The Power of Analogue-Digital Machines

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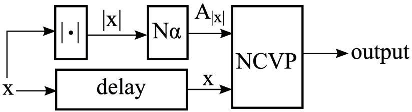
Overview



The *ARNN* model

Development of Physical Super-Turing Analog Hardware

A. Steven Younger[1], Emmett Redd[1], Hava Siegelmann[2]

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Abstract. In the 1930s, mathematician Alan Turing proposed a mathematical model of computation now called a Turing Machine to describe how people follow repetitive procedures given to them in order to come up with final calculation result. This extraordinary computational model has been the foundation of all modern digital computers since the World War II. Turing also speculated that this model had some limits and that more powerful computing machines should exist. In 1993, Siegelmann and colleagues introduced a Super-Turing Computational Model that may be an answer to Turing's call. Super-Turing computation models have no inherent problem to be realizable physically and biologically. This is unlike the general class of hyper-computer as introduced in 1999 to include the Super-Turing model and some others. This report is on research to design, develop and physically realize two prototypes of analog recurrent neural networks that are capable of solving problems in the Super-Turing complexity hierarchy, similar to the class BPP/log*. We present plans to test and characterize these prototypes on problems that demonstrate anticipated Super-Turing capabilities in modeling Chaotic Systems.

Analogue Recurrent Neural Net [SS94, SS95, Sie99]

System equation

$$x(t+1) \quad = \quad \sigma(Ax(t) + Bu(t) + c)\ .$$

Common sigmoids

Sigmoids [MP43], [SS94, SS95] and [Hay94]

(a) The McCulloch-Pitts sigmoid,

$$\sigma_d(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(b) The saturated sigmoid,

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 1 \\ x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x < 0 \end{cases}$$

(c) The analytic sigmoid of parameter k,

$$\sigma_k(x) = \frac{1}{1 + e^{-kx}}$$

Computing successor in unary

Example (Successor in unary)

$$
\begin{aligned}
y_1^+ &= \sigma(a) \\
y_a^+ &= \sigma(a + y_1)
\end{aligned}
$$

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Example (Successor in unary)

t	a	y_1	y_a
0	0	0	0
1	1	0	0
2	1	1	1
3	0	1	1
4	0	0	1
5	0	0	0

Computing addition in binary

Example (Adition in binary)

$$
\begin{aligned}
y_1^+ &= \sigma(a + b + v + y_1 - 2)\\
y_2^+ &= \sigma(a + b + v + y_1 - 3)\\
y_3^+ &= \sigma(a - 2b + v - 2y_1 - 1)\\
y_4^+ &= \sigma(-2a + b + v - 2y_1 - 1)\\
y_5^+ &= \sigma(-2a - 2b + v + y_1 - 1)\\
y_6^+ &= \sigma(-a - b - v + y_1)\\
y_7^+ &= \sigma(a + b + v - 3y_1)\\
y_8^+ &= \sigma(a - 3b + v + y_1)\\
y_9^+ &= \sigma(-3a + b + v + y_1)\\
y_{10}^+ &= \sigma(-a - b + v + y_1)\\
y_{a+b}^+ &= \sigma(y_2 + y_3 + y_4 + y_5 + y_6)\\
y_v^+ &= \sigma(y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10})
\end{aligned}
$$

Computing addition in binary

Example (Addition in binary)

t	a	b	v	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{a+b}	y_v
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	0	0	1	0	0	0	1	1	0	0	0	0
3	0	0	0	1	0	0	0	0	0	1	0	0	0	1	1
4	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Decidability

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Definition

A word $w \in \{0,1\}^+$ is said to be classified in time τ by a system $\mathcal{N}$ if the input streams are $\langle u_1, u_2 \rangle$, with $u_1 = 0w0^\omega$ and $u_2 = 01^{|w|}0^\omega$, and the output streams are $\langle v_1, v_2 \rangle$ with $v_2(t) \equiv (t = \tau)$. If $v_1(\tau) = 1$, then the word is said to be accepted, otherwise (if $v_1(\tau) = 0$) rejected.

Query tape

Definition

Let $\mathcal{B}$ be a class of sets and $\mathcal{F}$ a class of total functions of signature $\mathbb{N} \to \Sigma^\star$. The non-uniform class $\mathcal{B}/\mathcal{F}$ is the class of sets A for which some $B \in \mathcal{B}$ and some $f \in \mathcal{F}$ are such that, for every w, $w \in A$ if and only if $\langle w, f(|w|)\rangle \in B$. If we take $\mathcal{B}$ as P and $\mathcal{F}$ as poly, then we get class $P/$poly.

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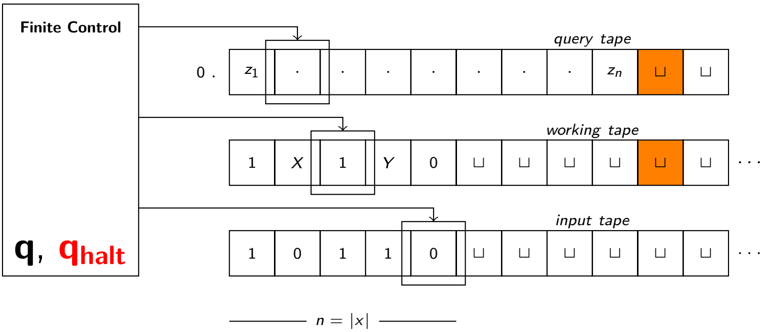
Vs. query tape



Lower and upper bounds in polynomial time

Proposition

The output of an ARNN after t steps is affected only by the first $O(t)$ digits in the expansion of the weights.

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Proposition

$ARNN[\mathbb{R}]P = P/poly$.

