Symmetric probes & Classification of toric fibres

Lisbon Geometry Seminar — April 18 2023

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"Hamiltonian classification of toric fibres and symmetric probes" — arXiv : 2302.00334

"On Lagrangian tori in $S^2 \times S^2$" — in preparation
joint w/ Joontae KIM.

Outline:

§0. Motivating example : $S^2 \times S^2$

§1. Toric geometry

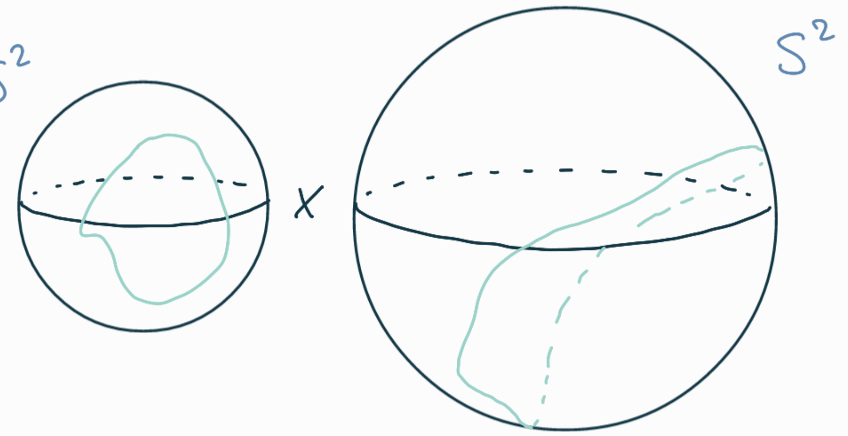
§2. Symmetric probes

§3. Obstructions : Chekanov's invariants

§0. Example

$\int_{S^2} \omega_{S^2} = 1$ $\quad a = b$: monotone

Take $S^2(a) \times S^2(b) = (S^2 \times S^2, a\omega_{S^2} \oplus b\omega_{S^2})$

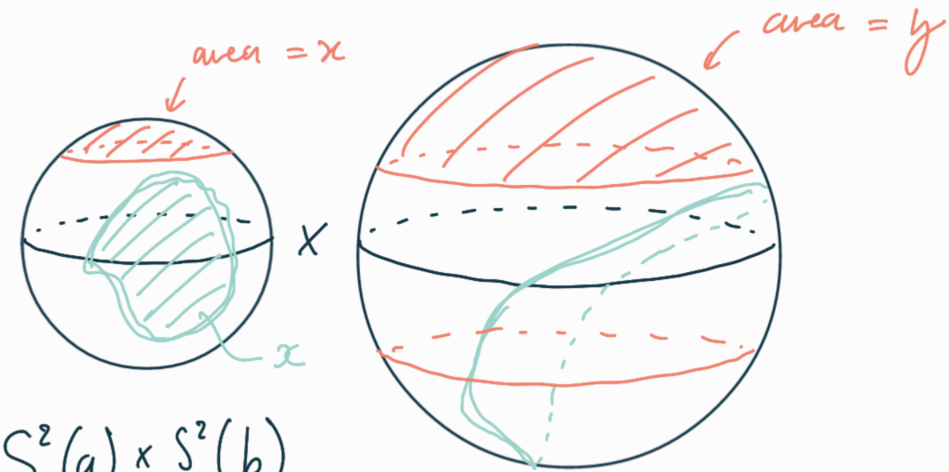


Taking circle x circle gives us a Lagrangian torus.

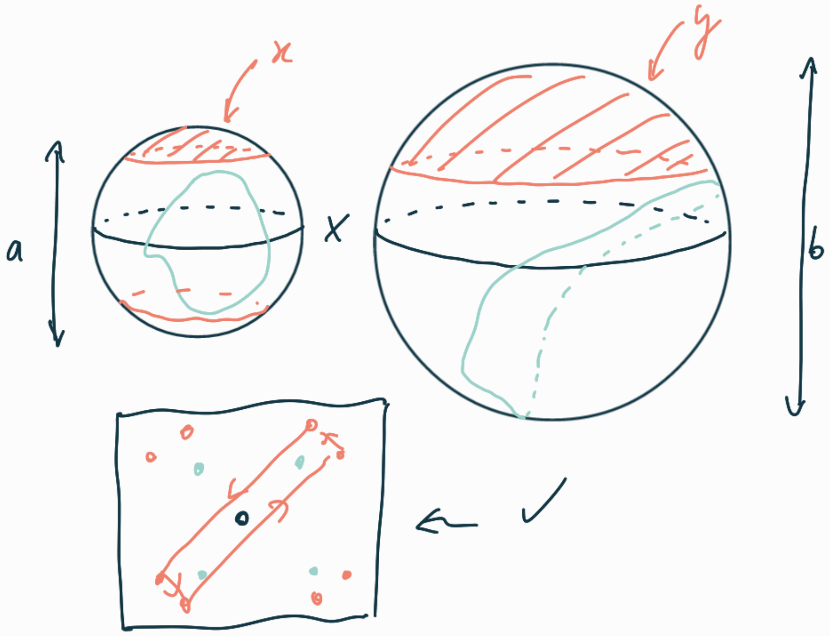
<u>Def</u>: Call a torus of this type <u>product torus</u>.

