TQFT club

Tuesday, May 2, 2023 9:51 AM

Witt groups of braided fusion categories and minimal non-degenerate extensions

by Dmitri Nikshych

A **fusion category** $(\mathcal{C}, \otimes, 1)$ is an Abelian semisimple category $\mathcal{C}$ along with $\mathbb{C}$-linear

$\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, a bifunctor,
1 – a unit object,
and constraints

$a_{X,Y,Z} : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$ associativity
$l_X : 1\otimes X \xrightarrow{\sim} X$, $r_X : X\otimes 1 \xrightarrow{\sim} X$ unit

that satisfy compatibility axioms (pentagon).

We will assume that there is a duality in $\mathcal{C}$, that is, for each $X\in\mathcal{C}$ there is X^* along with $ev_X : X^*\otimes X \to 1$, $coev_X : 1 \to X\otimes X^*$.

The term **fusion** corresponds to

$$X\otimes Y \simeq \sum_Z N_{XY}^Z Z, \quad N_{XY}^Z \in \mathbb{Z}_{\geq 0}.$$

Examples of fusion categories come from s/s Hopf algebras, affine Lie algebras/quantum groups, subfactors.

A **braided** fusion category is equipped with an additional constraint

$c_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X$, **braiding**,

again, satisfying certain compatibility.

Examples ① $\mathrm{Rep}(G)$, where G is a finite group. More generally, $\mathrm{Rep}(H)$, where H is a quasi-triangular Hopf algebra. R-matrix

②. Let A be a finite Abelian group, $q: A\to\mathbb{C}^\times$ be a quadratic form. Then there is a braided fusion category $\mathcal{C}(A,q)$, where A is the group of simple objects and

$x\otimes y = xy$ **pointed**

$$c_{x,x} = q(x), \quad x\in A.$$

③ For a simple Lie algebra $\mathfrak{g}$ and $\ell\in\mathbb{Z}_{>0}$ there is a braided fusion category $\mathcal{C}(\mathfrak{g},\ell)$ that can be defined by means of quantum groups or affine Lie algebras.

The center construction Given a fusion category $\mathcal{A}$ its **center**

$$\mathcal{Z}(\mathcal{A}) = \{(Z,\{\gamma_X\}) \mid Z\in\mathcal{A},\ \gamma_X : X\otimes Z \xrightarrow{\sim} Z\otimes X,\}_{X\in\mathcal{C}}$$

is a braided fusion category.

For example, $\mathcal{Z}(\mathrm{Rep}(H)) = \mathrm{Rep}(D(H))$, where $D(H)$ is the **Drinfeld double** of H.

Centralizers and non-degeneracy:

Note that $c_{YX}c_{XY} \neq \mathrm{id}_{X\otimes Y}$ in general.

We say that objects $X,Y\in\mathcal{C}$ **centralize** each other if

$$c_{YX}c_{XY} = \mathrm{id}_{X\otimes Y}.$$

For each $\mathcal{B}\subset\mathcal{C}$ let $\mathcal{B}' := \{X\in\mathcal{B} \mid c_{YX}c_{XY} = \mathrm{id}_{X\otimes Y} \text{ for all } Y\in\mathcal{B}\}$ denote its **centralizer**.

This is a categorical analogue of the **orthogonal complement**. For example, in $\mathcal{C}(A,q)$: if $B<A$ then $\mathcal{C}(B,q|_B)' = \mathcal{C}(B^\perp, q|_{B^\perp})$.

For any braided $\mathcal{C}$ let $\mathcal{Z}_{sym}(\mathcal{C}) := \mathcal{C}'$ denote its **symmetric center**.

‖ $\mathcal{C}$ is $\boxed{\text{symmetric}}$ if $\mathcal{Z}_{sym}(\mathcal{C}) = \mathcal{C}$: $c_{YX}c_{XY} = \mathrm{id}_{X\otimes Y}$ $\forall X,Y\in\mathcal{C}$.

By Deligne's Theorem any symmetric fusion category is equivalent to $\mathrm{Rep}(G,z)$, where G is a group and $z\in G$ is central, $z^2=1$.

Here $\mathrm{Rep}(G,z) = \mathrm{Rep}(G)$ as a fusion category with new braiding

$$\tilde{c}_{V,W} = \begin{cases} -c_{V,W} & \text{if } z|_V = z|_W = -1 \\ c_{V,W} & \text{otherwise,} \end{cases}$$

$v\otimes w \mapsto -w\otimes v$
$v\otimes w \mapsto w\otimes v$

where V,W are irreducible reps and $c_{V,W}$ is the "usual" braiding of $\mathrm{Rep}(G)$.

modular

vector spaces (multiples of 1)

‖ $\mathcal{C}$ $\boxed{\text{is non-degenerate}}$ if $\mathcal{Z}_{sym}(\mathcal{C}) = \mathrm{Vec}$ (trivial)

- categories $\mathcal{C}(\mathfrak{g},\ell)$ are non-degenerate
- $\mathcal{C}(A,q)$ is non-degenerate $\iff$ q is non-degenerate
- for any fusion $\mathcal{A}$ its center $\mathcal{Z}(\mathcal{A})$ is non-deg.

