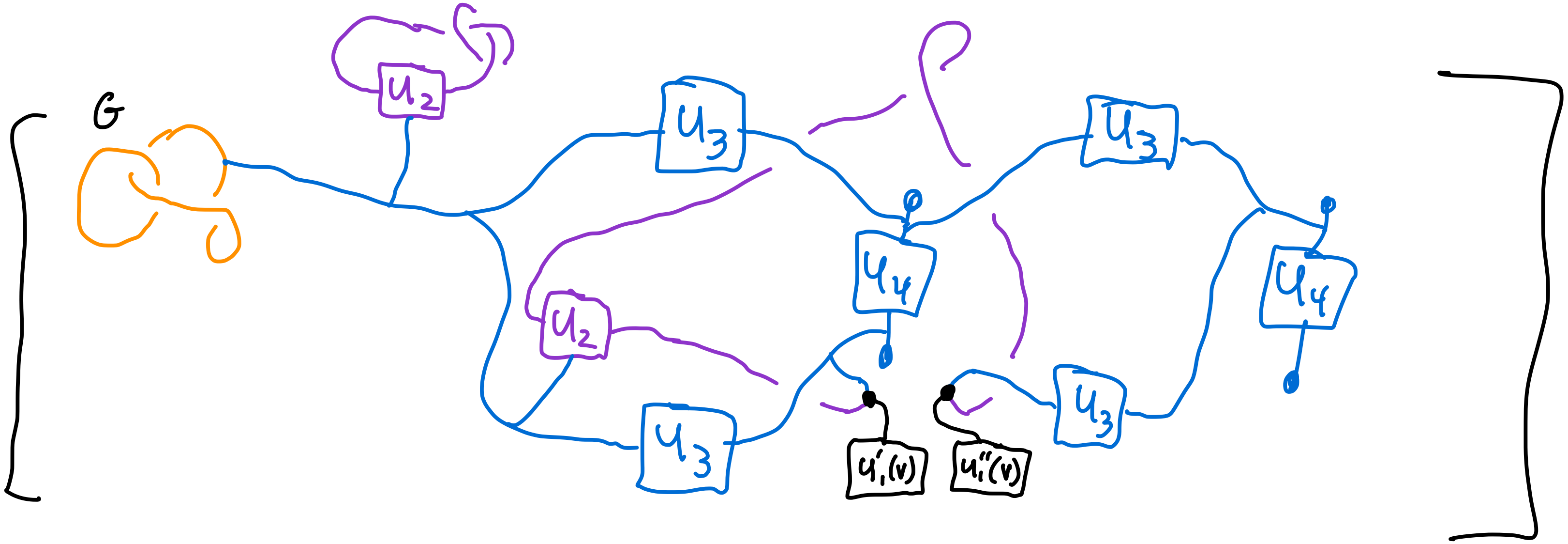
$= Z(W_2)$



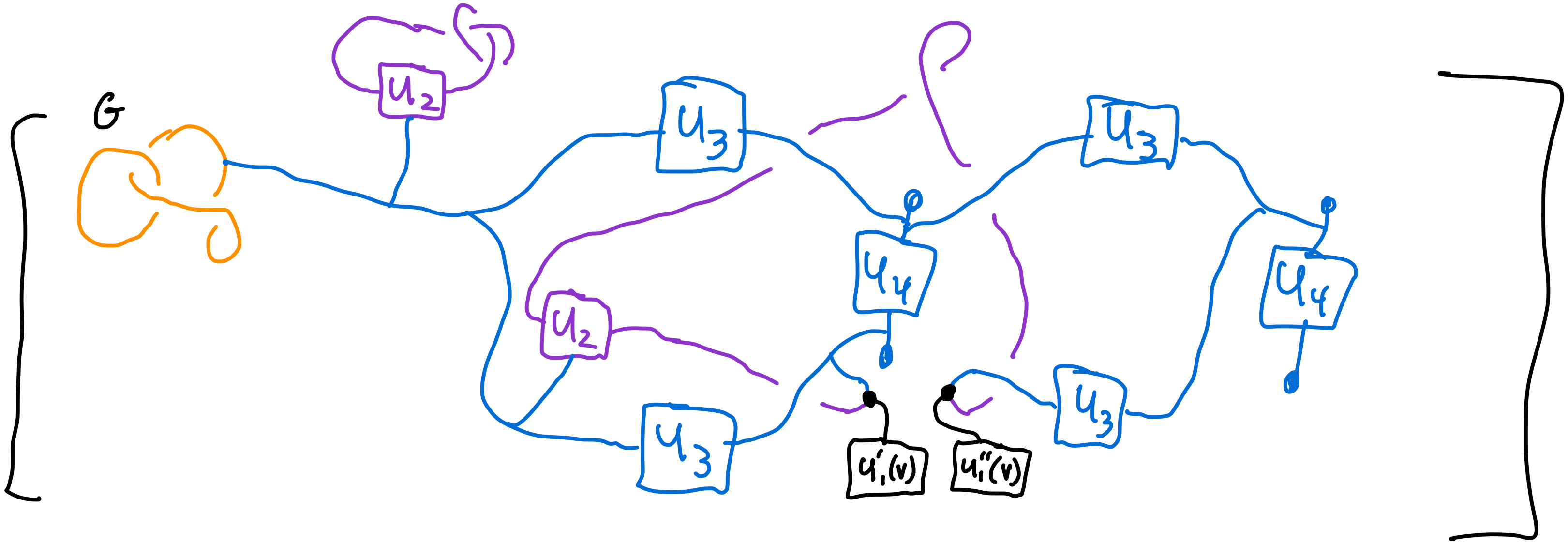
$= Z(W_1)$



$= Z(w_0)$



$=$ mtr



Thm (W, Reutter). Let $A(\cdot)$ be an $n+\varepsilon$-dimensional TQFT as above. Choose $Z(B^{n+1}) = Z(H_0) : A(S^n) \to \mathbb{K}$.

Then $Z(\cdots)$ extends to a full $n+1$-dim'l TQFT if and only if the inductively defined pairings

$$P_K : A(S^K \times B^{n-K}) \otimes A(S^K \times B^{n-K}) \to \mathbb{K} \quad , \quad 0 \leq K \leq n-1$$

are non-degenerate.

Remark 1: If $P_0, P_1, \dots, P_m$ are non-degenerate, then can define $Z(\cdots)$ on $n+1$-dim'l handlebodies, all handles of index $\leq m+1$, invariant under handle cancellations of index $\leq m+1$.

- very common for P_0 to be non-generate:
  - $n=2$ or $n=3$, $\mathrm{Rep}_q(\mathfrak{g})$, q generic. $\Rightarrow$ can define generalized Jone polynomials for links in $\partial(S^1\times B^3 \natural \cdots \natural S^1\times B^3)$

- $\{(n+1)\text{-dim'l } k\text{-handle bodies}\}/(\leq k)\text{-handle moves}$

$$\cong (n+1)\text{-dim'l manifolds w/ } (\leq k)\text{-handle structu}$$

<u>except</u> when $(n+1,k)=(4,2)$.