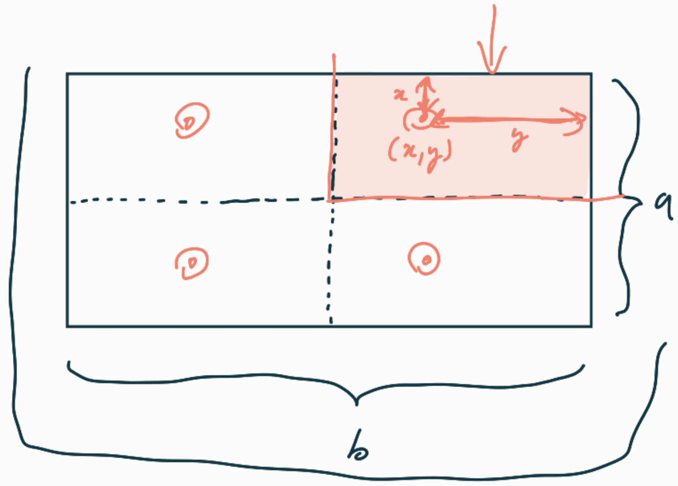
<u>Question</u>: Classify product tori up to Hamiltonian diffeomorphisms of the ambient space.

We can restrict our attention to products of circles of constant height.



$T(x,y) \subset S^2(a) \times S^2(b)$



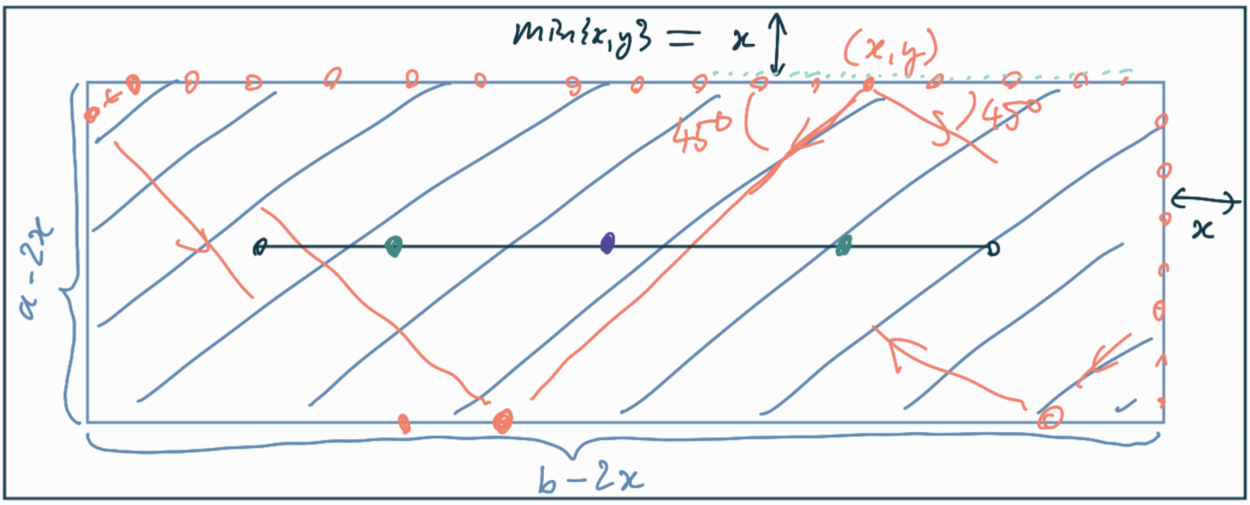


set of product tori.

Theorem: (joint w/ Joontae KIM)

$T(x,y) \cong T(x',y') \iff$ (x',y') is a bouncing point of a $45°$-billiard trajectory starting at (x,y), $(a-x,y)$, $(x,b-y)$ or $(a-x,b-y)$.