The Witt group of non-degenerate braided fusion categories

For any braided $\mathcal{C}$ let $\mathcal{C}^{rev}$ denotes the same category with the reverse braiding:

$$c^{rev}_{X,Y} = c^{-1}_{Y,X} : X\otimes Y \to Y\otimes X$$

Definition. Two non-degenerate braided fusion categories $\mathcal{C}$, $\mathcal{D}$ are **Witt equivalent** if

$$\mathcal{C}\boxtimes\mathcal{D}^{rev} \simeq \mathcal{Z}(\mathcal{A}) \quad \Big|\ \mathcal{C}\boxtimes\mathcal{C}^{rev} = \mathcal{Z}(\mathcal{C})$$

for some braided fusion category $\mathcal{A}$.

Let $[\mathcal{C}]$ denote the Witt equivalence class of $\mathcal{C}$.

The set $\mathcal{W}$ of these classes is a group:

$$[\mathcal{C}]\cdot[\mathcal{D}] := [\mathcal{C}\boxtimes\mathcal{D}], \quad [\mathcal{C}]^{-1} = [\mathcal{C}^{rev}]$$

and $[\mathrm{Vec}] = [\mathcal{Z}(\mathcal{A})]$ is the identity. $\mathcal{W}$ is called the **categorical Witt group**

$\mathcal{C}\boxtimes\mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{Z}(\mathcal{A}_2)$ $\Rightarrow$ $\mathcal{C} \simeq \mathcal{Z}(\mathcal{B})$

(= the quotient of the monoid of non-degenerate braided fusion categories by the Drinfeld centers)

Instead of trying to classify braided fusion categories up to equivalence (**hopeless!**) one can study the Witt group $\mathcal{W}$.

The classical Witt group — part of $\mathcal{W}$ that comes from $\mathcal{C}(A,q)$..

$$\mathcal{W}_{class} = \bigoplus_{p \text{ prime}} \mathcal{W}_{class}(p), \quad s\mathcal{W}_{class}(p) = \begin{cases} \mathbb{Z}_8\times\mathbb{Z}_2, & p=2 \\ \mathbb{Z}_2\times\mathbb{Z}_2, & p\equiv 1 \pmod 4 \\ \mathbb{Z}_4, & p\equiv 3 \pmod 4 \end{cases}$$

$\mathbb{Z}_2\times\mathbb{Z}_2$

of non-degenerate quadratic forms is a subgroup of $\mathcal{W}$.

So, what is a **non-classical** part of $\mathcal{W}$?

We will need a "super-version" of $\mathcal{W}$.

‖ A braided fusion category $\mathcal{C}$ is **slightly-degenerate** (or **super-nondegenerate**) if

$$\mathcal{Z}_{sym}(\mathcal{C}) = s\mathrm{Vec}, \quad \text{super vector spaces}$$

There is a **super version of Witt theory**, with centers replaced by super-centers, the latter being

$$\mathcal{Z}_s(\mathcal{A}) := (s\mathrm{Vec})' \subset \mathcal{Z}(\mathcal{A}).$$

One gets the **super Witt group** $s\mathcal{W}$ of classes of slightly degenerate braided fusion categories.

We have a homorphism

$$S : \mathcal{W} \to s\mathcal{W} : [\mathcal{C}] \mapsto [\mathcal{C}\boxtimes s\mathrm{Vec}]$$

that gives rise to a (non-split) short exact sequence:

$$\boxed{0 \to \mathbb{Z}_{16} \to \mathcal{W} \xrightarrow{S} s\mathcal{W} \to 0}$$

[Davydov-N-Ostrik, Johnson-Freyd - Reutter].

Here $\mathbb{Z}_{16}$ is generated by any Ising category, e.g. by $\mathcal{C}(sl_2, 2)$ — we will see that it is the group of minimal extensions of sVec. $\boxed{\mathcal{C}(\mathfrak{g},\ell)}$

The abstract structure of $s\mathcal{W}$ is known:

$$\boxed{s\mathcal{W} = s\mathcal{W}_{class} \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}^\infty}$$

[Davydov-Mueger-N-Ostrik, Ng-Rowell-Wang-Zhang]

In particular, $\mathcal{W}$ has no odd torsion and the largest finite order of its elements is **32**, e.g. for $\mathcal{C}(sl_2, 6)$.

Relative Witt groups and minimal extensions.