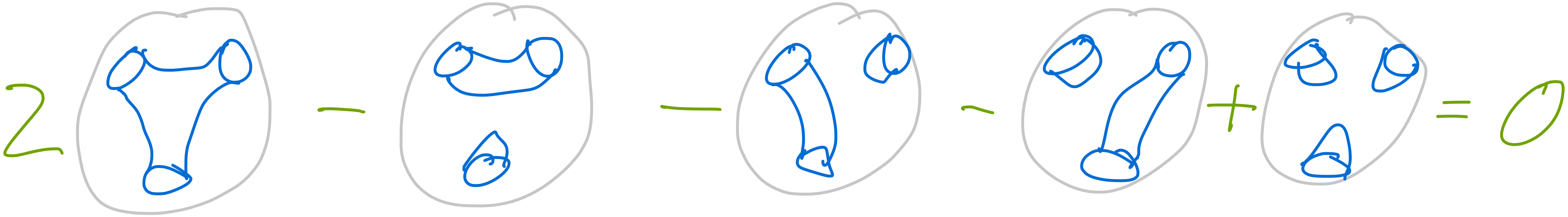
(related to Andrews-Curtis problem)

- interesting $(n+1,k)=(4,2)$ example:

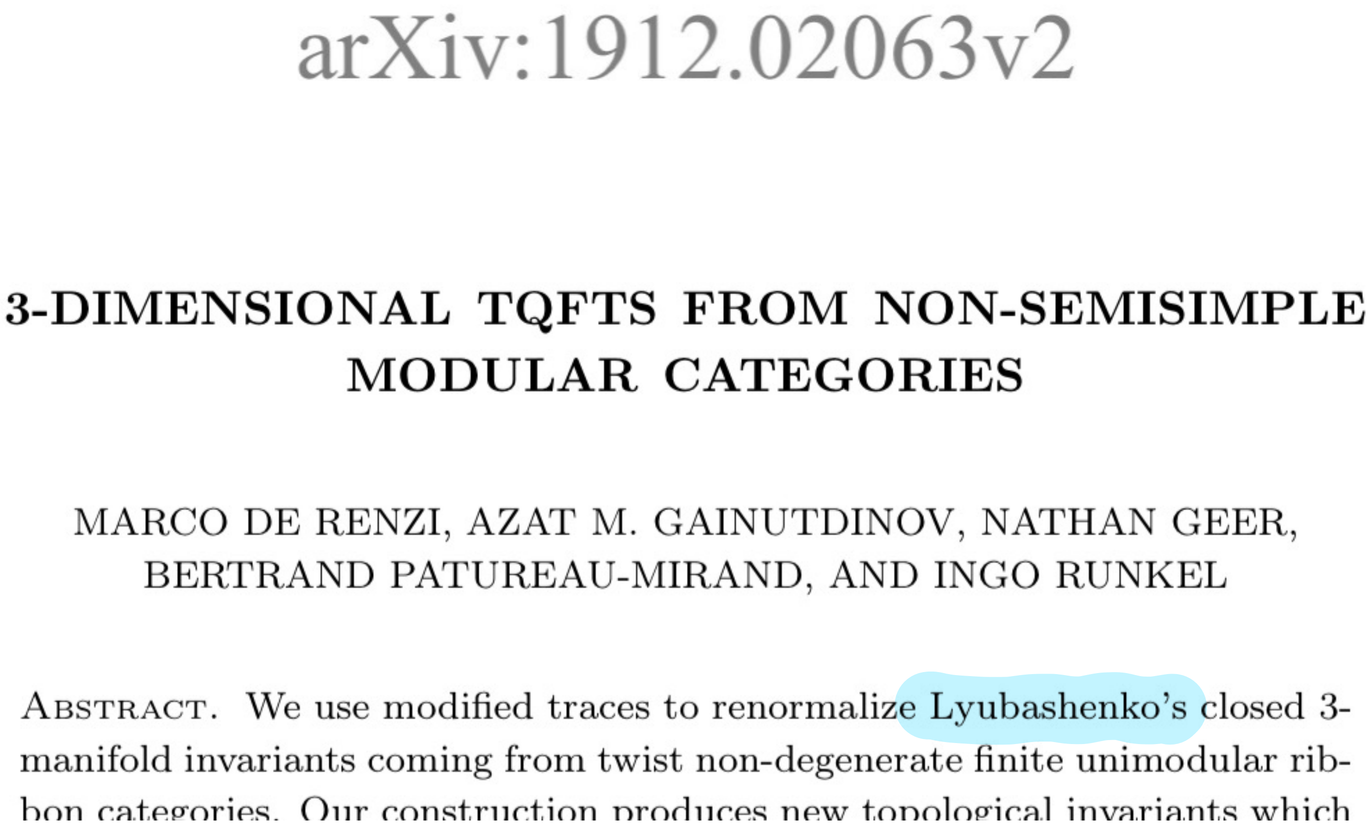
$$A(M^3) := \Bbbk\left[\{\text{unoriented surfaces in } M\}\right]/\sim$$

① partition relations

②



In this example, P_0 and P_1 are non-degenerate, but P_2 is degenerate.



A *trace* t *on a tensor ideal* $\mathscr{I} \subset \mathscr{C}$ is a family of linear maps

$$\{\mathrm{t}_X : \mathrm{End}_{\mathscr{C}}(X) \to \Bbbk\}_{X \in \mathscr{I}}$$

subject to the following conditions:

1) *Cyclicity*: For all $X, Y \in \mathscr{I}$ and $f : X \to Y$, $g : Y \to X$ we have

$$\mathrm{t}_Y(f \circ g) = \mathrm{t}_X(g \circ f);$$

2R) *Right partial trace*: For all $X \in \mathscr{I}$, $V \in \mathscr{C}$ and $h \in \mathrm{End}_{\mathscr{C}}(X \otimes V)$,

$$\mathrm{t}_{X \otimes V}(h) = \mathrm{t}_X(\mathrm{tr}_{\mathrm{R}}(h));$$

2L) *Left partial trace*: For all $X \in \mathscr{I}$, $V \in \mathscr{C}$ and $h \in \mathrm{End}_{\mathscr{C}}(V \otimes X)$,

$$\mathrm{t}_{V \otimes X}(h) = \mathrm{t}_X(\mathrm{tr}_{\mathrm{L}}(h)).$$

Since $\mathscr{C}$ is ribbon, conditions 2R) and 2L) above are equivalent [GKP10].