Structural complexity

Halting set

The sparse halting set is

$$\mathrm{HALT} = \{0^n : n \text{ codes for a TM that halts on input } 0\}$$

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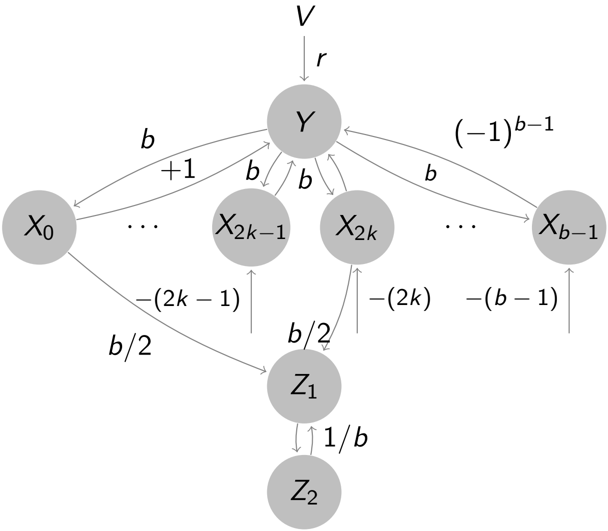
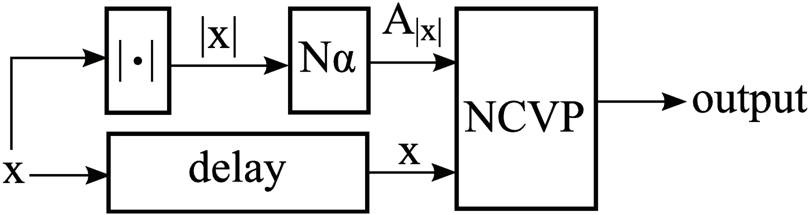
Computational power of *ARNN* under various restrictions

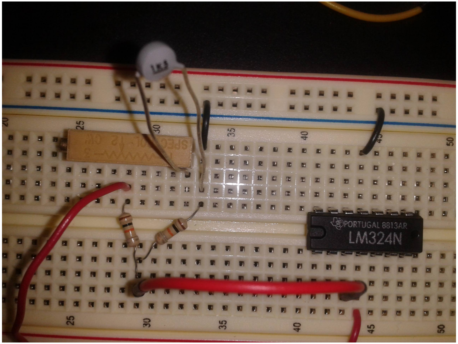
Weights	**Time restriction**	**Computational power**
$\mathbb{Z}$	none	Regular sets
$\mathbb{Q}$	none	Recursively enumerable sets
$\mathbb{R}$	polynomial	P/poly
$\mathbb{R}$	none	All sets

The BAM



The standard sigmoid





Measurement theory

Bachelard, Eddington

Gaston Bachelard

Let us briefly note that the behaviour of the precision balance, though it is faithful to the mass, is not always clear: many students are surprised and disturbed by the slowness of the measurement process. We can not say that, for everyone, there is a precise idea of measurement of mass.[a]

[a] Gaston Bachelard, *The Philosophy of No: A Philosophy of the New Scientific Mind*, Viking Press, 1968 (1940).

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Arthur Eddington

Yet space is a prominent feature of the physical world; and measurement of space — lengths, distances, volumes — is part of the normal occupation of a physicist. Indeed it is rare to find any quantitative physical observation which does not ultimately reduce to measuring distances.[a]

[a]Arthur Eddington, *The Expanding Universe*, Cambridge University Press, First published in 1933.

Measurement according to Hempel [Hem52, KSLT09]

Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-irrefexive* if, for all objects a and b in a set $\mathcal{O}$, if $a\mathcal{E}b$ is the case, then $a\mathcal{L}b$ does not hold.

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Definition

Given two binary relations $\mathcal{E}$ and $\mathcal{L}$ in a set $\mathcal{O}$, $\mathcal{L}$ is *$\mathcal{E}$-connected* if, for all objects a and b in $\mathcal{O}$, if $a\mathcal{E}b$ is not the case, then either $a\mathcal{L}b$ or $b\mathcal{L}a$ holds.

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Definition

Two binary relations $\mathcal{E}$ and $\mathcal{L}$ determine a *comparative concept*, or a *quasi-series*, for the elements of $\mathcal{O}$, if $\mathcal{E}$ is an equivalence relation and $\mathcal{L}$ is transitive, $\mathcal{E}$-irreflexive, and $\mathcal{E}$-connected.

Hempel: Measurement map [Hem52, KSLT09]

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Axiom 1 If $a\mathcal{E}b$, then $M(a) = M(b)$.

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Axiom 1 If $a\mathcal{E}b$, then $M(a) = M(b)$.

Axiom 2 If $a\mathcal{L}b$, then $M(a) < M(b)$.

Hempel: Propositional

Proposition

For all a, b in $\mathcal{O}$, one, and only one, of the following statements holds: (a) $a\mathcal{E}b$, (b) $a\mathcal{L}b$, or (c) $b\mathcal{L}a$.

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For all a, b in $\mathcal{O}$, one, and only one, of the following statements holds: (a) $a\mathcal{E}b$, (b) $a\mathcal{L}b$, or (c) $b\mathcal{L}a$.

Proposition

For all a, b in $\mathcal{O}$:

$$\begin{aligned} \text{If } M(a) &= M(b), \text{ then } a\mathcal{E}b \\ \text{If } M(a) &< M(b), \text{ then } a\mathcal{L}b \end{aligned}$$

Hempel: First order logic

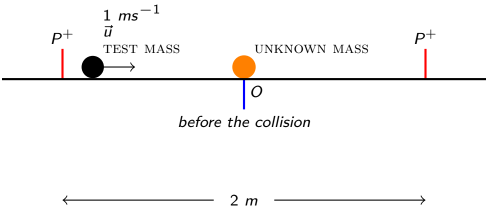
Proposition

$$\forall x\ \forall y\ (x\mathcal{E}y \ \Leftrightarrow\ \forall u\ ((x\mathcal{L}u \Leftrightarrow y\mathcal{L}u) \wedge (u\mathcal{L}x \Leftrightarrow u\mathcal{L}y)))$$

$$\forall x\ \forall y\ \forall z\ ((x\mathcal{E}y \wedge y\mathcal{L}z) \Rightarrow x\mathcal{L}z)$$

Timed measurement systems

Collider experiment



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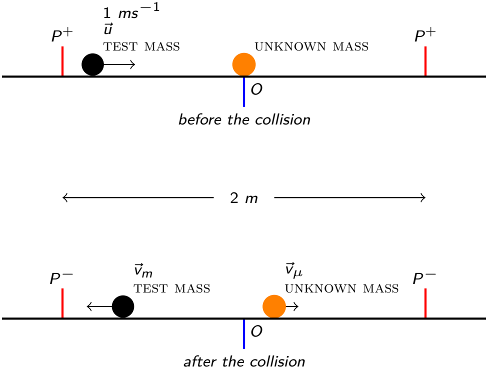

Collider experiment

Implementing a comparative concept

1. Test particle m is detected backward, in time t: $m\mathcal{L}_t\mu$;

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3. Test particle m not seen within time t: $m\mathcal{E}_t\mu$.

