The VOA $V^k(gl(1|1))$

Historically:

TFT's like Chern-Simons for compact G

↙ MTC ↘

Rational CFT at boundary ⟷ semi-simplification of quantum groups

Modernly: CFT's or better VOA's have non-finite and non-semisimple categories and associated quantum groups should be unrolled small quantum groups.

Progress:
- connection between categories of line operators in physical TQFT's (Dimofte, Niu, Garner, Bailin, ...
- TFT's

associated to locally-finite non-semisimple ribbon categories (Constantino, Geer, Patureau-Mirand, Young, ...

- Non-semisimple locally-finite ribbon categories of VOAs (McRae, Yang, TC, ...
- Logarithmic Kazhdan-Lusztig correspondences (Lentner, Rupert, TC, ...

Vertex operator algebras (VOAs)

A VOA $$V = (\underset{\substack{\uparrow\\ \text{vector space}}}{V}, \overset{\substack{\text{field}\\ \downarrow}}{Y}, \underset{\substack{\uparrow\\ \text{conformal}\\ \text{vector}}}{\omega}, \overset{\substack{\text{vacuum vector}\\ \downarrow}}{|0\rangle}, \underset{\substack{\uparrow\\ \text{translation}\\ \text{operator}}}{T})$$

formalises the notion of symmetry algebra of a 2-dim. CFT.

Most importantly fields

$$Y : V \longrightarrow \mathrm{End}(V)[[z^{\pm 1}]]$$

$$v \longmapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$$

that quantum commute

$$[Y(v,z), Y(v',z')](z-z')^N = 0 \qquad N \gg 0$$

- Modules and Intertwining operator are introduced in a similar spirit, e.g.

$$\mathcal{Y}\binom{S}{M\;N} : M \otimes N \longrightarrow S\{z\}[\log z]$$

$$\sum_{\substack{n\in\mathbb{C}\\ d\in\mathbb{Z}_{\geq 0}}} x_{n,d}\, z^{-n-1} [\log z]^d$$

Physics suggests that intertwining operators give suitable VOA categories the structure of ribbon categories.

There is a general theory due to Huang, Lepowsky, Zhang

Braiding automatic

Associativity requires analytic continuation of correlation functions : difficult

closure under tensor product difficult

rigidity difficult

We are interested in VOAs whose representation categories are locally finite, but neither finite nor semisimple.

The affine VOA of $gl(1|1)$ is one of the very few examples that we understand (TC-McRae-Yang)

$$\underline{V^k(gl(1|1))}$$

Criterion: (TC-Yang) If a VOA category satisfies a certain finiteness condition, called C_1-cofinite, plus some other mild assumptions and if this category is of finite length, then it is a vertex tensor category à la HLZ.

$\mathfrak{g} = \mathfrak{gl}(1|1)$

basis E, N , ψ^+, ψ^-
even odd

$$[N, \psi^\pm] = \pm \psi^\pm$$
$$[\psi^+, \psi^-] = E$$

invariant supersymmetric bilinear form

$$\kappa(N, E) = 1, \quad \kappa(\psi^+, \psi^-) = 1$$

$\hat{\mathfrak{g}} = \hat{\mathfrak{gl}}(1|1)$

basis $E_n, N_n, \psi^\pm_n$, $n \in \mathbb{Z}$, K, d

$$[d, X_n] = -n X_n \qquad X \in \{E, N, \psi^\pm\}$$
$$[N_n, E_m] = n K \delta_{n+m,0}$$
$$[\psi^+, \psi^-] \quad E$$

$${}_{n}, {}_{m} \} = {}_{n+m} + n \cdot K \delta_{n+m,0}$$

$$[N_n, \psi^\pm_m \} = \pm \psi^\pm_{n+m}$$

Representation Theory

Let M be a g-module, $k \in \mathbb{C} \setminus \{0\}$, require that K acts by $k \cdot Id$ on M.

$\uparrow$

level

$$\boxed{\hat{M}^k} := \mathcal{U}(\hat{g}) \otimes_{\mathcal{U}(\hat{g}_{\geq 0})} M \quad \text{Verma module}$$

Ex: $M = \mathbb{C}$ then $\hat{\mathbb{C}}^k = V^k(g)$

$KL^{wt}_k(g)$ Cartan subalgebra acts semi-simple

g-modules

$V_{e,n}$: basis v, w with $\psi^+ v = 0$, $E v = e v$, $N v = n v$

and $\psi^- v = w$

Then $\psi^+ w = \psi^+ \psi^- v = E v = e v$

- $V_{e,n}$ simple $\iff$ $e \neq 0$
  projective

- If $e = 0$ then P_n basis t, ℓ, r, s

$\psi^+ t = r \quad \psi^- t = \ell \quad \psi^- r = s \quad \psi^+ \ell = -s$

$N t = n t$

$$\begin{tikzcd}
 & t \arrow[dl, "\psi^-"'] \arrow[dr, "\psi^+"] & \\
\ell \arrow[dr, "\psi^+"'] & & r \arrow[dl, "\psi^-"] \\
 & s &
\end{tikzcd}$$