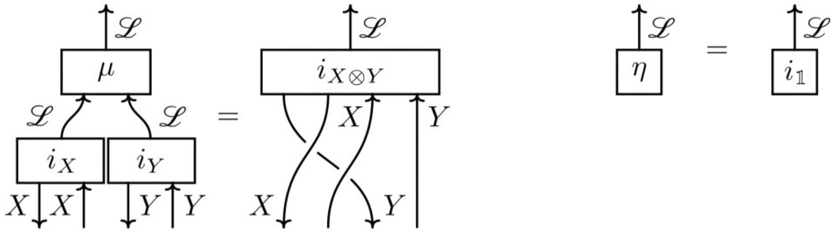
We say a trace t on an ideal $\mathscr{I} \subset \mathscr{C}$ is *non-degenerate* if for every $V \in \mathscr{I}$ and every $W \in \mathscr{C}$ the pairing $\mathrm{t}_V(\cdot \circ \cdot) : \mathscr{C}(W, V) \times \mathscr{C}(V, W) \to \Bbbk$ is non-degenerate. An important example of a tensor ideal is the projective ideal $\mathrm{Proj}(\mathscr{C})$. It is shown in Theorem 5.5 and Corollary 5.6 of [GKP18] that:

2.4. **Coends and ends.** We will now recall some well-known facts about the end of the functor $\mathscr{C} \times \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ sending every $(U,V) \in \mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ to $U \otimes V^* \in \mathscr{C}$ and about the coend of the functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ sending every $(U,V) \in \mathscr{C}$ to $U^* \otimes V \in \mathscr{C}$. We use the notation

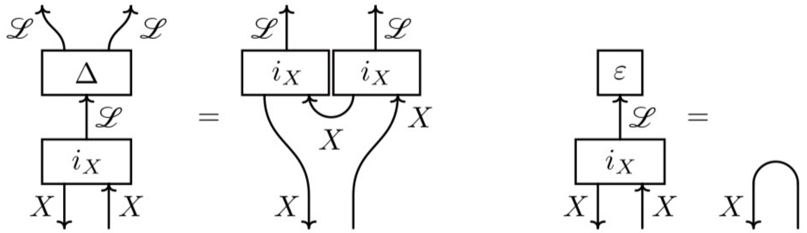
$$\mathscr{E} := \int_{X \in \mathscr{C}} X \otimes X^*, \qquad \mathscr{L} := \int^{X \in \mathscr{C}} X^* \otimes X,$$
$$j_X \colon \mathscr{E} \to X \otimes X^*, \qquad i_X \colon X^* \otimes X \to \mathscr{L},$$

for the end and the coend respectively, and for their corresponding dinatural transformations. See Sections IX.4–IX.6 of [M71] for a definition of dinatural

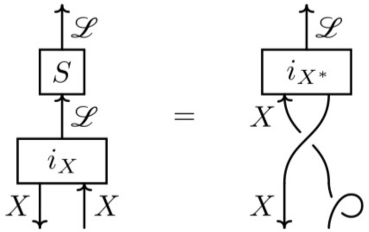
(3)



(4)



(5)



2.5. **Integrals and cointegrals.** Let us assume that $\mathscr{C}$ is in addition unimodular. A morphism $\Lambda \in \mathscr{C}(\mathbb{1}, \mathscr{L})$ is called a *right integral of* $\mathscr{L}$ if it satisfies

$$\mu \circ (\Lambda \otimes \mathrm{id}_{\mathscr{L}}) = \Lambda \circ \varepsilon. \tag{11}$$

A left integral of $\mathscr{L}$ is defined similarly[2]. It is known that right/left integrals of $\mathscr{L}$ exist and are unique up to scalar, see Proposition 4.2.4 of [KL01]. Furthermore,

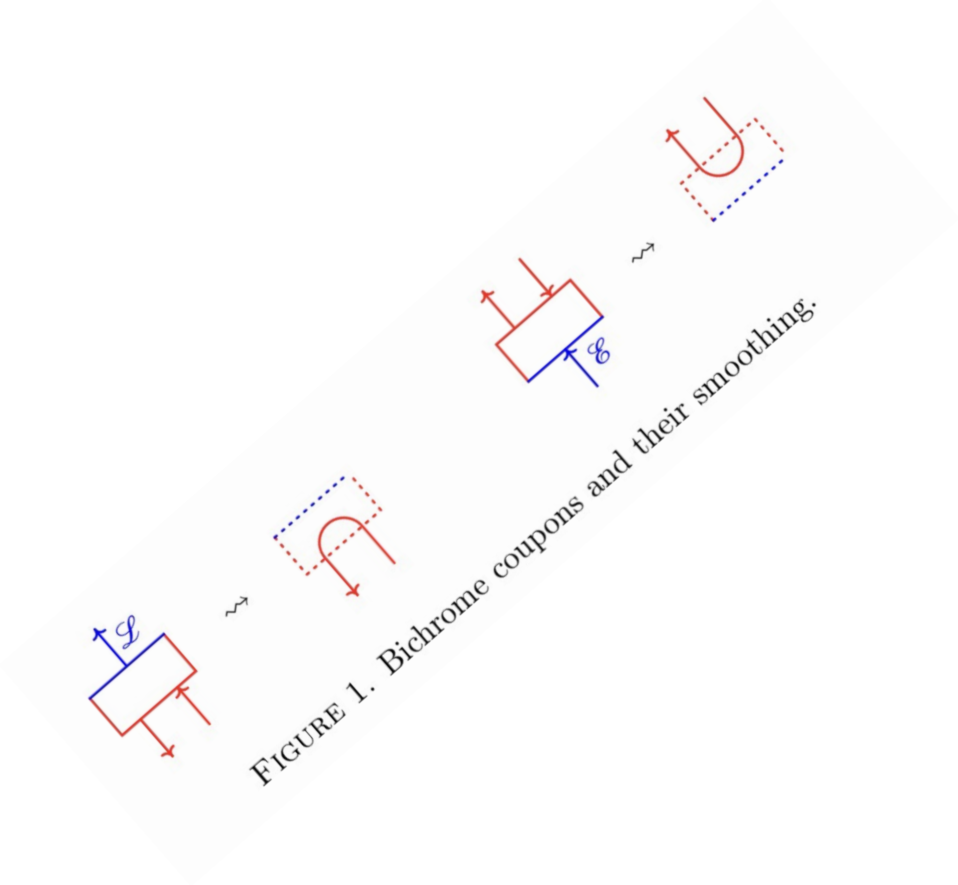


FIGURE 1. Bichrome coupons and their smoothing.

A 0-bottom graph is simply called a *bichrome graph*. See Figure 2 for an example of a 1-bottom graph together with its smoothing.

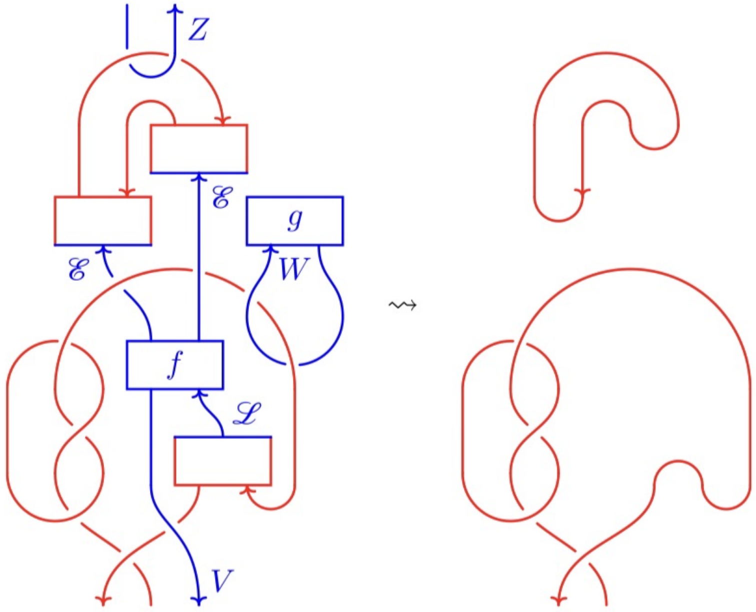


FIGURE 2. A 1-bottom graph and its smoothing.

