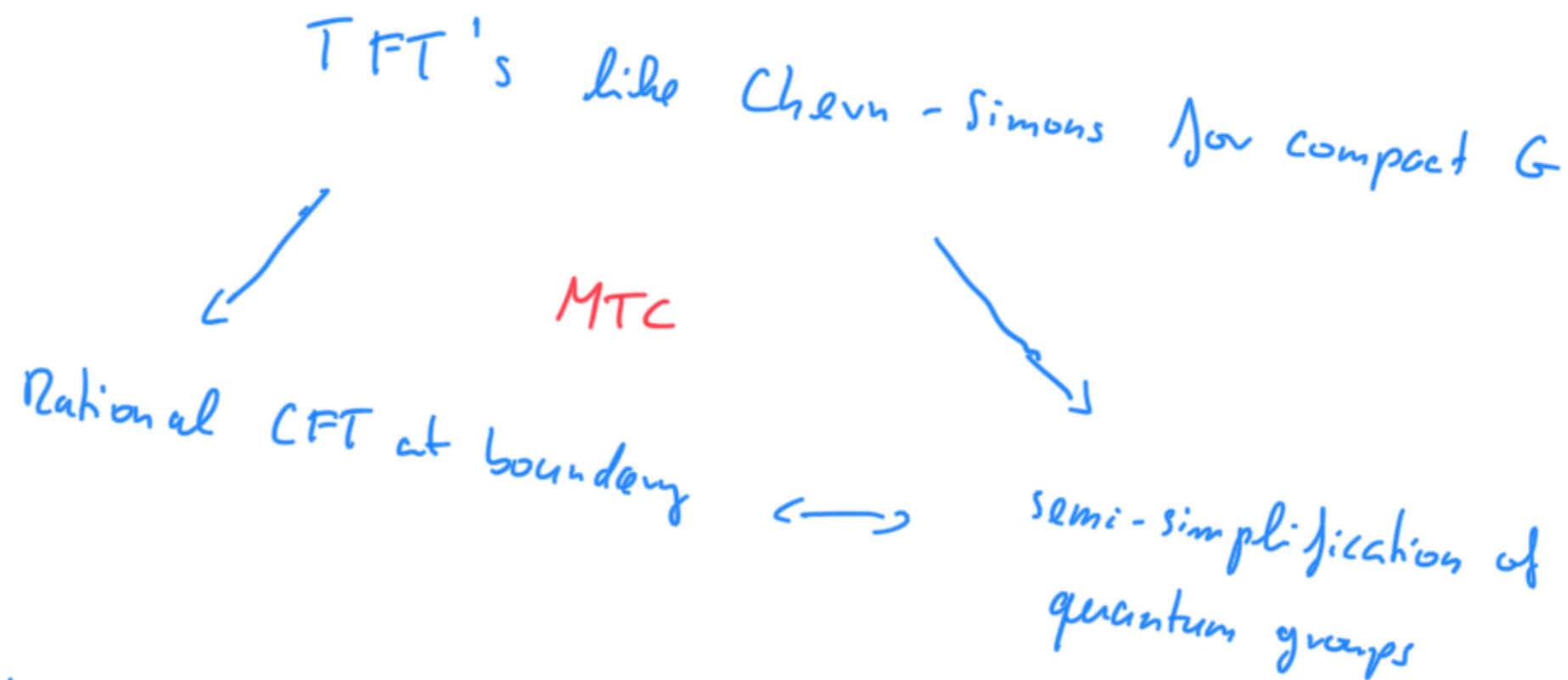


The VOA $V^h(\mathfrak{gl}(1|1))$

Historically:



Modernly:

CFT's or better VOA's have non-finite and non-semisimple categories and associated quantum groups should be unrolled small quantum groups.