$x < y$



Remark: If $\frac{a-2x}{b-2x} \in \mathbb{Q}$ the set of equiv. fibres is discrete.

If not, it has accumulation pts.

§1. Toric geometry.

$T^n = \underbrace{S^1 \times \dots \times S^1}_{n}$

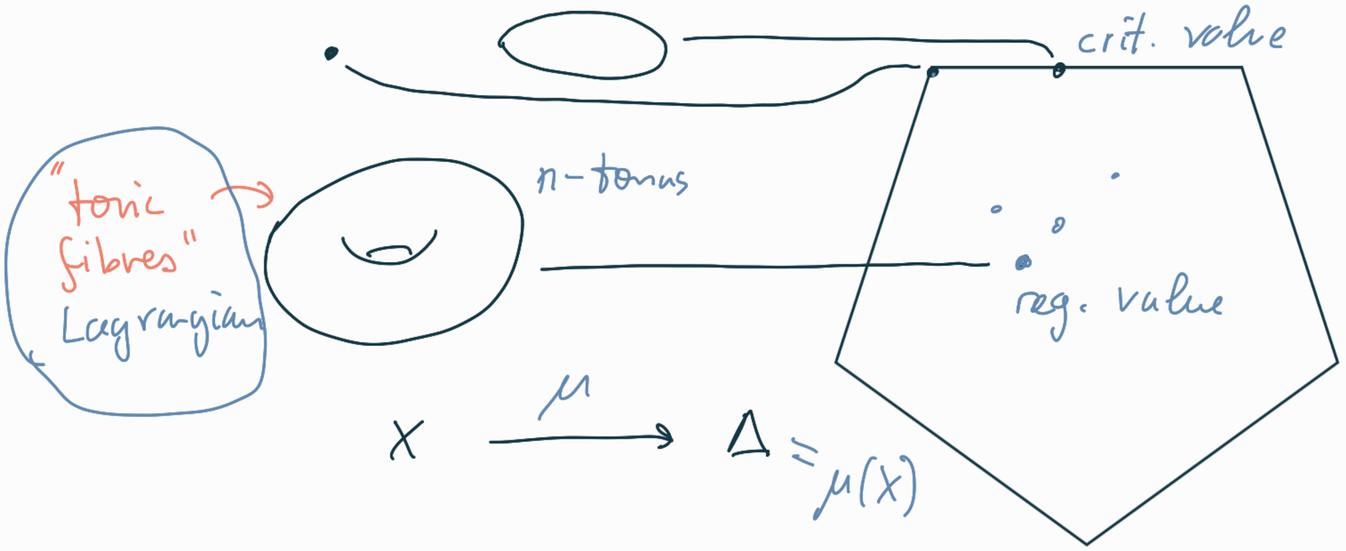
Let $((X^{2n}, \omega), \underbrace{\mu : X \to \mathbb{R}^n}_{\text{moment map}})$ be a toric symplectic manifold, i.e. the Hamiltonian flows of the $\mu_i : X \to \mathbb{R}$ generate an effective T^n-action.

1) generate S^1-actions 2) $\{\mu_i, \mu_j\} = 0$ 3)

Thm: (Delzant '88)

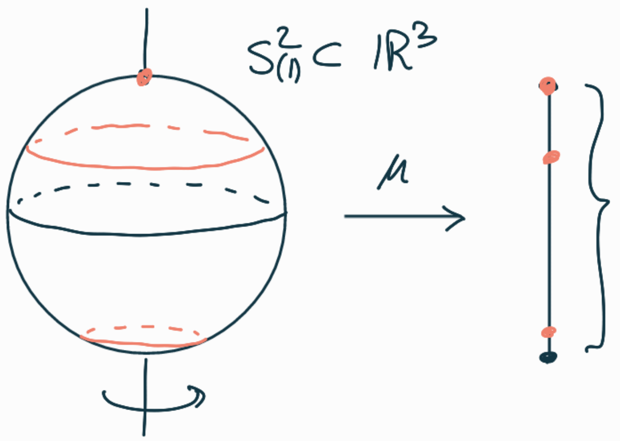
$\Delta := \mu(X)$ is a Delzant polytope which determines $(X, \omega) \circlearrowleft T^n$ up to equiv. symplectomorphisms.