$(X_1, Y_1, \ldots, X_n, Y_n) \in (\mathscr{C}^{\mathrm{op}} \times \mathscr{C})^{\times n}$. The n-dinatural transformation $\eta_{\tilde{T}}$ associates with every object $(X_1, \ldots, X_n) \in \mathscr{C}^{\times n}$ the morphism

$$F_{\mathscr{C}}(\tilde{T}_{(X_1,\ldots,X_n)}) \in \mathscr{C}(X_1^* \otimes X_1 \otimes \ldots \otimes X_n^* \otimes X_n \otimes F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V}), F_{\mathscr{C}}(\underline{\varepsilon'}, \underline{V'})),$$

where $\tilde{T}_{(X_1,\ldots,X_n)}$ is the ribbon graph obtained from the n-bottom graph $\tilde{T}$ by labeling its kth cycle with X_k, by labeling every bichrome coupon intersecting it with either i_{X_k} or j_{X_k}, the structure morphisms of $\mathscr{L}$ and $\mathscr{E}$ defined in Section 2.4, for every integer $1 \leqslant k \leqslant n$, and by forgetting the distinction between red and blue. The universal property defining $\mathscr{L}$ implies the object $\mathscr{L}^{\otimes n} \otimes F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})$ equipped with the dinatural transformation $i^{\otimes n} \otimes \mathrm{id}_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})}$ is the coend for the functor $H_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})} \circ \sigma$. This determines a unique morphism $f_{\mathscr{C}}(\eta_{\tilde{T}}) \in \mathscr{C}(\mathscr{L}^{\otimes n} \otimes F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V}), F_{\mathscr{C}}(\underline{\varepsilon'}, \underline{V'}))$ satisfying

$$f_{\mathscr{C}}(\eta_{\tilde{T}}) \circ (i_{X_1} \otimes \ldots \otimes i_{X_n} \otimes \mathrm{id}_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})}) = F_{\mathscr{C}}(\tilde{T}_{(X_1,\ldots,X_n)}). \tag{26}$$

Then we define $F_\Lambda(T) : F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V}) \to F_{\mathscr{C}}(\underline{\varepsilon'}, \underline{V'})$ as

$$F_\Lambda(T) := f_{\mathscr{C}}(\eta_{\tilde{T}}) \circ (\Lambda^{\otimes n} \otimes \mathrm{id}_{F_{\mathscr{C}}(\underline{\varepsilon}, \underline{V})}). \tag{27}$$

Proposition 3.1. *$F_\Lambda : \mathscr{R}_\Lambda \to \mathscr{C}$ is a well-defined monoidal functor.*

A universal state sum

K.W. – 2020-05-21

Very brief history:

∃ earlier?

- 1982 – Kauffman (Alexander-Conway polynomial)
- 1990 – Dijkgraaf-Witten (n+1-dim'l, finite group)
- 1992 – Turaev-Viro (2+1 dim'l oriented TQFT)
  (also Barret-Westbury 1996)
- 1993 – Crane-Yetter (3+1 dim'l)
- 2018 – Douglas-Reutter (3+1 dim'l)

State sum, rough idea:

combinatorial description of W^{n+1} ← e.g. triangulation or handle decomposition

$\rightsquigarrow$ set of labelings or "states"

$\rightsquigarrow$ local weights

$\rightsquigarrow Z(W) := \sum_{\text{labelings}} \prod_{\text{weights}} w_i$

$\rightsquigarrow$ equivalent to tensorial contraction

TQFT $\rightsquigarrow$ cut W into (combinatorial) pieces

$\rightsquigarrow$ reassemble pieces $\rightsquigarrow$ tensor contraction

$\rightsquigarrow$ state sum

$$\boxed{Z(W^{n+1}) = \sum_{\beta\in\mathcal{L}(\mathcal{H})} \prod_{j=0}^{n+1} \prod_{\beta\in j\text{-handles}} \frac{ev(\beta(\partial h))}{N(\beta(h))}}$$

H: SO or O or Spin or Pin$_\pm$

W: $n+1$-dim'l H-manifold

$\mathcal{H}$: handle/cell decomposition of W

C: (a) H-pivotal n-category ($\mathbb{K}$-linear)

(b) equipped with "conjugation" map

(c) finite, semisimple

(d) evaluation map $ev: A(S^n) \longrightarrow \mathbb{K}$

which induces non-degenerate pairings on n-morphisms

string diagrams on S^n

See below

$$\boxed{Z(W^{n+1}) = \sum_{\beta\in\mathcal{L}(\mathcal{H})} \prod_{j=0}^{n+1} \prod_{\beta\in j\text{-handles}} \frac{ev(\beta(\partial h))}{N(\beta(h))}}$$

$\mathcal{L}(\mathcal{H})$: the set of labelings of i-handles of $\mathcal{H}$ by minimal $(n+1-i)$-morphisms of C, compatibly with adjacent $(j>i)$-handles

minimal k-morphism y: $End(id^{n-1-k}(y))$ is a simple algebra. WLOG assume that any k-morphism of C is isomorphic to a sum of minimal k-morphisms ("weakly complete")

or minimal idem it $k=n-1$

$e \sim f \iff \exists\, u, v \ni$

$$\boxed{Z(W^{n+1}) = \sum_{\beta\in\mathcal{L}(\mathcal{H})} \prod_{j=0}^{n+1} \prod_{\beta\in j\text{-handles}} \frac{ev(\beta(\partial h))}{N(\beta(h))}}$$

$ev(\beta(\partial h))$ – cells $\cap\ \partial h \rightsquigarrow$ cell complex

$\beta \rightsquigarrow$ labeled cell complex $=: \beta(\partial h)$

(Now for the interesting part....)

$$N(x) := \sum_{\substack{y \text{ minimal} \\ y: x\to x}} \frac{tr_s(y)^2}{N(y)}$$

x: k-morphism

(inductive def'n)

when $k=n$, $N(x) := \langle x,x\rangle^2$

$= tr_s(x)^2$

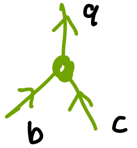
$$tr_s(y) := ev\left[y\times S^{n-k-1} \cup (\partial y)\times B^{n-k}\right]$$

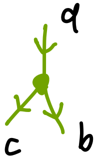
y: k+1-morphism

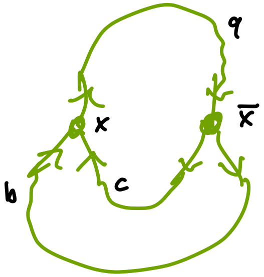
(examples below)