Progress:

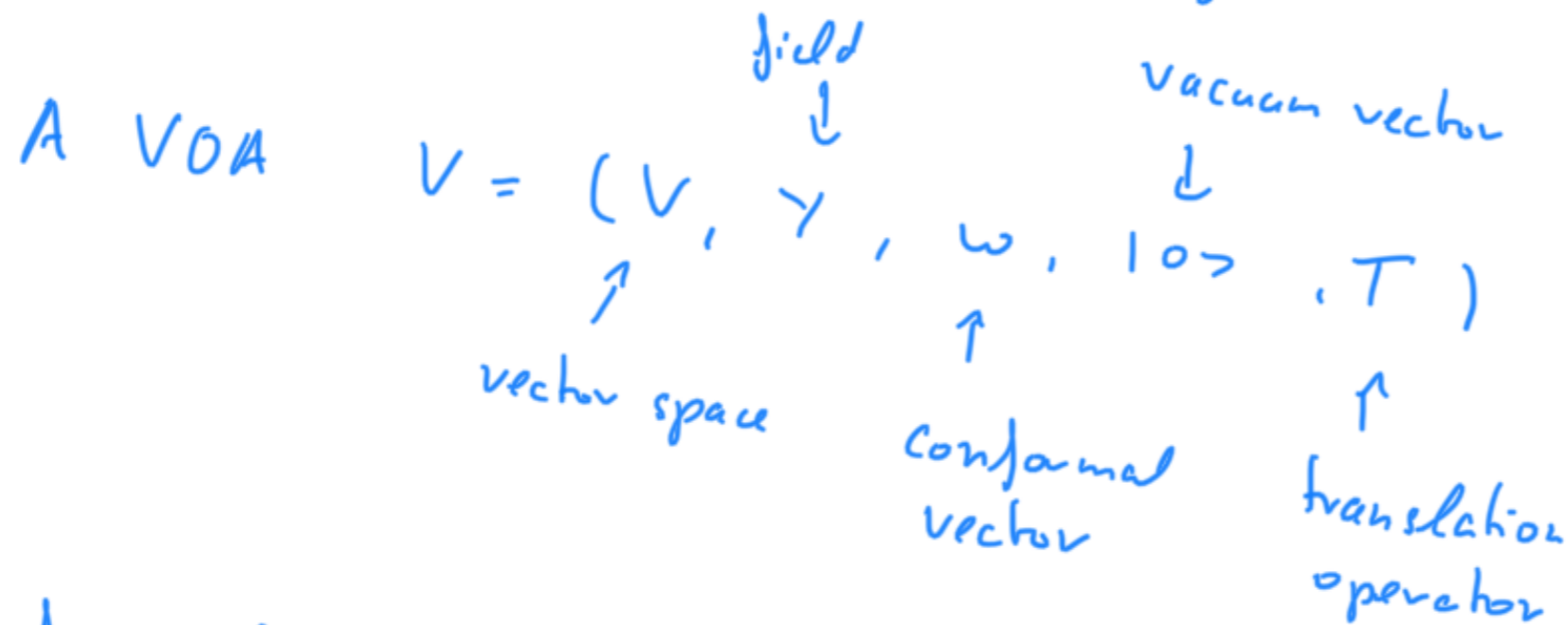
- connection between categories of line operators in physical TQFT's (Dimofte, Miemietz, Garner, Bailin, ...)
- TFT's ...

... associated to locally-finite non-semisimple ribbon categories (Constantin, Geyer, Paturneau-Miranda, Turaev, ...)

- Non-semisimple locally-finite ribbon categories of VOAs (McRae, Yang, TC, ...)

- Logarithmic Kazhdan-Lusztig correspondences (Tentner, Reupert, TC, ...)

Vertex operator algebras (VOAs)



formalises the notion of symmetry algebra of a 2-dim. CFT.