Given a degenerate braided fusion category $\mathcal{C}$ can we embed it in a non-degenerate category $\mathcal{D}$ of minimal possible size;

$\mathcal{C}\hookrightarrow\mathcal{D}$ such that $\mathcal{Z}_{sym}(\mathcal{C})$ coincides with the centralizer of $\mathcal{C}$ in $\mathcal{D}$?

Equivalently, $\dim(\mathcal{D}) = \dim(\mathcal{C})\dim(\mathcal{Z}_{sym}(\mathcal{C}))$?

Without the minimality condition — certainly possible, e.g. $\mathcal{C}\hookrightarrow\mathcal{Z}(\mathcal{C})$.

It turns that minimal non-degenerate extensions do not always exist.

Let $\mathcal{E}$ be a symmetric fusion category. We can define a $\boxed{\text{relative Witt group } \mathcal{W}(\mathcal{E})}$ of braided fusion categories with $\mathcal{Z}_{sym}\simeq\mathcal{E}$ (any symm. cat.) similarly to how this was done for $\mathcal{E} = s\mathrm{Vec}$, (so that $s\mathcal{W} = \mathcal{W}(s\mathrm{Vec})$).

We have a homomorphism

$$S_\mathcal{E} : \mathcal{W} \to \mathcal{W}(\mathcal{E}) : [\mathcal{C}] \mapsto [\mathcal{C}\boxtimes\mathcal{E}]_\mathcal{E}$$

[Ostrik-Yu]: A braided category $\mathcal{B}$ such that $\mathcal{Z}_{sym}(\mathcal{B})\simeq\mathcal{E}$ has minimal extension $\iff$

$[\mathcal{B}]$ in $\mathcal{W}(\mathcal{E})$ belongs to $\mathrm{Image}(\mathcal{W} \xrightarrow{S_\mathcal{E}} \mathcal{W}(\mathcal{E}))$

Note that a symmetric category $\mathcal{E}$ does have minimal extensions, e.g. $\mathcal{E}\hookrightarrow\mathcal{Z}(\mathcal{E})$.

[Lan-Kong-Wen]: the set $\mathcal{M}ext(\mathcal{E})$ of such extensions has a group structure.

Examples: ①. $\mathcal{M}ext(\mathrm{Rep}(G)) \simeq H^3(G,\mathbb{C}^\times)$,

Deligne's thm $\mathrm{Rep}(G,z)$

minimal extensions are $\mathrm{Rep}(G)\hookrightarrow\mathrm{Rep}(D^\omega(G))$ (twisted Drinfeld doubles)

② $\mathcal{M}ext(s\mathrm{Vec}) \simeq \mathbb{Z}_{16}$, same group we have seen before.

③ In non-Tannakian case (i.e. when $z\neq 1$) partial results are known; e.g:

$$\mathcal{M}ext(\mathbb{Z}_4, z)\simeq\mathbb{Z}_8, \quad \mathcal{M}ext(\mathbb{Z}_2\times\mathbb{Z}_2, z)\simeq\mathbb{Z}_{16}\times\mathbb{Z}_8.$$

If a braided fusion category $\mathcal{B}$ has minimal extensions, they form a torsor over $\mathcal{M}ext(\mathcal{Z}_{sym}(\mathcal{B}))$.

The homomorphism $S_\mathcal{E} : \mathcal{W}\to\mathcal{W}(\mathcal{E})$ is, in fact, a fibration with the fiber $\mathcal{M}ext(\mathcal{E})$, so

$$\mathcal{M}ext(\mathcal{E}) \to \mathcal{W} \xrightarrow{S_\mathcal{E}} \mathcal{W}(\mathcal{E})$$

is a part of a long exact sequence that can be used to compute the groups involved.

For example, when $\mathcal{E} = \mathrm{Rep}(G)$ this yields

$$0 \to \mathcal{W} \to \mathcal{W}(\mathrm{Rep}(G)) \to H^4(G,\mathbb{C}^\times) \to 0$$

a split short exact sequence.

An application to fusion 2-categories and 4-dimensional topological field theory

[Douglass - Reutter]: fusion 2-categories provide a framework for 4-dimensional semisimple TFT.

Examples of such categories are $\mathrm{Mod}(\mathcal{B})$, 2-category of module categories over a braided fusion category $\mathcal{B}$. degenerate.

Decopette

If $\mathcal{B}_1, \mathcal{B}_2$ are 2 such categories, then the corresponding field theories are equivalent $\iff$

$\mathrm{Mod}(\mathcal{B}_1)$, $\mathrm{Mod}(\mathcal{B}_2)$ are 2-Morita equivalent $\iff$

$\mathcal{Z}_{sym}(\mathcal{B}_1)\simeq\mathcal{Z}_{sym}(\mathcal{B}_2)$ and $\mathcal{B}_1, \mathcal{B}_2$ are Witt equivalent.

Thank you!