More on n-categories

- "H-pivotal" = string diagrams on H-manifolds make sense (strict version of "H(n)-fixed point", presumably)
- "finite, semisimple" = $\dim(A(M^n;c)) < \infty$ $\forall M, c$
  $A(Y^{n-1};c)$ semisimple $\forall Y, c$
- "conjugation": reflection in "time" direction. If x is a k-morphism, then can form string diagram $x \cup \bar{x}$ on S^k

$x =$ 

$\bar{x} =$ 

$x \cup \bar{x} =$ 

- pairing $A(B^n;c) \otimes A(B^n;c) \to \mathbb{K}$, $x \otimes y \mapsto ev(\bar{x} \cup y)$

↖ assumed non-degenerate

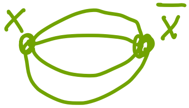
n	SO-pivotal	also finite, semisimple, etc.
2	• pivotal ⊗-cat • pivotal 2-cat	• fusion cat, • subfactor planar alg. • multi-fusion cat
3	• ribbon cat • $\mathrm{Rep}_q(\mathfrak{g})$, q generic • contact 3-cat	• premodular cat • Fusion 2-cat (Douglas-Reutter)
4	• Kh-4-cat (Morrison-Wedrich-W)	
n	• $\pi_{\le n}(X)$, X any space • symmetric monoidal ribbon cat • disk-like n-cat (Morrison-W)	• $\pi_{\le n}(X)$, $\lvert\pi_i(X)\rvert<\infty$ $\forall i$ • Rep(G), G: finite group

$\mathrm{tr}_s(x)$

C: n-cat, $x \in C^k$

$n=2,\ k=0$ $ev(\emptyset_x)$

$n=2,\ k=1$ $ev($  $_x) = d_x$

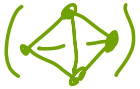
$n=2,\ k=2$ $ev($ 

$) = \langle x, x \rangle$

$n=3,\ k=0$ $ev(\emptyset_x)$

$n=3,\ k=1$ $ev($  $_x) = ev(S^2_x)$

$n=3,\ k=2$ $ev\Big[$ "spun  " $\Big]$

$n=3,\ k=3$ $ev\Big[$ double-cone() $\Big]$

$$N(x) = \sum_{\substack{y : x \to x \\ y \text{ minimal}}} \frac{tr_s(y)^2}{N(y)}$$

C: n-cat, $x \in C^k$

$k = n$ $\quad N(x) = ev(x \cup \overline{x})^2$

$k = n-1$ $\quad N(x) = \dim(End(x)) \leftarrow \begin{cases} = 1 \text{ for normal simple obj} \\ = 2 \text{ for Majorana simple obj.} \end{cases}$

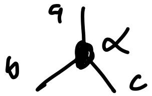
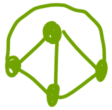
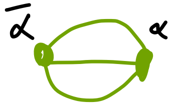
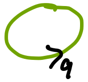
$k = n-2$ $\quad N(x) = GD(End(x)) = \sum_y \frac{d_y^2}{N(y)} \xleftarrow{\text{fun}} \bullet\, ev(S^2_x)^2$

$k = n-3$ $\quad N(x) = \sum_y \frac{1}{GD(End(y))}$

Turaev-Viro

input: fusion category C

assume generic cell decomp.

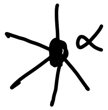
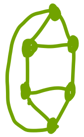
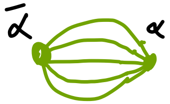
	0h	1h	2h	3h
label	—		simple ob a	$* \in C^0$
$ev(\beta(\partial h))$	 $= Tet$	 $= \Theta_{abc\alpha}$	 $= d_a$	$\emptyset = 1$
$N(\beta(h))$	1	$\Theta_{abc\alpha}^2$	1	$GD = \sum_x d_x^2$

$Z(M^3) = \sum_{\text{labelings}} \prod_{0h} ev($  $) \prod_{1h} \frac{1}{\Theta_{abc\alpha}} \prod_{2h} d_a \prod_{3h} \frac{1}{GD}$

Turaev-Viro

input: fusion category C

general cell decomp.

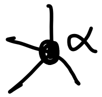
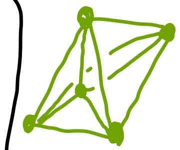
	0h	1h	2h	3h
label	—		simple ob a	$* \in C^0$
$ev(\beta(\partial h))$	 = Link(h)	 = Θ_α	 $= d_a$	$\emptyset = 1$
$N(\beta(h))$	1	Θ_α^2	1	$GD = \sum_x d_x^2$

$$Z(M^3) = \sum_{\text{labelings}} \prod_{0h} ev(Link(h)) \prod_{1h} \frac{1}{\Theta_\alpha} \prod_{2h} d_a \prod_{3h} \frac{1}{GD}$$

Crane-Yetter

input: premodular cat C

assume generic cell decomp.

	0h	1h	2h	3h	4h
label	–	 (Tet)	simple obj q	$*_1 \in C^1$	$*_0 \in C^0$
ev(β(∂h))	 4-simplex		 $= d_q$	$\emptyset = 1$	$\emptyset = 1$
N(β(h))	1	θ_α^2	1	GD(C)	$GD(C)^{-1}$

$$Z(W^4) = \sum_{\text{labelings}} \prod_{0h} ev(4\text{-simplex}) \prod_{1h} \frac{1}{\theta_\alpha} \prod_{2h} d_a \prod_{3h} \frac{1}{GD} \prod_{4h} GD$$

Crane-Yetter

input: premodular cat $\mathcal{C}$

$W^4 = 0h \cup \{2h\}$

along framed link $L \subset S^3$

	0	2
labels	—	simple obj a
$ev(\beta(\partial h))$	$J(L,\beta)$	$\bigcirc_a = d_a$
$N(\beta(h))$	1	1

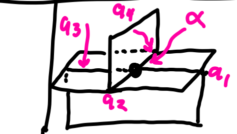
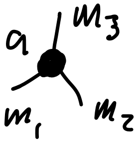
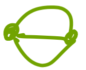
$$Z(W^4) = \sum_{\substack{\text{labelings}\\ \beta}} J(L,\beta) \cdot \prod_{2h} d_{\beta(h)}$$

RT Dehn surjery formula, up to Euler char. normalization

$\mathcal{C}$ modular $\Rightarrow$ depends only on ∂W

Douglas-Reutter

input: fusion 2-cat
(finite s.s. 3-cat, one 0-mor.)
generic cell decomp.