Timed relation [BCT10a]

Definition

A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a *timed equivalence relation* if there is a $\kappa \geq 1$ so that

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2. $\mathcal{E}_t$ is *timed symmetric*: for every a, b in $\mathcal{O}$, if $a\mathcal{E}_t b$, then $b\mathcal{E}_{t/\kappa}a$;

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3. $\mathcal{E}_t$ is *timed transitive*: for every a, b, and c in $\mathcal{O}$, if $a\mathcal{E}_t b$ and $b\mathcal{E}_t c$, then $a\mathcal{E}_{t/\kappa} c$;

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4. if $t < t'$ and $a\mathcal{E}_{t'}b$, then $a\mathcal{E}_t b$.

Timed relation [BCT10a]

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Two binary relations $\mathcal{E}_t$ and $\mathcal{L}_t$ $(t > 0)$ determine a *timed comparative concept* for the elements of $\mathcal{O}$, if

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1. $\mathcal{E}_t$ is a timed equivalence relation;

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2. There is a $\kappa \geq 1$ so that for every a, b, c in $\mathcal{O}$, if $a\mathcal{L}_t b$ and $b\mathcal{L}_t c$, then $a\mathcal{L}_{t/\kappa} c$;

Timed relation [BCT10a]

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3. For all $t > 0$ and $a, b \in \mathcal{O}$, exactly one of $a\mathcal{E}_t b$, $a\mathcal{L}_t b$, $b\mathcal{L}_t a$ holds;

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4. If $t < t'$ and $a\mathcal{L}_t b$, then $a\mathcal{L}_{t'} b$.

Definition

Let $\mathcal{E}_t$ and $\mathcal{L}_t$ be timed comparative relations on the set $\mathcal{O}$ of objects. Suppose there exists an *experimental apparatus* to witness these relations. Then the map $M : \mathcal{O} \to \mathbb{R}$ is said to be a *measurement map* if

$$\exists_{t>0}\ a\mathcal{L}_t b \quad \Rightarrow \quad M(a) < M(b)$$

Separation axiom

Axiom

The apparatus satisfies the *separation property* for the measurement map $M : \mathcal{O} \to \mathbb{R}$ if, for every objects a and b in $\mathcal{O}$, if $M(a) < M(b)$, then there exists a time bound t such that $a\mathcal{L}_t b$.

Limit timed relations

Definition

Given the timed comparative concept $\mathcal{E}_t$ and $\mathcal{L}_t$, for some time bound t, we define the following relations $\mathcal{E}_{lim}$ and $\mathcal{L}_{lim}$:

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2. for every a and b in $\mathcal{O}$, $a\mathcal{L}_{lim}b$ if there exists a time bound t such that $a\mathcal{L}_t b$

Proposition

If the two relations $\mathcal{E}_t$ and $\mathcal{L}_t$ define a timed comparative concept and the physical apparatus witnessing the relations satisfies the separation property, then the two relations $\mathcal{E}_{lim}$ and $\mathcal{L}_{lim}$ define a comparative concept and M is a measurement map in the sense of Hempel.

Collider experiment

Proposition

The collider experiment is a measurement procedure in the sense of Hempel, once we move from concept $\langle \mathcal{E}_t, \mathcal{L}_t, M \rangle$ *to the concept* $\langle \mathcal{E}_{lim}, \mathcal{L}_{lim}, M \rangle$.

Complexity of the measurement map

Definition

The *complexity of a measurement map* $M : \mathcal{O} \to \mathbb{R}$, given the timed comparative relations $\mathcal{E}_t$ and $\mathcal{L}_t$ on the set $\mathcal{O}$ of objects, is the map $T : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$
\begin{aligned}
T(n) \quad = \quad & \min\{t \in \mathbb{N} - \{0\} : a_n \mathcal{L}_t a \\
& \text{for some } a, a_n \in \mathcal{O} \text{ with } M(a_n) = M(a){\downharpoonleft}_n\}.
\end{aligned}
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Definition

We say that a measurement in physical theory $\mathcal{T}$ has complexity T if the associated measurement map M has a computable complexity T.

BCT Conjecture

Conjecture

No reasonable physical measurement has an associated measurement map with polynomial time complexity.

Geroch and Hartle [GH86]

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We propose, in parallel with the notion of a computable number in mathematics, that of a measurable number in a physical theory. The question of whether there exists an algorithm for implementing a theory may then be formulated more precisely as the question of whether the measurable numbers of the theory are computable.[a]

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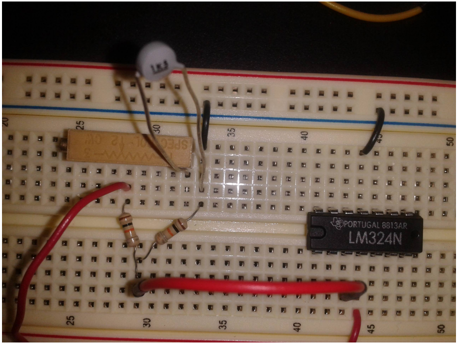
Geroch and Hartle [GH86]

Regard number w as measurable if there exists a finite set of instructions for performing an experiment such that a technician, given an abundance of unprepared raw materials and an allowed error ε, is able by following those instructions to perform the experiment, yielding ultimately a rational number within ε of w. [a]

[a] Robert Geroch and James B. Hartle, *Computability and Physical Theories*, Foundations of Physics, 16(6), 1986.



The three types of measurements

Three cases of measurability [BCT10c, BCT14]

The vertical axis measures the outcome of the experiment; we have to find the first zero x by trial and error on the value a:

Three cases of measurability [BCT10c, BCT14]

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Type I

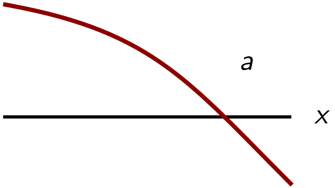

Figure: Measure both $a < x$ and $x < a$.

Three cases of measurability [BCT10c, BCT14]

Type I

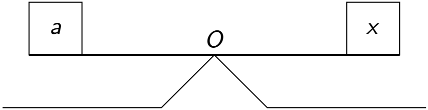

Figure: Balance.

Type II

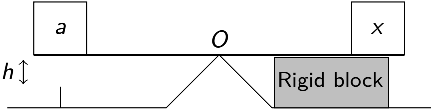

Figure: Broken balance.

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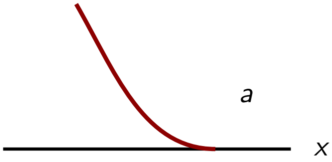

Figure: Can only measure $a < x$.