Most importantly fields

$$\gamma : V \longrightarrow \text{End}(V) [[z^{\pm 1}]]$$

$$v \longmapsto \gamma(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

that quantum commute

$$[\gamma(v, z), \gamma(v', z')] (z - z')^N = 0 \quad N \gg 0$$

- Modules and Intertwining operator are introduced in a similar spirit, e.g.

$$\gamma_{(M, N)}^s : M \otimes N \longrightarrow S\{z\} [[\log z]]$$

$$\sum_{\substack{n \in \mathbb{C} \\ d \in \mathbb{Z}_{\geq 0}}} x_{n,d} z^{-n-1} [[\log z]]^d$$

Physics suggests that intertwining operators give suitable VOA categories the structure of ribbon categories.

There is a general theory due to Huang, Lepowsky, Zhang

Braiding automatic

Associativity requires analytic continuation of correlation functions: difficult

closure under tensor product difficult

rigidity difficult

We are interested in VOAs whose representation categories are locally finite, but neither finite nor semisimple.

The affine VOA of $\mathfrak{gl}(1|1)$ is one of the very few examples that we understand (TC-McRae-Yang)

$V^h(\mathfrak{gl}(1|1))$

- review: (TC-Yang) If a VOA category satisfies a certain finiteness condition, called C_1 -cofinite, plus some other mild assumptions and if this category is of finite length, then it is a vertex tensor category à la HLZ.

$\mathfrak{g} = \mathfrak{gl}(1|1)$

basis E, N , ψ^+, ψ^-
 even odd

$[N, \psi^\pm] = \pm \psi^\pm$

$[\psi^+, \psi^-] = \mathbb{1}$

invariant supersymmetric bilinear form

$\kappa(N, E) = 1, \quad \kappa(\psi^+, \psi^-) = 1$

$\hat{\mathfrak{g}} = \hat{\mathfrak{gl}}(1|1)$

basis E_n, N_n, ψ_n^\pm , $n \in \mathbb{Z}$, K, d

$[d, X_n] = -n X_n \quad X \in \{E, N, \psi^\pm\}$

$[N_n, E_m] = n \kappa \delta_{n+m, 0}$

$[\psi^+, \psi^-] = \mathbb{1}$

$$[n, m] = \sigma_{n+m} + n \cdot k \delta_{n+m, 0}$$

$$[N_n, \psi_m^\pm] = \pm \psi_{n+m}^\pm$$

Representation Theory

Let M be a \mathfrak{g} -module, $h \in \mathbb{C} \setminus \{0\}$, require that

K acts by $h \cdot \text{Id}$ on M .

↑
level

$$\hat{M}^h := U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}_{\geq 0})} M \quad \text{Verma module}$$

Ex: $M = \mathbb{C}$ then $\hat{\mathbb{C}}^h = V^h(\mathfrak{g})$

$KL^{\text{wt}}_h(\mathfrak{g})$

Cartan subalgebra acts semi-simple

\mathfrak{g} -modules

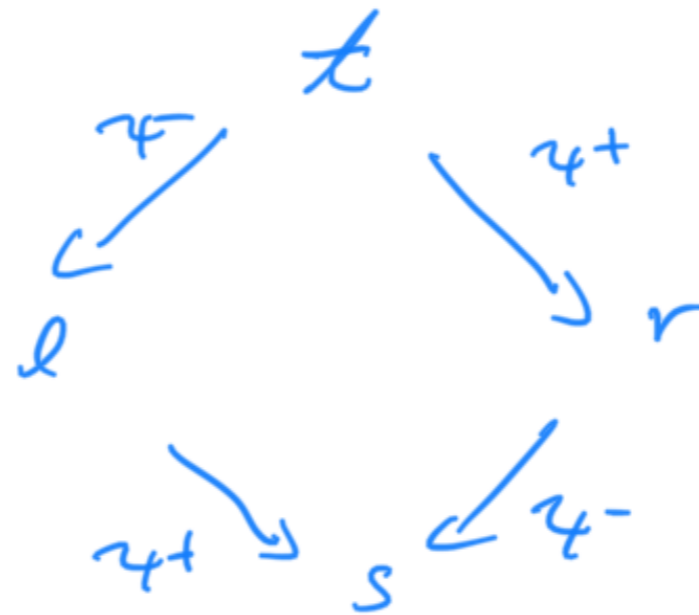
$V_{e,n}$: basis v, w with $\psi^+ v = 0$, $E v = e v$, $N v = n v$
and $\psi^- v = w$