	0h	1h	2h	3h	4h
label	–	 (Tet)	 simple obj a	minimal $m \in C^1$	$* \in C^0$
$ev(\beta(\partial h))$	4-simplex	d-cone(Tet)	span()	 $_m = S^2_m$	$\emptyset = 1$
$N(\beta(h))$	1	$d\text{-cone}(Tet)^2$	1	$GD_m \cdot ev(S^2_m)^2$	$\underbrace{\sum_m \frac{1}{GD_m}}_{GD_3(C)}$

$$Z(W^4) = \sum_{\text{labelings}} \prod_{0h} ev(4\text{-simplex}) \prod_{1h} \frac{1}{ev(d\text{-cone}(Tet))} \prod_{2h} ev\Big(span\big($$

$$\big)\Big)$$

$$\prod_{3h} \frac{1}{GD_m \, ev(S^2_m)} \quad \prod_{4h} \frac{1}{GD_3(C)}$$

Proof

Outline:

① $(n+\varepsilon)$-dim'l TQFT

② Inductive construction of path integral

$$\cdots \to Z(S^k \times B^{n-k+1}) \rightsquigarrow A(S^k \times B^{n-k}) \text{ pairing}$$

$$\rightsquigarrow A(S^k \times B^{n-k}) \text{ copairing} \rightsquigarrow Z(S^{k+1} \times B^{n-k}) \rightsquigarrow \cdots$$

③ Compute P.I. $Z(W^{n+1})$ in terms of handle decomposition $\mathcal{H}$

④ Gluing associativity lemma ⇒ indep. of choice of $\mathcal{H}$

⑤ Observe that ③ is a state sum

$(n+\varepsilon)$-dim'l TQFT

- $\mathcal{C}$: H-pivotal n-category

- $\mathcal{C}(X^k; b)$: $\mathcal{C}$-string diagrams on X, $0 \le k \le n$.
  with $\partial = b$

- $A(M^n; b)$: $\quad \mathbb{K}[\mathcal{C}(M;b)]/\sim \quad$ "(generalizd) skein module"

- $A(Y^{n-1}; b)$: 1-cat $\left[\begin{array}{ll} \text{0-mor:} & \mathcal{C}(Y;b) \\ \underset{x\to y}{\text{1-mor}} & A(Y\times I; \bar{x}\cup y) \end{array}\right.$

- $A(V^{n-2}; b)$: 2-cat $\left[\begin{array}{ll} \text{0-mor} & \mathcal{C}(V;b) \\ \underset{x\to y}{\text{1-mor}} & \mathcal{C}(V\times I; \bar{x}\cup y) \\ \underset{p\to q}{\text{2-mor}} & A(V\times B^2; \bar{p}\cup q) \end{array}\right.$

$\vdots$

- $A(X^{n-k}; b)$: k-cat with j-morphisms $\begin{cases} \mathcal{C}(X\times B^j; \dots) & j<k \\ A(X\times B^k; \dots) & j=k \end{cases}$

Path Integral Axioms

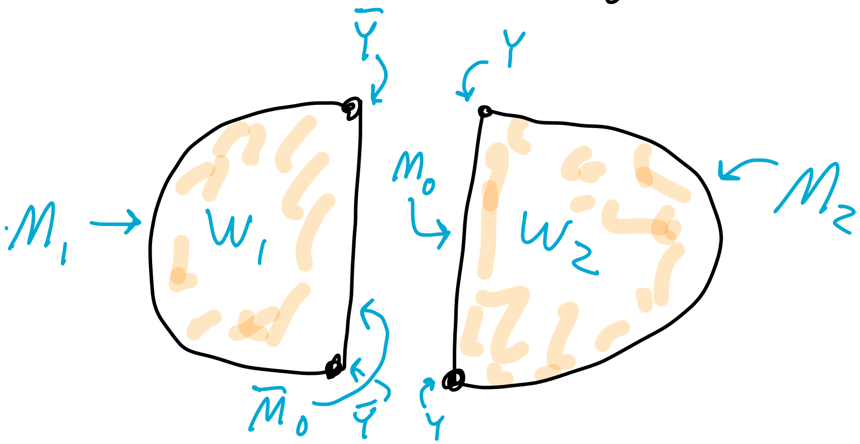
$\boxed{0}$ $Z(W^{n+1}) : A(\partial W) \longrightarrow \mathbb{k}$

$\boxed{1}$ pairing $P_{(M,b)} : A(\bar{M};\bar{b}) \otimes A(M;b) \longrightarrow \mathbb{k}$

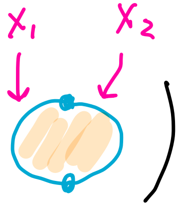
assumed non-degenerate $\bar{x} \otimes y \longmapsto Z(M\times I)(\bar{x} \cup y)$

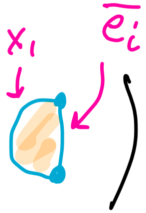
$\Rightarrow$ copairing $Q_{(M,b)} : \mathbb{k} \to A(M;b) \otimes A(\bar{M},\bar{b})$

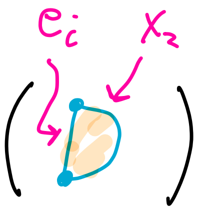
$\boxed{2}$ gluing $W = W_1 \cup_{Y_0} W_2$



$$\xrightarrow{A(M_1)} Z(W) \xleftarrow{A(M_2)} \;=\; \xrightarrow{A(M_1)} Z(W_1) \xleftarrow{A(\bar{M}_0)} Q_{M_0} \xrightarrow{A(M_0)} Z(W_2) \xleftarrow{A(M_2)}$$

$Z(W)\Big($


$\Big) = \sum_i Z(W_1)\Big($


$\Big) \cdot Z(W_2)\Big($


$\Big) \cdot \dfrac{1}{\langle \bar{e}_i, e_i \rangle}$

where $b = \partial x_2 = \overline{\partial x_1}$, $\{e_i\}$ orthog. basis of $A(M_0, \breve{b})$

$$p_{M_0,\bar{b}}(\bar{e}_i, e_j) =: \langle \bar{e}_i, e_j \rangle = \delta_{ij} \cdot \lambda_i$$

Def. "(m,k)-handlebody": an m-manifold built out of $(0 \le i \le k)$-handles

Inductive assumptions (k)

① $Z(W^{n+1})$ defined $\forall$ $(n+1,k)$-handlebodies
invariant under handle slides and index $\leq k$ handle cancellations