(Singular) fibration structure of μ:



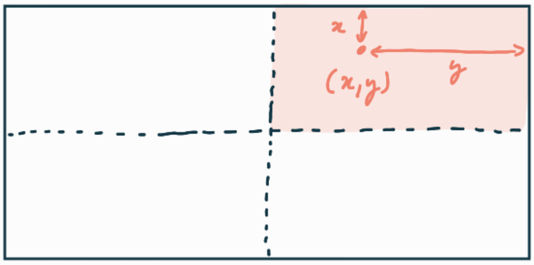
with fibres of μ = orbits of T^n-action.

Example:



$\mu(x,y,z) = 2\pi z$

The discussion in §0, for $S^2 \times S^2$ is a special case of toric geometry:

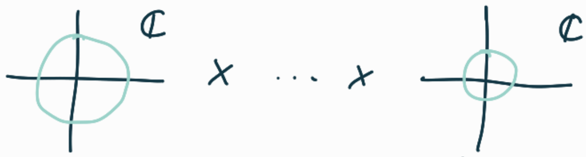
moment polytope = 

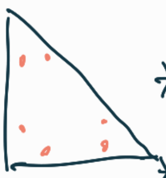
toric fibres = product tori

Main question:

Classify toric fibres up to Hamiltonian diffeomorphisms.
(symplectom.)

Answered for: $\mu(z_1, \dots, z_n) = (\pi |z_1|^2, \dots, \pi |z_n|^2)$

*) $\mathbb{R}^{2n} = \mathbb{C}^n$ by Chekanov ('96)
here toric fibres are 



*) $\mathbb{CP}^2$ by Shelukhin - Tonkonog - Vianna ('19)

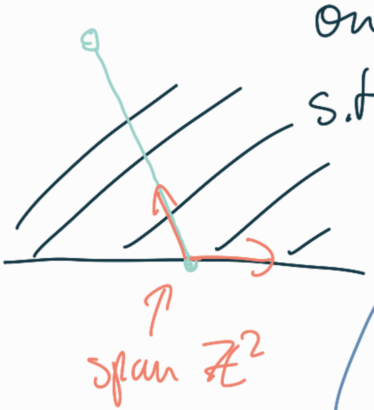
*) Many 4-dimensional examples (B. '23)
Constructions & Obstructions in the general case.
(for dim ≥ 6 still open...)

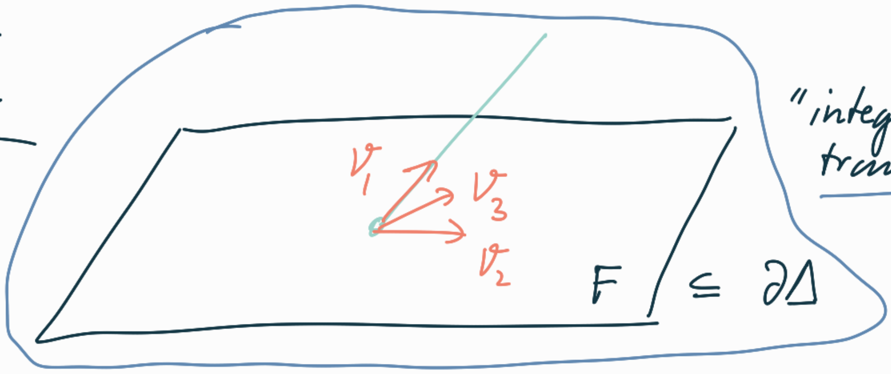
*) $S^2 \times S^2$ (B. - KIM '23)
non-mon.

§2. Symmetric probes

(way to construct equivalences of toric fibres)

<u>Def</u>: (McDuff) A *probe* is a rational segment $\sigma \subset \Delta$ intersecting $\partial\Delta$ in one point contained in a facet F s.th. $\sigma \cup F$ contains a basis of $\mathbb{Z}^n$.