Three cases of measurability [BCT10c, BCT14]

Type I

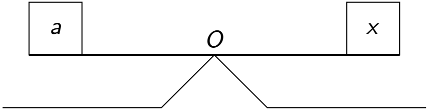

Figure: Balance.

Type II

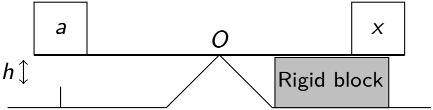

Figure: Broken balance.

Three cases of measurability [BCT10c, BCT14]

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Three cases of measurability [BCT10c, BCT14]

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Type III

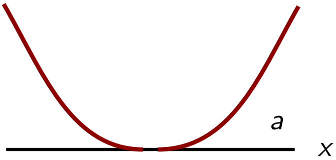
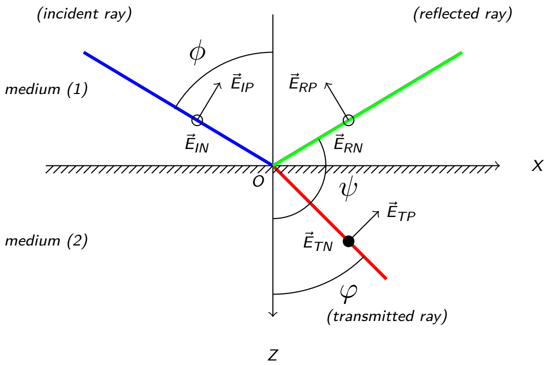
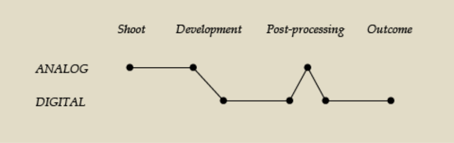


Figure: Can only measure ($a < x$ or $x < a$).

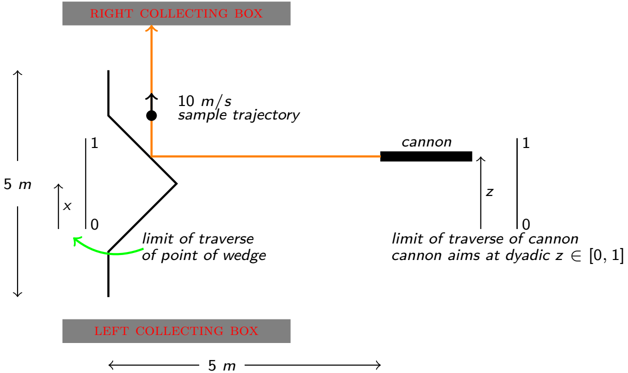
Brewster angle





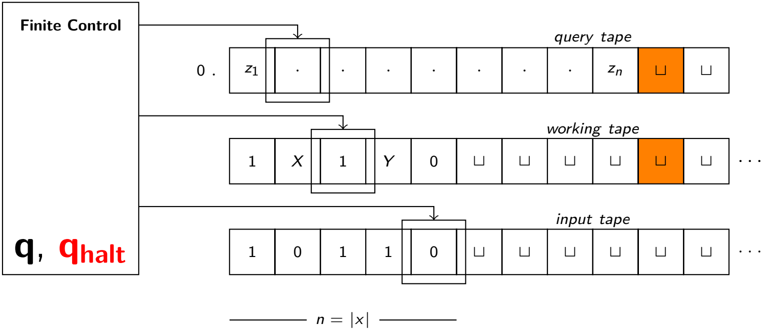
The scatter machine model I

The scatter machine [BT07]



Query tape [BCLT08]

Query tape



Analog-digital scatter machine: decidability

Error-free analog-digital scatter machine

Let $A \subseteq \Sigma^\star$ be a set of words over Σ. We say that an error-free analog-digital scatter machine $\mathcal{M}$ decides A if, for every input $w \in \Sigma^\star$, w is accepted if $w \in A$ and rejected if $w \notin A$. We say that $\mathcal{M}$ decides A in polynomial time, if $\mathcal{M}$ decides A, and there is a polynomial p such that, for every input $w \in \Sigma^\star$, the number of steps of the computation of $\mathcal{M}$ on w is bounded by $p(|w|)$.

Analog-digital scatter machine: decidability

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Error-prone analog-digital scatter machine

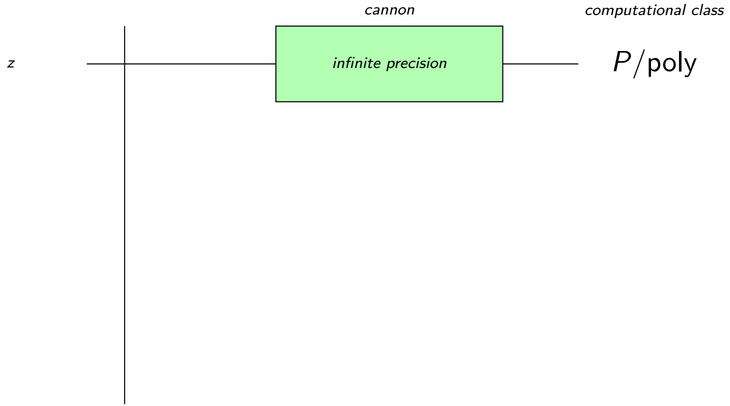
Let $A \subseteq \Sigma^\star$ be a set of words over Σ. We say that an error-prone analog-digital scatter machine $\mathcal{M}$ decides A if there is a number $\gamma < \frac{1}{2}$, such that the error probability of $\mathcal{M}$ for any input w is smaller than γ. We say that $\mathcal{M}$ decides A in polynomial time, if $\mathcal{M}$ decides A, and there is a polynomial p such that, for every input $w \in \Sigma^\star$, the number of steps in every computation of $\mathcal{M}$ on w is bounded by $p(|w|)$.

