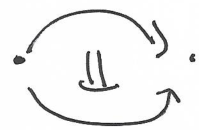
Θ_n-spaces with discreteness conditions

Idea of (∞,n)-categories: structure with

objects: $\bullet$

1-morphisms: $\bullet \longrightarrow \bullet$

2-morphisms: 

$\vdots$

n-morphisms

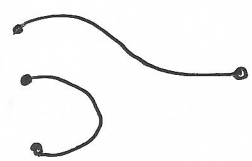
(n+1)-morphisms that are invertible

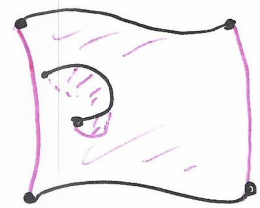
$\vdots$

Another perspective: have <u>sets</u> of objects, 1-morphisms, ..., (n-1)-morphisms, then a <u>space</u> of n-morphisms.

Ex: Cobordism (∞,n)-categories

objects: 0-manifolds 

1-morphisms: cobordisms (1-dim'l) 

2-morphisms: cobordisms between cobordisms 

$\vdots$

n-morphisms: n-dim'l cobordisms

(n+1)-morphisms: diffeomorphisms

(n+2)-morphisms: isotopies between diffeomorphisms

$\vdots$

Used to define extended TQFTs.

Q: How do we give precise definitions ("models") of these structures as concrete mathematical objects?

$n=0$: $(\infty,0)$-categories $=$ ∞-groupoids

Can take topological spaces: objects $=$ points

1-morphisms $=$ paths (invertible)

2-morphisms $=$ homotopies

3-morphisms $=$ homotopies between homotopies

$\vdots$

General principle for (∞,n)-categories: should be categories enriched in $(\infty,n-1)$-categories

$\hookrightarrow$ structure with objects, morphism $(\infty,n-1)$-categories

So, one approach to $(\infty,1)$-categories is taking categories enriched in spaces. (have mapping spaces)

Ex: Top given X, Y, have a space $\mathrm{Map}(X,Y)$

It is convenient to work instead with simplicial sets: $K: \Delta^{op} \to \mathrm{Sets}$

$\Delta^{op} =$ opposite of the category of finite ordered sets

$$K_0 \leftleftarrows K_1 \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, K_2 \quad \cdots\cdots$$

geometrically realize to spaces

Other approaches to $(\infty,1)$-categories are simplicial spaces $=$ bisimplicial sets

$X: \Delta^{op} \longrightarrow \mathrm{SSets}$ $\quad (X: \Delta^{op} \times \Delta^{op} \to \mathrm{Sets})$

Defn: A <u>Segal space</u> is a simplicial space X such that the Segal maps $X_n \longrightarrow \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$ are weak equivalents for all $n \geq 2$.

Idea: X_0 "objects" $\quad X_1$ "morphisms"

Condition says that any $\bullet \to \bullet \to \bullet \cdots \bullet \to \bullet$ has an essentially unique composite e.g. $n=2$

$G(2)$

$$\begin{array}{ccc} \mathrm{Map}(\Delta[2], X) & \xrightarrow{\simeq} & \mathrm{Map}(G(2), X) \\ \| & & \| \\ X_2 & & X_1 \times_{X_0} X_1 \end{array}$$

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, dashed, bend right] & X_2 \arrow[l, "(d_0, d_2)"', "\simeq"] \arrow[r, "d_1"] & X_1 \end{tikzcd}$$

What is the difference between a Segal space and a category enriched in spaces?

- Composition of mapping space only defined up to homotopy
- X_0 is a space, not a set.

Quick fix:

Defn: A <u>Segal category</u> is a Segal space X with X_0 discrete.

Awkward for doing homotopy theory.

Alternative: Let $X_{heq} \subseteq X_1$ be the subspace of homotopy equivalences

$$\begin{tikzcd} s_0 : X_0 \arrow[rr] \arrow[dr] & & X_1 \\ & X_{heq} \arrow[ur] & \end{tikzcd}$$

Defn: A Segal space is <u>complete</u> if $s_0 : X_0 \to X_{heq}$ is a weak equivalence.

Idea: Have a space of objects, but encoded in the space of morphisms, so no extra data.

Thm: Segal categories and complete Segal spaces are equivalent models for $(\infty,1)$-categories; also equivalent to categories enriched in simplicial sets / spaces.