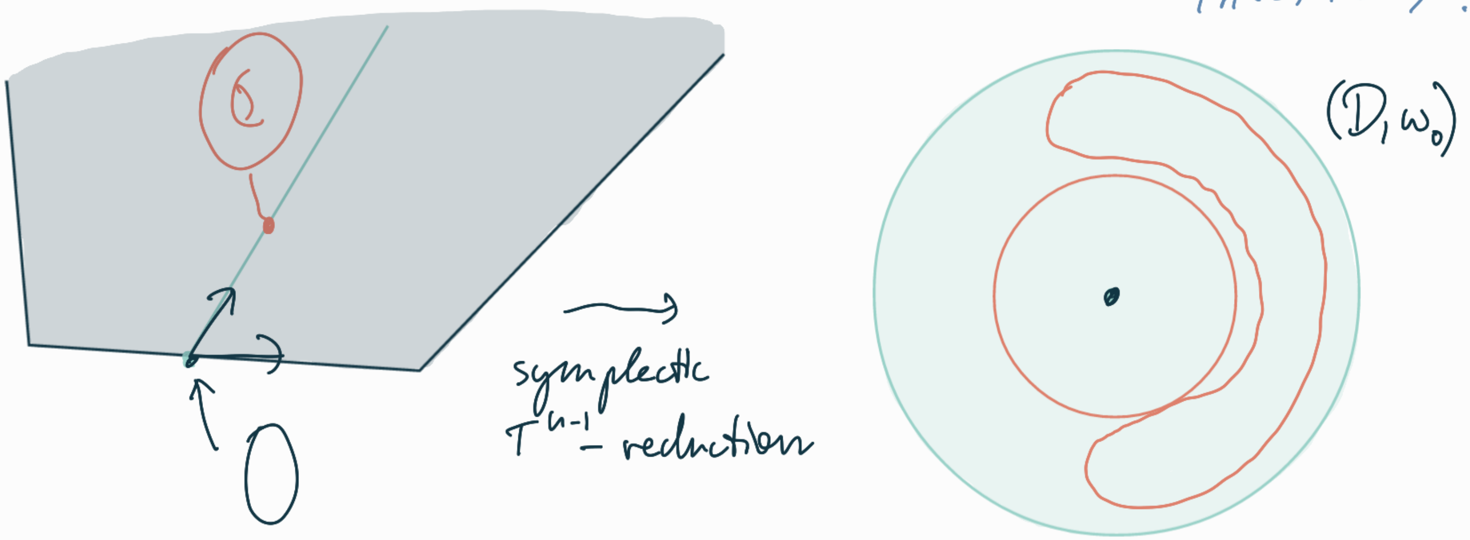


"integral transversality"

$$\mathrm{span}_{\mathbb{Z}}\{v_1, v_2, v_3\} = \mathbb{Z}^3.$$

<u>Application</u>: Displacing toric fibres by Hamiltonian isotopies.

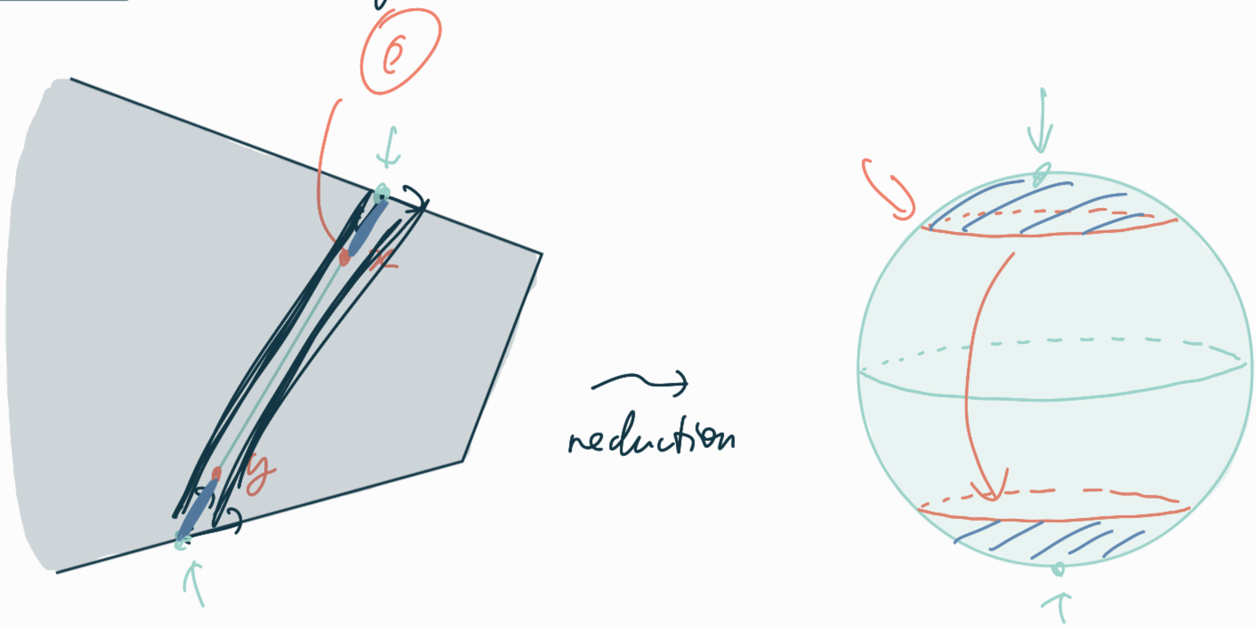
Given L, $\exists ?\ \phi_H$ $\phi_H(L) \cap L = \emptyset$?