② Pairings for (M,b) nondegenerate $\forall$ b and
$\forall$ (n,k)-hbodies M

Start of induction (k=0)

$$Z(B^{n+1}) = ev : A(S^n) \longrightarrow \mathbb{K}$$

I.P. $$A(\overline{B^n};\bar{c}) \otimes A(B^n;c) \longrightarrow \mathbb{K}$$

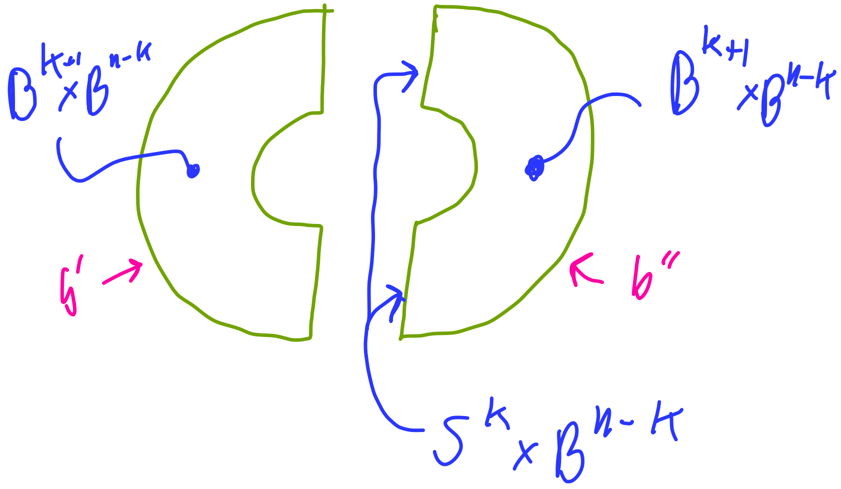
$$\bar{x} \otimes y \longmapsto Z(B^{n+1})(\bar{x} \cup y)$$

assumed nondegenerate $\forall$ $c \in C(\partial B^n)$

Inductive step $(K \to k+1)$

$b = b' \cup b''$

$\boxed{A}$ Compute $Z(S^{k+1} \times B^{n-k})(b)$, $b \in C(\partial(S^{k+1} \times B^{n-k}))$



$$Z(S^{k+1} \times B^{n-k})(b) = \sum_i \frac{Z(B^{k+1} \times B^{n-k})(b' \cup \bar{e}_i) \cdot Z(B^{k+1} \times B^{n-k})(b'' \cup e_i)}{\langle \bar{e}_i, e_i \rangle}$$

Special case: $b = S^{k+1} \times c \quad , \quad c \in C(S^{n-k-1})$

$\star$ $\{m_i\}$ orthogonal basis of $A(S^k \times B^{n-k} ; S^k \times c)$

$\uparrow$ minimal $(n-k)$-morphisms with $\partial m_i = c$

$$Z(S^{k+1} \times B^{n-k})(S^{k+1} \times c) = \sum_{m_i} \frac{\left[Z(B^{k+1} \times B^{n-k})(B^{k+1} \times c \cup m_i) \right]^2}{\langle \bar{m}_i , m_i \rangle}$$

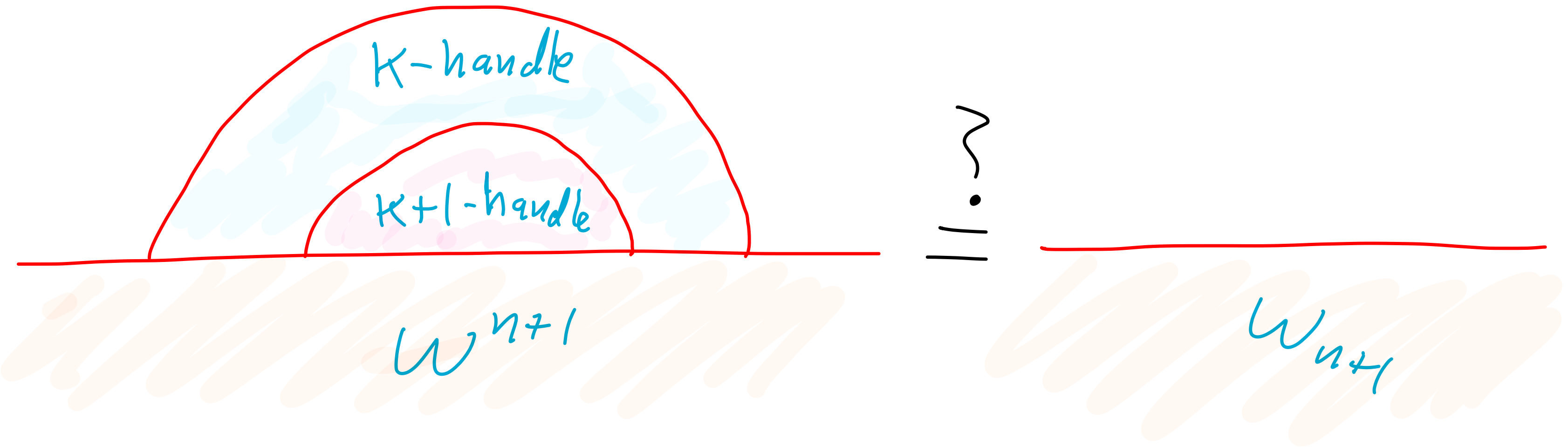
$$= \sum_{m_i} \frac{tr_s(m_i)^2}{\langle \bar{m}_i , m_i \rangle}$$