$\underline{KL^{wt}_{h}(g)}$:

$\hat{V}_{e,n}$ simple $\iff$ $e \neq m h \quad m \in \mathbb{Z}$

$e = m h$: $\quad 0 \to \hat{A}^{h}$

$$\to A_{n+1,e} \to V_{e,n} \to A_{n,e} \to 0 \qquad e>0$$

$\hat{P}^{e}_{n}$:

$$\begin{tikzcd}
 & \hat{A}^{e}_{n} \arrow[dl] \arrow[dr] & \arrow[l, Rightarrow] \\
\hat{A}^{e}_{n+1} \arrow[dr] & & \hat{A}^{e}_{n+1} \arrow[dl] \\
 & \hat{A}^{e}_{n} &
\end{tikzcd}$$

Reason: spectral flow : $\sigma^{m}(N_n) \Rightarrow N_n$

$$\sigma^{m}(E_n) = E_n - \boxed{mk\,\delta_{n,0}}$$

$$\sigma^{m}(\psi^{\pm}_{n}) = \psi^{\pm}_{n\mp m}$$

$\hat{V}^{e}_{e,n}$, $\hat{P}_{n}$ complete list of indecomposable projective modules.

$$\hat{V}^{e}_{n,e} \otimes \hat{V}^{e}_{n',e'} = \begin{cases} \hat{V}^{e}_{n+n',e+e'} \oplus \hat{V}^{e}_{n+n'-1,e+e'} & e+e' \notin k\mathbb{Z} \\ \hat{P}_{n+n'+\bullet,\, e+e'} & e+e' \in k\mathbb{Z} \end{cases}$$

Thm C-Mc

Rae - Yang

$\mathfrak{g} = \mathfrak{gl}(1|1)$, $k \in \mathbb{C} \setminus \{0\}$

$KL_k^{wt}(\mathfrak{g})$ the category of $\hat{\mathfrak{g}}$-modules of level k with semi-simple action of E_0, N_0

$KL_k^{wt}(\mathfrak{g})$ is a ribbon tensor category

=

3 known VOA categories that are not finite and not semisimple

- $\beta\gamma$-VOA Allen-Wood
- $V^k(\mathfrak{gl}(1|1))$
- singlet VOAs $\mathcal{M}(p)$ $p \in \mathbb{Z}_{\geq 2}$

} with McRae - Yang

=

There exists a VOA embedding

$$V^k(\mathfrak{gl}(1|1)) \hookrightarrow \mathcal{F}^2 \otimes \pi^2$$

$\mathcal{F}^2 \otimes \pi^2\text{-mod} =$ ^{Q}

$- \mathrm{sVect} \boxtimes \mathrm{Vect}_{\mathbb{C}^2}$

$\uparrow$ 2 - free fermions

$\uparrow$ 2 free bosons

$$\mathcal{F}^L \otimes \Pi^2 = \hat{V}_0^{Q*}$$

Thm (C - Lentner - Rupert)

- C commutative Hopf $\quad \mathcal{C} = \mathrm{Rep}(C)$
- U algebra $U \supset C \quad \mathcal{U} = \mathrm{Rep}(U)$
- V, A VOAs $\quad V \hookrightarrow A$ + many assumptions $\swarrow$ difficult

if A - mod $\cong \mathcal{C}$ as braided TC

V - mod $\cong \mathcal{U}$ as abelian cat.

Then V-mod $\cong \mathcal{U}$ as braided TC

Ex: $KL_Q^{wt}(g) \cong u_q^B(gl(1|1))$ - wtmod

$$V\text{-mod} \xrightarrow{\text{Schauenburg}} \text{rel. Drinfeld center} \leftarrow \mathcal{Z}_{\mathcal{C}}(B^+)$$

$$\xrightarrow{\text{Laugwitz}} \text{Yetter-Drinfeld modules} \quad \boxed{{}^{B^+}_{B^+}\mathcal{YD}(\mathcal{C})}$$

$$\xrightarrow{\text{well-known}} \text{quantum group}$$

$$\mathcal{U} = QG = \mathcal{U}^- \otimes \underbrace{\mathcal{C} \otimes \mathcal{U}^+}_{B^+}$$

$$\text{CFT} \neq \text{Hilbert space: } \mathcal{H} = \bigoplus \underset{\uparrow}{\overset{V}{\downarrow}}{} \; \overset{\overset{V}{\downarrow}}{M_i} \otimes \overset{\overset{\overline{V}}{\downarrow}}{\overline{M}_i}$$

Fields of CFT