<u>Fact</u>: Hamiltonian isotopies in reduced spaces can be "lifted" to Hamiltonian isotopies in the initial space. (<u>Abreu-Macarini</u> '11)

Def: (Abreu - Borman - McDuff "Extended probes")
A symmetric probe is a segment $\sigma \subset \Delta$ satisfying integral transversality at both endpoints.

Application: Finding Hamiltonian equivalent fibres.

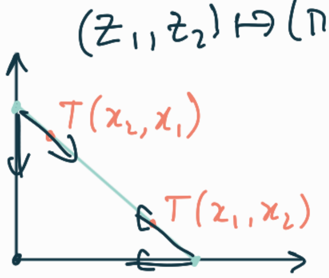


Observation: If $x, y \in \sigma$ in a symmetric probe at equal distance to $\partial\sigma$, then

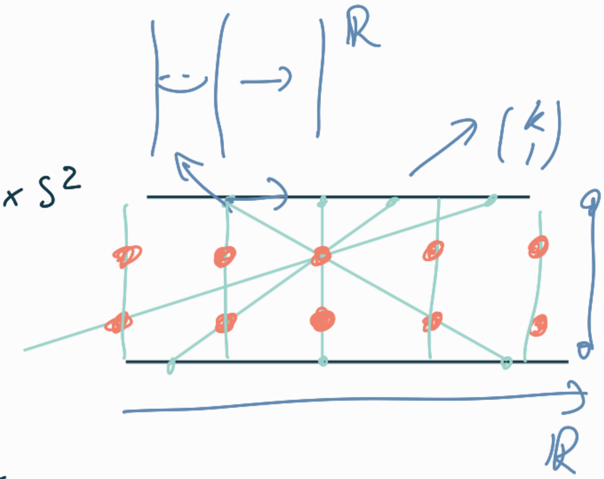
$$T(x) \stackrel{H}{\simeq} T(y)$$

Some examples:

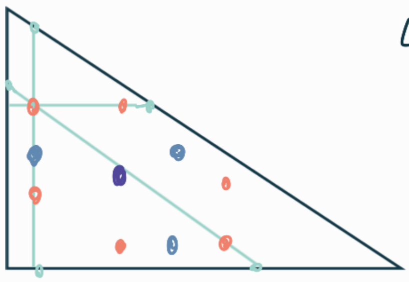
1) $\mathbb{C}^2$ $\quad (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$



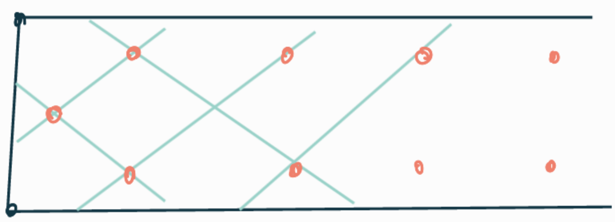
2) $T^*S^1 \times S^2$



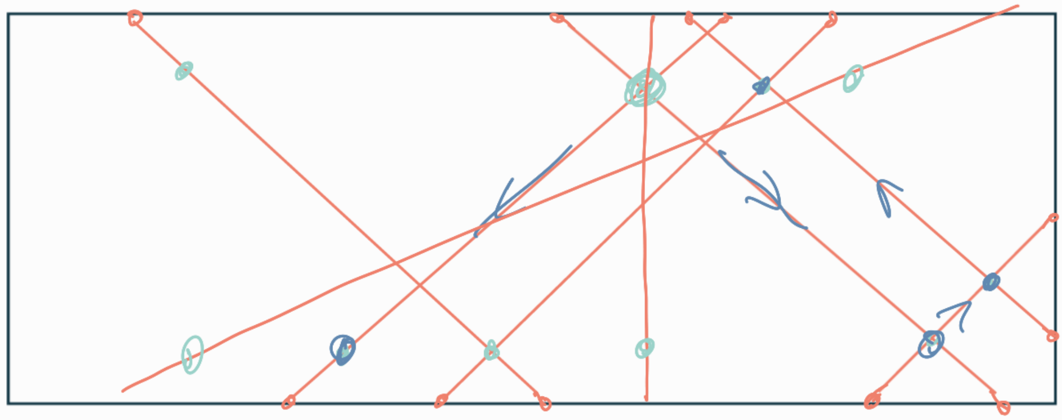
3) $\mathbb{CP}^2$



4) $\mathbb{C} \times S^2$



5) $S^2 \times S^2$: "billiard trajectories" can be realized by symmetric probes:



<u>Rk</u>: In all of the above examples, this is the actual classification.