(Quillen equivalences of model categories)

$$\underline{\text{Segal categories}} \qquad\qquad\qquad\qquad \underline{\text{complete Segal spaces}}$$

$$\left(\begin{array}{c} X : \Delta^{op} \to SSets \\ X_0 \text{ discrete} \end{array} \right) \underset{\text{discretization}}{\overset{\text{inclusion}}{\rightleftarrows}} \left(X : \Delta^{op} \to SSets \right)$$

collapse X_0 to $\pi_0 X_0$

SSets-Cat

$\Big\}$ nerve

$X : \Delta^{op} \to SSets$
with $X_n \xrightarrow{\simeq} \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}$

What about higher (∞, n)-categories?

For $n=2$, one approach takes collections of spaces indexed by diagrams like

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \bullet \end{tikzcd}$$

$[3]([1],[0],[2])$

Defn: The category Θ_2 has objects $[m]([c_1], \ldots, [c_m])$ where $[m]$, $[c_i]$ objects of Δ

$$\underset{0}{\bullet} \xrightarrow{c_1} \underset{1}{\bullet} \xrightarrow{c_2} \underset{2}{\bullet} \xrightarrow{c_3} \cdots \underset{m-1}{\bullet} \xrightarrow{c_m} \underset{m}{\bullet}$$

morphisms $[m]([c_1], \ldots, [c_m]) \to [p]([d_1], \ldots, [d_p])$ given by $[m] \to [p]$ in Δ and $[c_i] \to [d_j]$ when appropriate.

ex:
$$\begin{tikzcd}[column sep=small] 0 \arrow[r, "{[c_1]}"] \arrow[d] & 1 \arrow[r, "{[c_2]}"] \arrow[d] & 2 \arrow[r, "{[c_3]}"] \arrow[drr] & 3 \arrow[d] & \\ 0 \arrow[r, "{[d_1]}"'] & 1 \arrow[r, "{[d_2]}"'] & 2 \arrow[r, "{[d_3]}"'] & 3 \arrow[r, "{[d_4]}"'] & 4 \end{tikzcd}$$

$\Theta_2 = \Theta\Delta \qquad \Theta_0 = *$

$\Theta_n = \Theta\Theta_{n-1}$

$\Theta\mathcal{C} \qquad [m](\underbrace{c_1, \ldots, c_m}_{ob(\mathcal{C})})$

Consider $X : \Theta_2^{op} \to SSets$

Want Segal conditions

$$X([3]([1],[0],[2])) \simeq X([1]([1])) \times_{X(\bullet)} X(\to) \times_{X(\bullet)} \boxed{X([1]([2]))}$$

$$\wr\!\wr$$

$$X([1]([1])) \times_{X(\to)} X([1]([1]))$$

Defn: A <u>Θ_2-Segal space</u> is a functor $X : \Theta_2^{op} \to SSets$ satisfying these two "levels" of Segal conditions.

Completeness: Before, we had $X(\cdot) \simeq X(\xrightarrow{\cong})$.
Also want $X(\to) \simeq X\left(\overset{\cong}{\Downarrow}\right)$.

Thm: (B-Rezk)
Θ_2-complete Segal spaces are equivalent to categories enriched in complete Segal spaces, i.e., are models for $(\infty,2)$-categories.

Can generalize to Θ_n.
What about discreteness instead?

Defn: A <u>Θ_2-Segal category</u> is a ~~functor~~ Θ_2-Segal space $X : \Theta_2^{op} \to SSets$ such that the spaces $X(\cdot)$ and $X(\to)$ are discrete.

$$X(\cdot) = X[0] \qquad X(\to) = [1]([0])$$

$\cdot$

Q: Can we mix the conditions?
We can consider Θ_2-Segal spaces $X : \Theta_2^{op} \to SSets$ with $X(\cdot)$ discrete and $X(\to) \simeq X\left(\overset{\cong}{\Downarrow}\right)$.

Thm: There are equivalences

$$\begin{pmatrix} \Theta_2\text{-Segal} \\ \text{categories} \end{pmatrix} \rightleftarrows \begin{pmatrix} \Theta_2\text{-Segal spaces} \\ \text{with } X(\cdot) \text{ discrete} \\ X(\to) \simeq X\left(\overset{\cong}{\Downarrow}\right) \end{pmatrix} \rightleftarrows \begin{pmatrix} \Theta_2\text{-complete} \\ \text{Segal spaces} \end{pmatrix}$$

If we had $X(\to)$ discrete, then via the degeneracy map $X(\cdot) \to X(\to)$ we get $X(\cdot)$ is a retract of $X(\to)$, hence also discrete, so we recover Θ_2-Segal categories.

For general Θ_n-Segal spaces, can get different models by taking discreteness up to level k, and the completeness up to level n.

Application: Models for monoidal $(\infty,1)$-categories. (Zapata Castro)