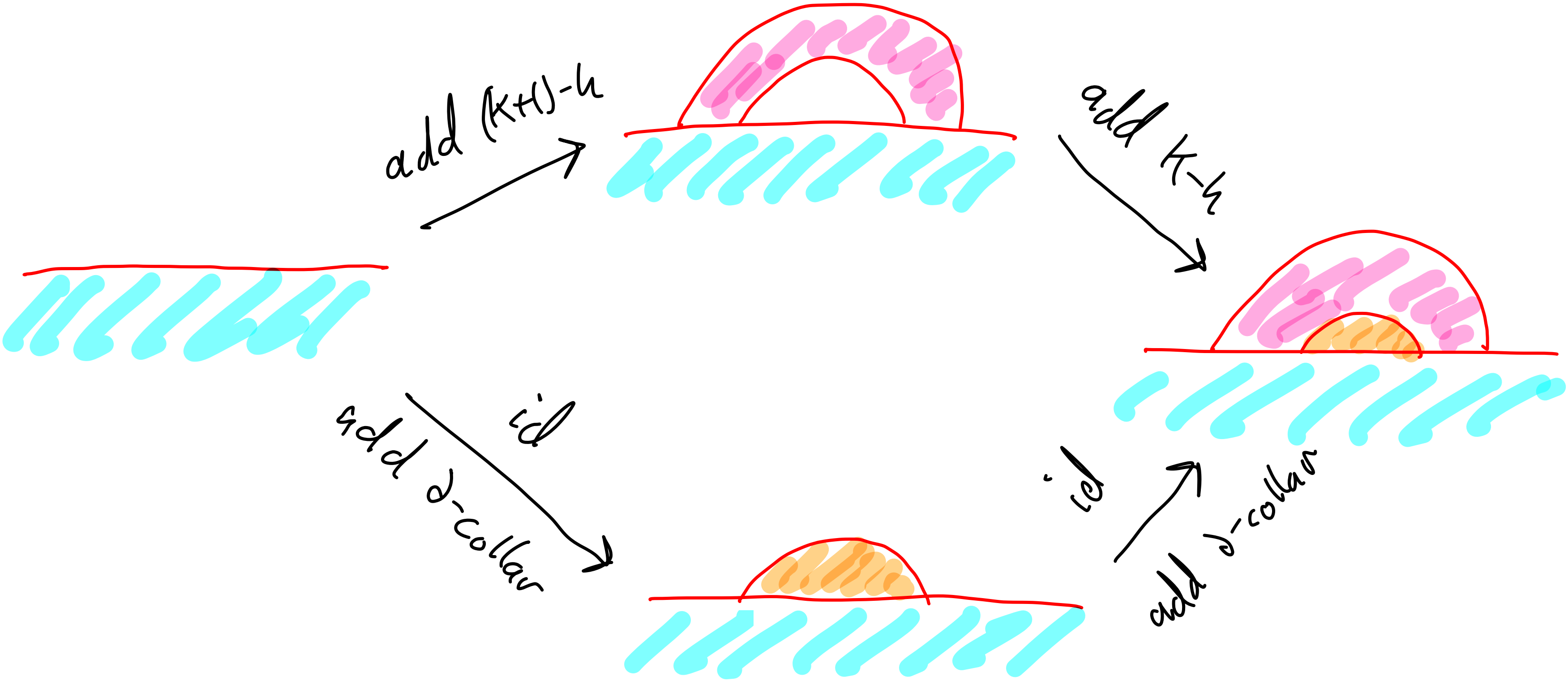
$\Downarrow$

$$\boxed{\langle \underset{\substack{\uparrow \\ N(x)}}{S^{k+1} \times \bar{x}} , S^{k+1} \times x \rangle_{S^{k+1} \times B^{n-k-1}} = \sum_{\substack{\text{minimal} \\ y \in End(x)}} \frac{tr_s(y)}{\underset{\substack{\uparrow \\ N(y)}}{\langle S^k \times \bar{y} , S^k \times y \rangle_{S^k \times B^{n-k}}}}}$$

$\boxed{B}$ Can now compute $Z(W^{n+1})$ for any $(n+1, k+1)$-handlebody W^{n+1}. Independent of handle decomp?

- Invariance w.r.t. handle slides: trivial
- Invariance w.r.t. hande cancellations:





★ So associativity of gluing formula implies invariance w.r.t. handle cancellation

Thm. (W, 2006) If C is an n-category satisfying (a)-(d) far above (pivotal, finite, semisimple, non-degenerate eval. map), then the "easy" $(n+\varepsilon)$-dim'l TQFT extends uniquely to an $(n+1)$-dim'l TQFT with $Z(B^{n+1}) = ev$.

The proof of the above thm provides an algorithm for computing $Z(W^{n+1})$ in terms of a handle decomposition. To go from the algorithm to the state sum formula, observe that minimal j-morphisms give an orthogonal basis of $A(S^{n-j} \times B^j; S^{n-j} \times c)$.

Brown-Arf $(\mathrm{Pin}_-, n=1)$

$ev(\emptyset) = \lambda \in \mathbb{C}$

input $\mathbb{C}l(1)$, s odd, $s^2=1$

reflection of β' $\longrightarrow$ $r(s) = \alpha \cdot s \quad \alpha^4 = 1$

	0	1	2
label	—	id or S	$*$
$ev(\beta(\partial h))$	$\lambda \cdot \alpha^k$ or zero	λ	λ
$N(\beta(h))$	1	λ^2	2

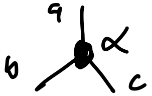
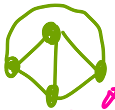
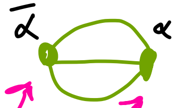
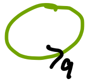
$$Z(W^2) = \lambda^{\chi(W)} \sum_{\text{labelings}} \prod_{0h} \alpha^k \prod_{2h} \frac{1}{2}$$

$$Z(W^2) = \lambda^{\chi(W)}\, 2^{-\#2h} \sum_{\text{1-cycles } x} \alpha^{q(x)} \longleftarrow q(x) \in \mathbb{Z}/4$$

cf. Kirby-Taylor

Fermionic TV

input: super fusion cat $\mathcal{C}$
$(n=2, H=\mathrm{Spin})$

	0h	1h	2h	3h
label	–		simple ob a maybe Majorana	$* \in C^0$
$ev(\beta(\partial h))$	 $= \mathrm{Tet}$	 $= \theta_{abc\alpha}$	 $= d_a$	$\emptyset = 1$
$N(\beta(h))$	1	$\theta^2_{abc\alpha}$	1 or 2	$GD = \sum_x d_x^2 / N(x)$

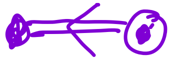
maybe fermionic

Simplest super fusion categories: (2 simple objects)

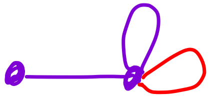
$\{\mathbb{1}, x\}$ $\{\mathbb{1}, m\}$

↑ normal ↑ Majorana

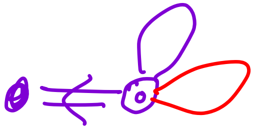
C_2 : $m \otimes m \cong \mathbb{C}^{1/1} \cdot \mathbb{1}$



$SO(3)_6/4$: $x \otimes x \cong \mathbb{1} \oplus \mathbb{C}^{1/1} \cdot x$



$\frac{1}{2}E_6/4$: $m \otimes m \cong \mathbb{C}^{1/1} \cdot \mathbb{1} \oplus \mathbb{C}^{1/1} \cdot m$



If $A(Y^2) \cong \mathbb{C}^{p/q}$, then

$Z(Y \times S^1_B) = p+q$ and $Z(Y \times S^1_N) = p-q$

	C_2	$SO(3)_6/\psi$	$\frac{1}{2}\mathrm{E}_6/y$
$g = 1, \mathrm{Arf} = 0$	$3 \mid 0$	$4 \mid 0$	$3 \mid 0$
$g = 1, \mathrm{Arf} = 1$	$0 \mid 3$	$2 \mid 2$	$1 \mid 2$
$g = 2, \mathrm{Arf} = 0$	$10 \mid 0$	$40 \mid 24$	$19 \mid 8$
$g = 2, \mathrm{Arf} = 1$	$0 \mid 10$	$32 \mid 32$	$11 \mid 16$
$g = 3, \mathrm{Arf} = 0$	$36 \mid 0$	$1184 \mid 1120$	$281 \mid 232$
$g = 3, \mathrm{Arf} = 1$	$0 \mid 36$	$1152 \mid 1152$	$241 \mid 272$
$g = 4, \mathrm{Arf} = 0$	$136 \mid 0$	$51328 \mid 51072$	$5755 \mid 5504$
$g = 4, \mathrm{Arf} = 1$	$0 \mid 136$	$51200 \mid 51200$	$5531 \mid 5728$
$g = 5, \mathrm{Arf} = 0$	$528 \mid 0$	$2368000 \mid 2366976$	$126449 \mid 125056$
$g = 5, \mathrm{Arf} = 1$	$0 \mid 528$	$2367488 \mid 2367488$	$125137 \mid 126368$

Figure 4.5.1: Hilbert space dimensions for closed surfaces in various theories.

For $C = C_2$ above, state sum labelings are

$$\{(\mathbb{Z}/2\text{-}2\text{-cycle } S,\ 1\text{-cycle } J \subset S)\}$$

Guess $$Z_{C_2}(M^3) = \frac{1}{2} \sum_{S \in H_2(M;\mathbb{Z}/2)} (-1)^{\mathrm{Arf}(S)}$$