<u>Conjecture</u>: $T(x) \cong T(x') \iff x$ and x' can be connected by iterated symmetric probes

(optimistic?)

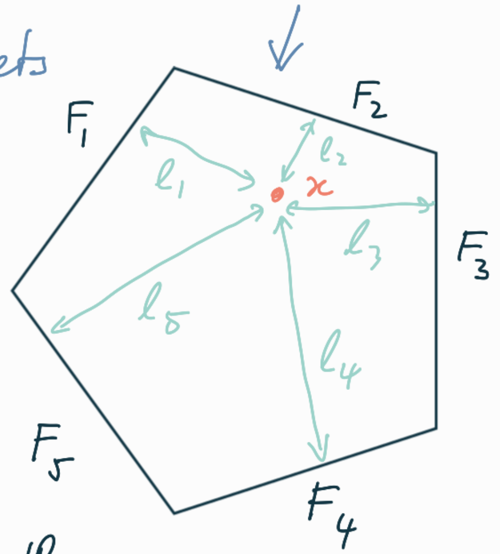
<u>Thm.</u> (B. '23)

This is true for product tori in $\mathbb{C}^n$.

($\hookrightarrow$ Chekanov)

§3. Obstructions.

Some notation: $N = \#\text{facets}$

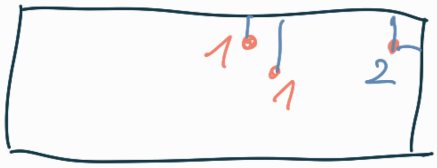


$\ell_i(x)$ = integral affine distance of x to F_i. ($GL(n;\mathbb{Z})$-inv.)

(1) $d(x)$ = integral affine distance of x to $\partial\Delta$
$= \min\{\ell_1(x), \dots, \ell_N(x)\} \in \mathbb{R}$

(2) $\#_d(x) = \#\{i \mid \ell_i(x) = d(x)\} \in \{1, \dots, N\}$

(3) $\Gamma(x) = \mathbb{Z}\langle \ell_1(x) - d(x), \dots, \ell_N(x) - d(x)\rangle \subset \mathbb{R}$



Theorem: (B. '23)

$$T(x) \cong T(x') \implies \left.\begin{aligned} d(x) &= d(x') \\ \#_d(x) &= \#_d(x') \\ \Gamma(x) &= \Gamma(x') \end{aligned}\right\} \text{ "Chekanov invariants"}$$

proof: Relies heavily on Chekanov's classification of product tori in $\mathbb{C}^n$:

Thm: (Chekanov '96) $d, \#_d, \Gamma$ are complete invariants for product tori in $\mathbb{C}^n$.

Fact: ("Delzant construction")

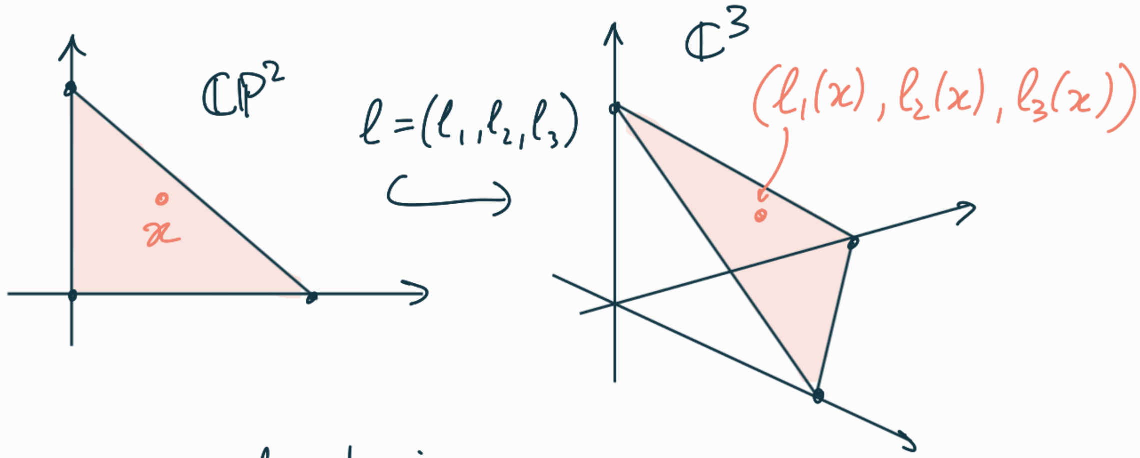
Every compact toric X is a symplectic reduction of some $\mathbb{C}^N$:

$$\begin{array}{ccccc} \text{product torus} & \subset & \mu^{-1}(c) & \hookrightarrow & \mathbb{C}^N \\ \downarrow & & \downarrow & & \\ \text{toric fibre} & \subset & X & & \end{array}$$

If two toric fibres are Ham.-equivalent, then so are the corresponding toric fibres.