$BPP//\log\star$

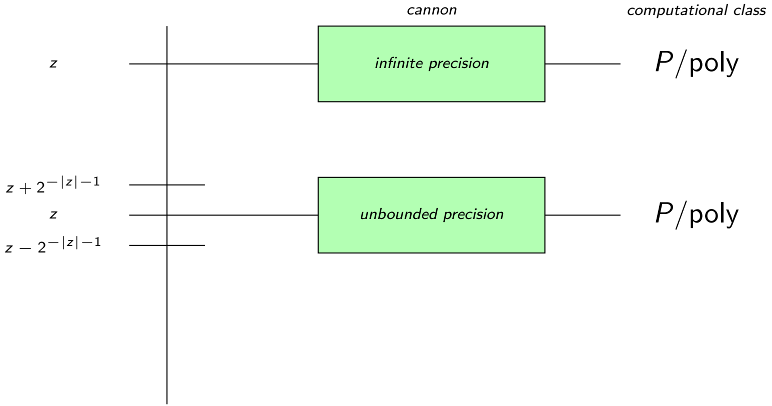
Definition

$BPP//\log\star$ is the class of sets $A \subseteq \Sigma^\star$ for which a probabilistic Turing machine $\mathcal{M}$, clocked in polynomial time, a prefix function $f \in \log$, and a constant $\gamma < \frac{1}{2}$ exist such that, for every length n and input w with $|w| \leq n$, $\mathcal{M}$ rejects $\langle w, f(n)\rangle$ with probability at most γ if $w \in A$ and accepts $\langle w, f(n)\rangle$ with probability at most γ if $w \notin A$.

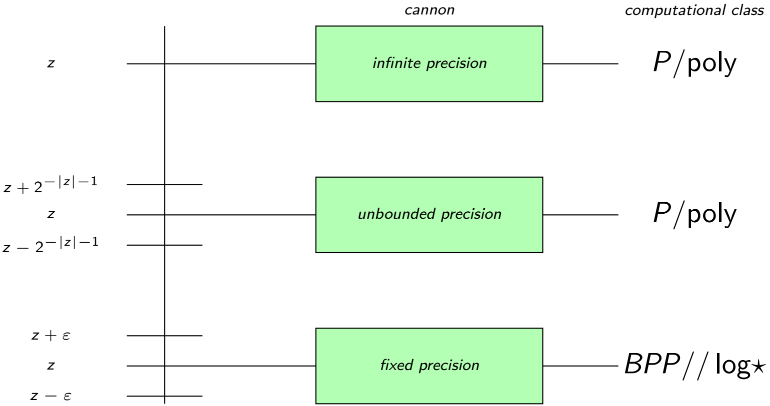
ARNN case and the sharp scatter machine



ARNN case and the sharp scatter machine



ARNN case and the sharp scatter machine



We describe two more scenarios, for the vertex position:

1. The wedge can be placed at the real x — infinite precision.

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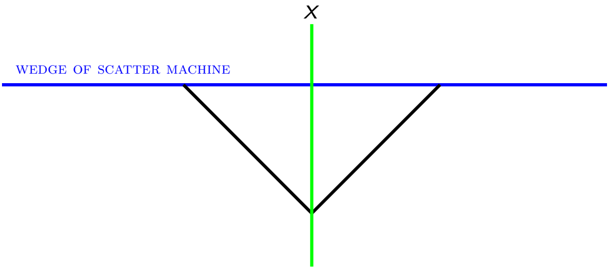

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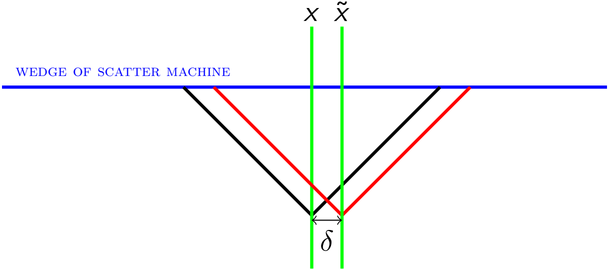
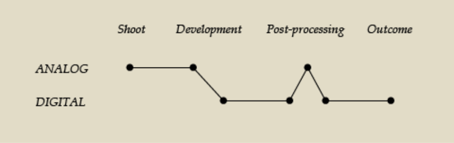
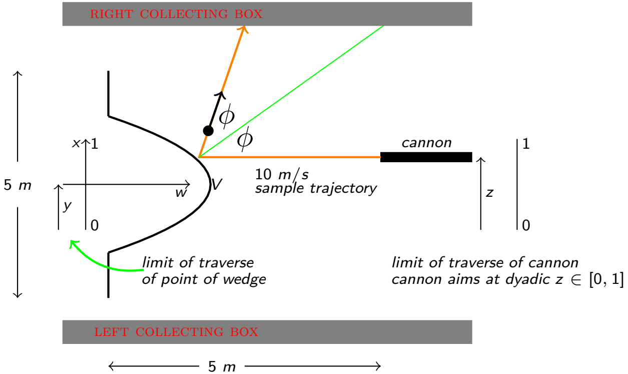




The scatter machine model II

Generalised scatter machine [BCT12b]



Complexity of the vertex position [BCT12b, BCT12a]

Proposition

Any particle hitting horizontally, sufficiently closer to the vertex V, will bounce back covering an horizontal distance before detection that goes to infinity as $O(\frac{1}{|z-y|})$.

Complexity of the vertex position [BCT12b, BCT12a]

Proposition

Any particle hitting horizontally, sufficiently closer to the vertex V, will bounce back covering an horizontal distance before detection that goes to infinity as $O(\frac{1}{|z-y|})$.

Proposition

The protocol that processes queries between a Turing machine and the generalised scatter machine takes a time that is at least exponential in the size of the dyadic rational specified by the query during the binary search procedure.

Complexity of the vertex position [BCT12b, BCT12a]

Proposition

Consider that $g(x)$ is the function describing the shape of the wedge of a SmSE. Suppose that $g(x)$ is n times continuously differentiable near $x = 0$, all its derivatives up to $(n-1)$-th vanish at $x = 0$, and the n-th derivative is nonzero. Then, when the SmSE, with vertex position y, fires the cannon at position z, the time needed to detect the particle in one of the boxes is $t(z)$, where:

$$\frac{A}{|y-z|^{n-1}} \leq t(z) \leq \frac{B}{|y-z|^{n-1}} \ , \qquad (1)$$

for some $A, B > 0$ and for $|y - z|$ sufficiently small.

Protocol [BCLT08, BCLT09]

The cannon can be placed at the dyadic rational z — infinite precision

Algorithm 1: Measurement algorithm for infinite precision.

Data: Positive integer ℓ representing the desired precision

```
x_0 = 0 ;
x_1 = 1 ;
z = 0 ;
while x_1 − x_0 > 2^(−ℓ) do
    z = (x_0 + x_1)/2 ;
    s = Prot_IP(z↾ℓ) ;
    if s == "q_r" then
        x_1 = z ;
    if s == "q_l" then
        x_0 = z ;
    else
        x_0 = z ;
        x_1 = z ;
return Dyadic rational denoted by x_0
```

Protocol [BCLT08, BCLT09]

The cannon can be placed at the dyadic rational z, but only with unbounded but finite precision, say $2^{-|z|-1}$, i.e., the cannon can be set at position $z \pm 2^{-|z|-1}$

Algorithm 5: Measurement algorithm for unbounded precision.