Then $\psi^+ w = \psi^+ \psi^- v = \epsilon v = \epsilon v$

• $V_{\epsilon \hbar}$ simple $\Leftrightarrow \epsilon \neq 0$
projective

• If $\epsilon = 0$ then P_n basis t, l, r, s

$\psi^+ t = r$ $\psi^- t = l$ $\psi^- r = s$ $\psi^+ l = -s$
 $N t = \hbar t$



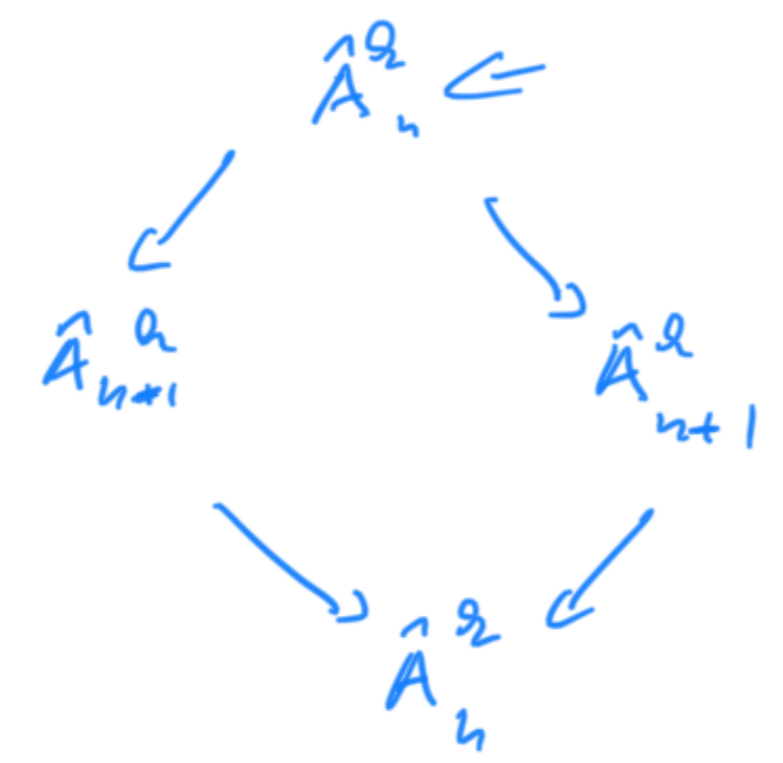
$K(\mathbb{Q}^{\text{cut}})$:

$\hat{V}_{\epsilon \hbar}$ simple $\Leftrightarrow \epsilon \neq m \hbar$ $m \in \mathbb{Z}$

$\epsilon = m \hbar$: $0 \rightarrow \hat{\mathbb{A}}^{\epsilon} \rightarrow \dots \rightarrow \hat{\mathbb{A}}^{\epsilon}$

$$V_{n+1, \ell} \rightarrow V_{n, \ell} \rightarrow A_{n, \ell} \rightarrow 0 \quad \ell > 0$$

$\hat{P}_n^{\mathbb{Q}}$:



Reason: spectral flow: $\sigma^m(N_n) \cong N_n$

$$\sigma^m(E_n) = E_n - m \kappa S_{n,0}$$

$$\sigma^m(\psi_n^{\pm}) = \psi_{n \mp m}^{\pm}$$

$\hat{V}_{\ell, n}^{\mathbb{Q}}, \hat{P}_n^{\mathbb{Q}}$

complete list of indecomposable projective modules.