$\square$

Example:



corresponds to:

$$\mathbb{CP}^2 \twoheadleftarrow H^{-1}(c) \subseteq \mathbb{C}^3$$

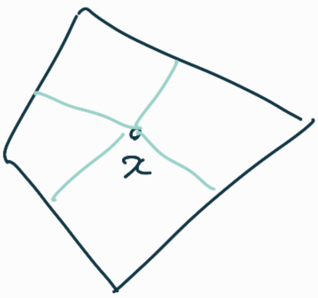
for $H(z_1, z_2, z_3) = \pi \left(|z_1|^2 + |z_2|^2 + |z_3|^2 \right)$

Remark: The Chekanov invariants are not complete for toric fibres, even for e.g. $\mathbb{CP}^2$.

Instead: Suppose $\phi(T(x)) = T(y)$

$\rightsquigarrow$ $\boxed{\phi_* : H_2(X, T(x); \mathbb{Z}) \to H_2(X, T(y); \mathbb{Z})}$

w/ :
1) ϕ_* preserves Maslov & area class
2) ϕ_* acts by permutations on classes of minimal area.



$\rightsquigarrow$ Find which ϕ_* are allowed.

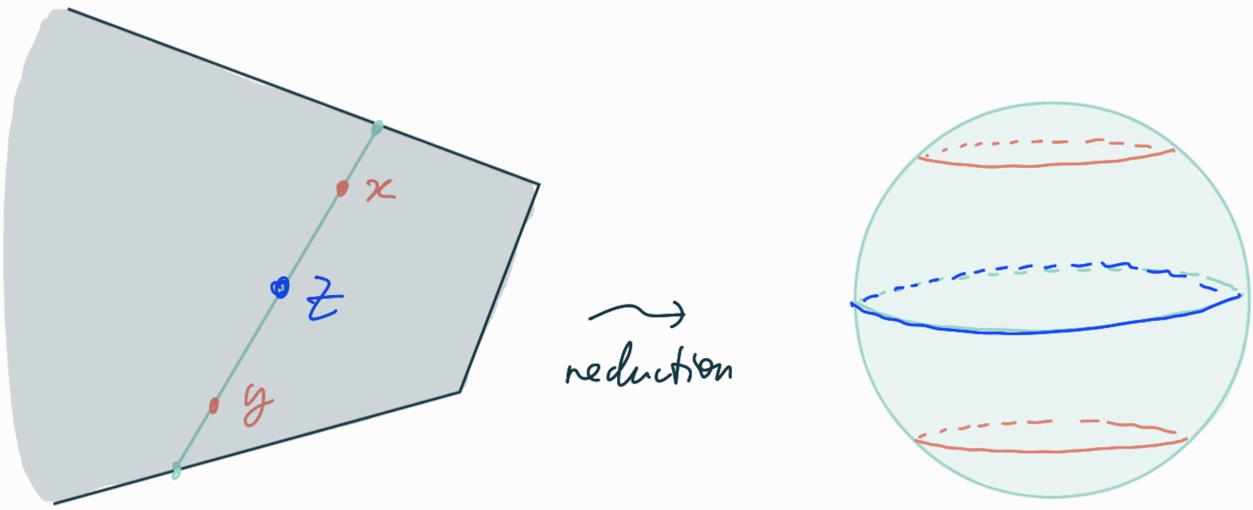
Note: Knowing its area class determines $x \in \Delta$.

Related question: "Hamiltonian monodromy"

$$\mathcal{H}_L = \left\{ (\phi|_L)_* \in \operatorname{Aut} H_1(L) \;\middle|\; \begin{array}{l} \phi \in \mathrm{Ham} \\ \phi(L) = L \end{array} \right\}$$

Studied by: M.-L. Yau, Ono, J. Smith, ...
Hu-Lalonde-Leclerq, Porcelli, ...

Can determine $\mathcal{H}_{T(x)}$ for many (non-monotone!) fibres.



$\Longrightarrow$ $\mathcal{H}_{T(z)}$ is non-trivial.

Thank you!