Data: Positive integer ℓ representing the precision

$x_0 = 0$;
$x_1 = 1$;
$z = 0$;
while $x_1 - x_0 > 2^{-\ell}$ **do**
  $z = (x_0 + x_1)/2$;
  $s = Prot\_UP(z\rfloor_{\ell})$;
  if $s ==$ "q_r" **then**
    $x_1 = z$;
  if $s ==$ "q_l" **then**
    $x_0 = z$;
  else
    $x_0 = z$;
    $x_1 = z$;
return *Dyadic rational denoted by* x_0

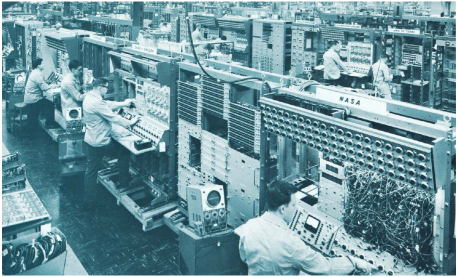
Protocol [BCLT08, BCLT09]

The cannon can be placed at the dyadic rational z, but only with fixed a priori precision ε (dyadic rational), i.e., the cannon can be set at position $z \pm \varepsilon$

Algorithm 9: Measurement algorithm for fixed precision.

Data: Integer ℓ representing the precision

$c = 0$;
$i = 0$;
$\xi = 2^{2\ell+h}$;
while $i < \xi$ **do**
    $s = Prot\_FP(1\rfloor_{\ell})$;
    if $s ==$ "q_l" **then**
        $c = c + 2$;
    if $s ==$ "q_t" **then**
        $c = c + 1$;
    i++ ;
return $c/(2\xi)$



The Power of Analogue-Digital Machines

The digital-analog device as a biased coin

Proposition

Given an error-prone smooth scatter machine, vertex position at y, experimental time t, and time schedule T, there is a dyadic rational z and a real number $\delta \in]0, 1[$ such that the outcome of Prot_UP on z is a random variable that produces left with probability δ.

The digital-analog device as a biased coin

Proposition

Given an error-prone smooth scatter machine, vertex position at y, experimental time t, and time schedule T, there is a dyadic rational z and a real number $\delta \in]0,1[$ such that the outcome of Prot_UP on z is a random variable that produces left with probability δ.

Proposition

Given a biased coin with probability of heads $q \in]\delta, 1-\delta[$, for some $0 < \delta < 1/2$, and $\gamma \in]0,1[$, we can simulate, up to probability $\geq \gamma$, a sequence of independent fair coin tosses of length n by doing a linear number of biased coin tosses.

The digital-analog device as a biased coin

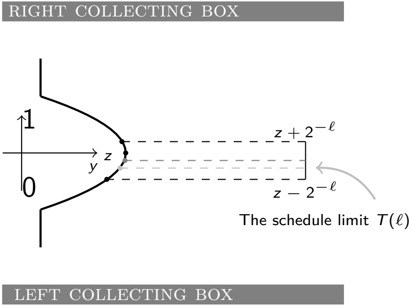


Figure: The *SmSE* with unbounded precision as a coin.

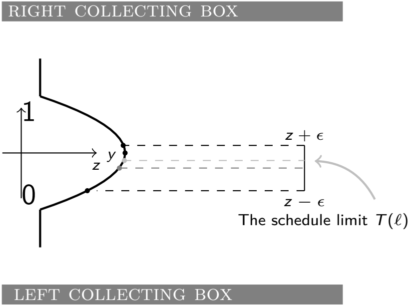
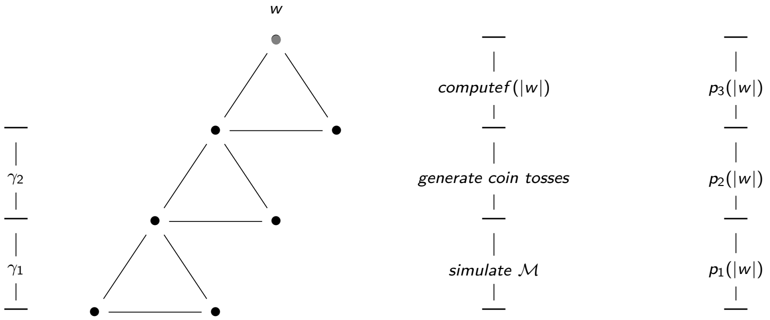
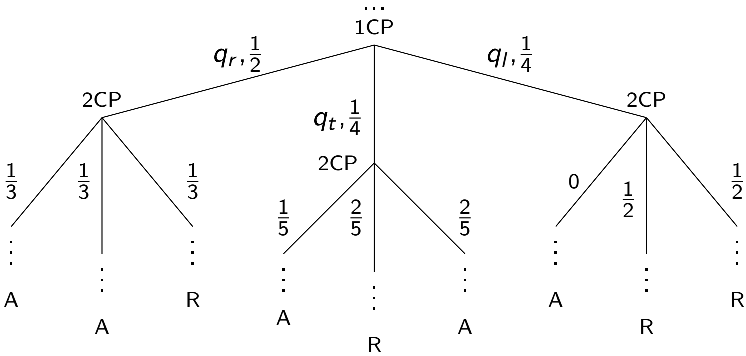


Figure: The *SmSE* with fixed precision as a coin.

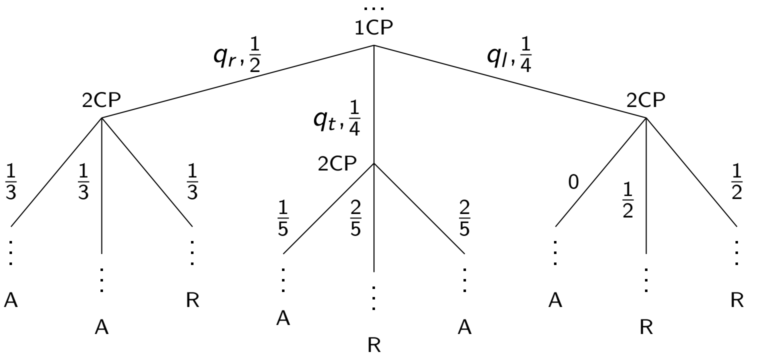
Lower bounds



Upper bounds



Upper bounds



Proposition

For any $m \in \mathbb{N}$, $s \in [0, 1]$, *and any number out* $\in \mathbb{N}$ *of children in the tree,* $\mathcal{A}_{out}(m, s) \leq (out - 1)ms$.