$$\hat{V}_{n, \ell}^{\mathbb{Q}} \otimes \hat{V}_{n', \ell'}^{\mathbb{Q}} = \left\{ \begin{array}{l} \hat{V}_{n+n', \ell+\ell'}^{\mathbb{Q}} \oplus \hat{V}_{n+n'-1, \ell+\ell'}^{\mathbb{Q}} \quad \ell+\ell' \in 2\mathbb{Z} \\ \hat{P}_{n+n'+1, \ell+\ell'}^{\mathbb{Q}} \quad \ell+\ell' \in 2\mathbb{Z} \end{array} \right.$$

Then C-M... ..

McKee - Yang

$\mathfrak{g} = \mathfrak{gl}(1|1)$, $h \in \mathbb{C} \setminus \{0\}$

$\text{KLa}_h^{\text{ut}}(\mathfrak{g})$ the category of $\hat{\mathfrak{g}}$ -modules of level h
with semi-simple action of E_0, \mathcal{N}_0

$\text{KLa}_h^{\text{ut}}(\mathfrak{g})$ is a ribbon tensor category

=

3 known VOA categories that are not finite and not semi-simple

• $\beta\gamma$ -VOA Allen-Wood

• $V^h(\mathfrak{gl}(1|1))$

• singlet VOAs $\mathcal{M}(p)$ $p \in \mathbb{Z}_{\geq 2}$

} with McKee - Yang

=

There exists a VOA embedding

$$V^h(\mathfrak{gl}(1|1)) \hookrightarrow \mathbb{F}^2 \otimes \pi^2$$

$$\mathbb{F}^2 \otimes \pi^2\text{-mod} = \dots \oplus \dots \oplus \dots$$

\uparrow \uparrow
 2-free 2 free bosons
 fermions

$$\mathbb{F}^L \otimes \pi^2 = \bigvee_0^{\infty} \mathbb{Q}^*$$

Thm (C- Kuntzner - Rupert)

- C commutative Hopf $\mathcal{E} = \text{rep}(C)$
 - \mathcal{U} algebra $\mathcal{U} \supset C$ $\mathcal{U} = \text{rep}(\mathcal{U})$
 - V, A VOA's $V \leftrightarrow A$ + many assumptions ← difficult
- if $A\text{-mod} \cong \mathcal{E}$ as braided TC
- $V\text{-mod} \cong \mathcal{U}$ as abelian cat.

Then $V\text{-mod} \cong \mathcal{U}$ as braided TC

Ex: $\xrightarrow{\text{wt}}$ $KL_{\mathfrak{g}}(\mathfrak{g}) \cong \mathcal{U}_{\mathfrak{g}}^{\mathbb{B}}(\mathfrak{gl}(1,1)) - \text{wt mod}$

$V\text{-mod}$ $\xrightarrow{\text{Schauenburg}}$ rel. Drinfeld center $\leftarrow \begin{matrix} \mathbb{Z}(\mathbb{B}^+) \\ \mathcal{L} \end{matrix}$
 $\xrightarrow{\text{Lusztig}}$ Yetter-Drinfeld modules $\left(\begin{matrix} \mathbb{B}^+ \\ \mathcal{YD}(\mathcal{L}) \\ \mathbb{B}^+ \end{matrix} \right)$
 $\xrightarrow{\text{well-known}}$ quantum group

$$U = \mathbb{Q}G = \mathcal{U}^- \oplus \underbrace{\mathcal{L} \oplus \mathcal{U}^+}_{\mathbb{B}^+}$$

CFT \cong Hilbert space: $\mathcal{H} = \bigoplus \begin{matrix} \mathcal{U} \\ \downarrow \\ \mathcal{H}_i \end{matrix} \oplus \begin{matrix} \overline{\mathcal{U}} \\ \downarrow \\ \overline{\mathcal{H}}_i \end{matrix}$

Fields of CFT