Computational power ([BCPT13, ABC$^+$16])

	Infinite	**Unbounded**	**Fixed**
Lower Bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Upper Bound Exponential schedule	$P/\log\star$	$BPP//\log^2\star$	$BPP//\log^2\star$
Upper Bound Explicit Time	—	$BPP//\log\star$ Exponential schedule	$BPP//\log\star$ Exponential schedule

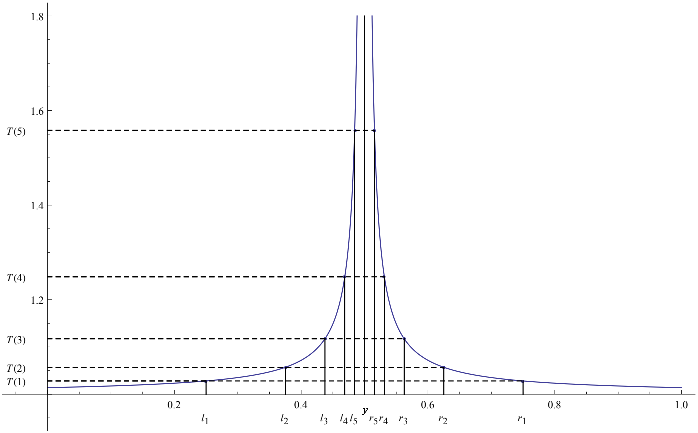
Computational power ([BCPT13, ABC+16])

Proposition

If B is decidable by a smooth scatter machine with infinite precision and exponential protocol, clocked in polynomial time, then $B \in P/\log^2\star$.

Computational power ([BCPT13, ABC+16])

Boundary numbers



Computational power ([BCPT13, ABC+16])

Proof

1. $\mathcal{M}$ only queries the oracle with words of size less or equal to $\ell = a\lceil \log(n) \rceil + b$;
2. $f(n)$ encodes the concatenation of boundary numbers needed to answer to all the queries of size ℓ:
   $l_1\rfloor_1 \# r_1\rfloor_1 \# l_2\rfloor_2 \# r_2\rfloor_2 \# \cdots \# l_\ell\rfloor_\ell \# r_\ell\rfloor_\ell \#$;
3. $|f(n)| \in \mathcal{O}(\log^2(n))$;
4. B is decided in polynomial time with prefix advice $f \in \log^2$; $\mathcal{M}$ is simulated on the input word but now the Turing machine compares the query z with the boundary numbers $l_{|z|}$ and $r_{|z|}$.

Computational power ([BCPT13, ABC+16])

Proposition

Given the boundary numbers for a smooth scatter machine with time schedule $T(k) \in \Omega(2^k)$ *it is possible to define a prefix advice function* f *such that* $f(n)$ *encodes all the boundary numbers with size up to* n *and* $|f(n)| \in \mathcal{O}(n)$.

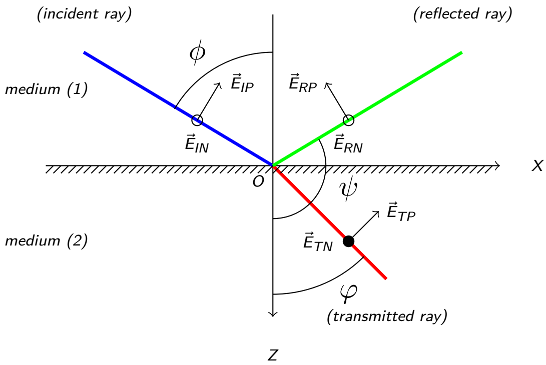
Computational power ([BCPT13, ABC+16])

Proposition

If B is decidable by a smooth scatter machine with infinite precision and exponential protocol $T(k) \in \Omega(2^k)$, clocked in polynomial time, then $B \in P/\log\star$.

Vanishing experiments

Brewster angle



Vanishing experiments, [BCT14, BCT10c, BCPT17]

Parallel strategy

To perform two experiments simultaneously, that is, to use two copies of the balance with the same unknown mass y in the right pan. We can place masses z_1 and z_2 at the left pans of the balances and start both experiments at the same time. If $T_{\mathrm{exp}}(z_1, y) < T_{\mathrm{exp}}(z_2, y)$, then the experiment with test mass z_1 sends a first signal and if $T_{\mathrm{exp}}(z_1, y) > T_{\mathrm{exp}}(z_2, y)$, then the experiment with test mass z_2 calls back first.

Vanishing experiments, [BCT14, BCT10c, BCPT17]

Parallel strategy

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Clock strategy

Suppose we only have one balance, but now we can count the machine steps during an experiment until the end. In this way we can begin by performing an instance of the experiment for test mass z_1, and counting the number T_1 of machine transitions that the experiment takes. Then repeat the experiment for test mass z_2, obtaining a number T_2 of machine transitions. Finally, compare T_1 and T_2. If $T_1 < T_2$, then we conclude that $T_{\mathrm{exp}}(z_1) < T_{\mathrm{exp}}(z_2)$; if $T_1 > T_2$, then $T_{\mathrm{exp}}(z_1) > T_{\mathrm{exp}}(z_2)$.

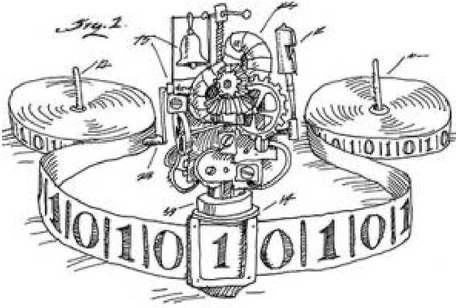
Time precision, [BCT14, BCPT17]

Types of precision

1. *Infinite precision*: ...;
2. *Unbounded precision*: ...;
3. *Fixed precision*: ...;
4. *Time precision g*, given a map $g : \mathbb{N} \to \mathbb{N}$: when an experiment settled for the query word z takes an amount of time t, the number of machine transitions counted is T_1, where T_1 is a natural number uniformly sampled in $[\lceil t \rceil - g(|z|), \lceil t \rceil + g(|z|)]$.

Vanishing experiments, [BCT14, BCPT17]

Type of Oracle		Infinite	Unbounded	Finite
Two-sided	lower bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
	upper bound	P/poly	P/poly	P/poly
	upper bound (w/ exponential T)	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Threshold	lower bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
	upper bound	——	——	——
	upper bound (w/ exponential T)	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Vanishing Type 1 (Parallel)	lower bound	P/poly	P/poly	$BPP//\log\star$
	upper bound	P/poly	P/poly	$BPP//\log\star$
	upper bound (w/ exponential T)	——	——	——
Vanishing Type 2 (Clock)	lower bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
	upper bound	P/poly	P/poly	$BPP//\log\star$
	upper bound (w/ exponential T)	——	$BPP//\log\star$	——



Space bounded AD machines

Space bounded AD machines, [AC18]

	Infinite	Arbitrary	Fixed
Lower Bound	*PSPACE / poly*	*BPPSPACE / / poly*	*BPPSPACE / / poly*
Upper Bound	*PSPACE / poly*	*BPPSPACE / / poly*	*BPPSPACE / / poly*

	Infinite	Arbitrary	Fixed
Lower Bound	*PSPACE / poly*	*BPPSPACE / / poly*	*BPPSPACE / / poly*
Upper Bound	*PSPACE / poly*	*BPPSPACE / / poly*	*BPPSPACE / / poly*

Table: Standard communication protocol for the sharp (above) and smooth (below) scatter machines.

Space bounded AD machines, [AC18]

	Infinite	Arbitrary	Fixed
Lower Bound	$2^{\Sigma^\star}$	$2^{\Sigma^\star}$	$BPPSPACE//poly$
Upper Bound	$2^{\Sigma^\star}$	$2^{\Sigma^\star}$	$BPPSPACE//poly$

	Infinite	Arbitrary	Fixed
Lower Bound with time schedule	$PSPACE/poly$	$BPPSPACE//poly$	$BPPSPACE//poly$
Lower Bound without time schedule	$2^{\Sigma^\star}$	$2^{\Sigma^\star}$	—
Upper Bound	$2^{\Sigma^\star}$	$2^{\Sigma^\star}$	$BPPSPACE//poly$

Table: Generalized communication protocol for the sharp (above) and smooth (below) scatter machines.

Concept of a measurable quantity

Geroch and Hartle [GH86]

Geroch and Hartle [GH86]

Every computable number is measurable. This is easy to see: Let the instructions direct that the raw materials be assembled into a computer, and that a certain [...] program — one specified in the instructions — be run on that computer. That is, every digital computer is at heart an analog computer. [a]

[a] Robert Geroch and James B. Hartle, *Computability and Physical Theories*, Foundations of Physics, 16(6), 1986.

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Geroch and Hartle [GH86]

We now ask whether, conversely, every measurable number is computable — or, in more detail, whether current physical theories are such that their measurable numbers are computable. This question must asked with care. [a]

[a] Robert Geroch and James B. Hartle, *Computability and Physical Theories*, Foundations of Physics, 16(6), 1986.

Concept of measurable [BCT10b]

Definition

A distance y is said to be measurable if there exists a Turing machine, equipped with a computable schedule T, such that it prints the first n bits of y on the output tape in less than $T(n)$ time steps without timing out in any query.

Concept of measurable [BCT10b]

Definition

A distance y is said to be measurable if there exists a Turing machine, equipped with a computable schedule T, such that it prints the first n bits of y on the output tape in less than $T(n)$ time steps without timing out in any query.

Proposition

There are programs N_k (with integer $k \geq 1$), with specified waiting times (say T_k), so that the following is true: For any non-dyadic value $y \in (0, 1)$ and any $n > 0$, there is a k so that the program will find the first n binary places of y.

Measuring distance [BCT10b]

Proposition

There are uncountable many $y \in [0, 1]$ *so that, for any program* P *with specified waiting times, there is a* n *so that* P *can not determine the first* n *binary places of* y.

Measurable distances [BCT10b]

Proposition

For the SmSM with vertice at y (not a dyadic rational), written according to the pattern:

$$y = 0{\cdot}\underbrace{1\ldots1}_{u_1}\underbrace{0\ldots0}_{u_2}\underbrace{1\ldots1}_{u_3}\underbrace{0\ldots0}_{u_4}\underbrace{1\ldots1}_{u_5}\underbrace{0\ldots0}_{u_6}\ldots$$

where $u_1 \geq 0$, $u_i \geq 1$ ($i \geq 2$).

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1. *If y is measurable by any program, then the sequence u_k is bounded by a computable function.*

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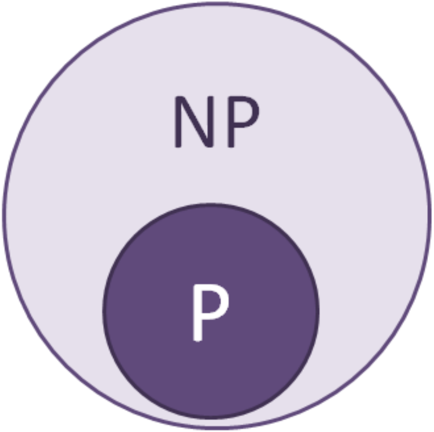
1. *If y is measurable by any program, then the sequence u_k is bounded by a computable function.*
2. *If the sequence u_k is bounded by a computable function, then y is measurable by the linear search method.*



Open problems

Open problems

Examples of open problems

1. Infinite precision: do the lower and the upper bound coincide without assumptions on the time schedule?
2. Error-prone: do the lower and the upper bounds coincide without using the explicit time technique? Namely, it is not known if there exists a set not belonging $BPP//\log\star$ decidable by a two-sided machine in polynomial time.

Class of advice functions

Definition (Ordinal iterate of the logarithm)

For each ordinal α, we define inductively the class $\log^{(\alpha)}$:

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A first hierarchy of scales

Proposition (A first hierarchy of scales)

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Consider $\log\star$, defined by (a) $\log\star(t) = 0$, for $t = 0$, and (b) $\log\star(t) = \min\{k : \log^{(k)}(t) \leq 1\}$, for $t > 0$.

A second hierarchy of scales

Proposition (A second hierarchy of scales)

$$\log^{(2\omega)} \prec \cdots \prec \log^{(\omega+1)} \prec \log^{(\omega)} \prec \cdots \prec \log^{(2)} \prec \log^{(1)} \prec \text{poly}.$$

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Consider $\log \star\star = \log \star \circ \log \star$.

Example (Non-emptyness of limit classes)

We can continue descending by setting $\log^{(2\omega+k)}$ to be the class generated by $\log^{(k)} \circ \log \star\star$...

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