Frobenius objects in the category of spans and the symplectic category

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TQFT Seminar, IST Lisbon
April 12th, 2023

Goals

Based on joint work with Rajan Mehta and Molly Keller (*Rev. in Math. Phys (34) 10 (2022)*), Mehta, Adele Long and Sophia Marx (https://arxiv.org/abs/2208.14716 to appear in *Contemp. Math.* (2023)), and ongoing work with Mehta and Walker Stern.

Objectives of the Talk

1. Frobenius algebras: 2D TQFT and symplectic geometry
2. A toy (simplicial) model of the Wehrheim-Woodward construction
3. Ongoing: The role of the symplectic groupoid in field theory: the Poisson sigma model
4. 2-Segal (higher categorical) picture

Motivation

Poisson Geometry

* Hamiltonian formulation of mechanics
* Structure of classical observables

Topological Field Theory

* TQFT (Atiyah & Segal '80)
* Poisson Sigma Model (Schaller-Strobl-Ikeda '94) Cattaneo-Felder '00

Symplectic Geometry

* Symplectic groupoids (Weinstein '87, Karasëv '89, Zakrzewski '90)

Symplectic Realization

Symplectic Reduction (Reduced phase space)

Also

1. Correspondence between 2D TQFT and commutative Frobenius algebras
2. An intermediate step in quantization:

$$\mathrm{Cob} \longrightarrow \mathrm{Symp} \longrightarrow \mathrm{Hilb}$$

The symplectic category

Definition

- Objects: Symplectic manifolds (M, ω)
- Morphisms: Lagrangian relations/correspondences
  $L: M \longrightarrow N \quad (L \subseteq M \times \bar{N})$

- Issue: Composition is only partially defined (strong transv)
- Possible solution: Wehrheim-Woodward '07

Wehrheim-Woodward's Construction '07

- Objects: Symplectic Manifolds
- Morphisms: (Formal) sequences of Lagrangian relations / strong transv. compositions

What happens in Set?

Rel

Objects: Sets

Morphisms: Relations $R: X \longrightarrow Y$

$R \subseteq X \times Y$

Span

Objects: Sets

Morphisms: Isomorphism classes of spans

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Composition: Pullback

$$\begin{tikzcd} & & A \times_Y B \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

Theorem (Li-Bland, Weinstein '14)

$$WW(Rel) = Span$$

- Idea: **Span** is a good set-theoretic model for **Symp**.
- Question: Can we study TQFTs with values in Span?

Frobenius Objects in $\mathcal{C}$

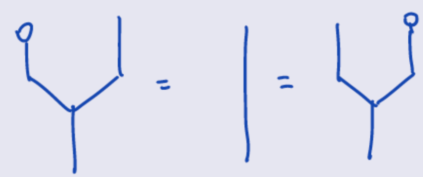
Let $\mathcal{C}$ be a monoidal category.

Definition

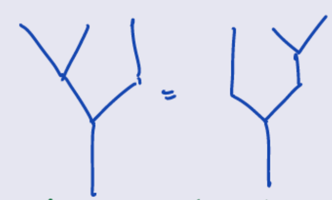
A Frobenius object in $\mathcal{C}$ is an object $X \in Ob(\mathcal{C})$ and the following morphisms:

- $\eta : 1 \longrightarrow X$ (unit)
- $\mu : X \otimes X \longrightarrow X$ (multiplication)
- $\varepsilon : X \longrightarrow 1$ (counit)

s.t

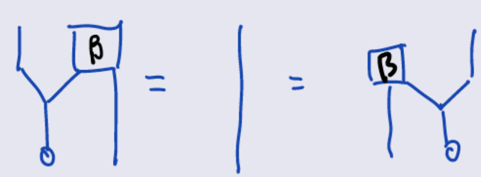
(1) 

(Unitality)

(2) 

(Associativity)

(3) $\exists \beta : 1 \longrightarrow X \otimes X$ s.t
(trace)

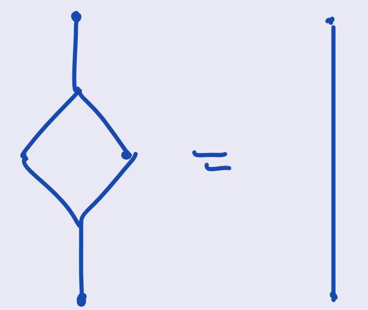


(Non-degeneracy)

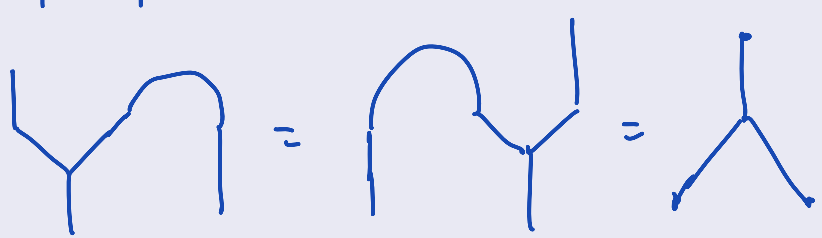
Frobenius Objects II

Definition

A Frobenius object is special if



Some results about Frobenius objects

- β is unique: $\boxed{\beta} :=$ 
- (comultiplication): 
- The natural co-unitality/co-associativity follows.

Classification of Frobenius objects

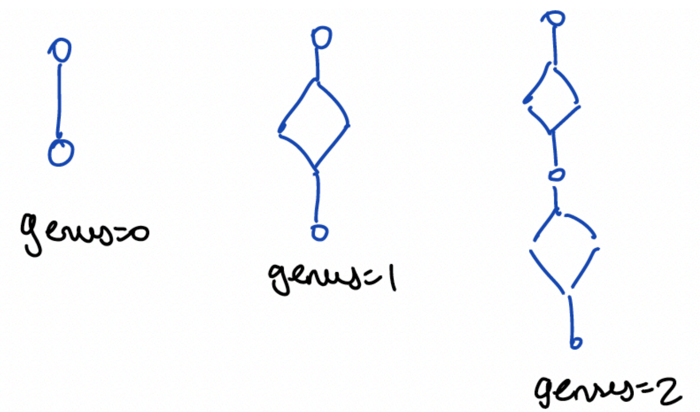
Theorem (Dijkgraaf '89–Abrams '96)

Commutative Frobenius objects in $\mathcal{C} \longleftrightarrow \mathcal{C}$*-valued 2D TQFTs where*
$\mathcal{C}$*-valued 2D TQFTs=symmetric monoidal functors* $2Cob \longrightarrow \mathcal{C}$

Theorem (Cattaneo,C-,Heunen '13)

Special Frobenius objects in Rel $\longrightarrow$ *Groupoid objects in Set*

Here, Rel is considered as a dagger symmetric monoidal category. Also, one can recover topological invariants of surfaces via $Hom_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$



What happens when $\mathcal{C}$=Span?

- Monoidal structure: Cartesian Product
- Monoidal unit: $\{\bullet\}$
- $Hom_{\mathcal{C}}(\{\bullet\},\{\bullet\}) = \{$iso-classes of sets$\}=\{$cardinalities $\}$

Theorem (C-, Keller, Mehta '21)

Frobenius objects in Span $\longleftrightarrow$ *simplicial sets* $X_\bullet$ *with conditions*

Conditions on the simplicial sets I

- (Unitality):

$$\begin{tikzcd}
 & X_0 \arrow[dl] \arrow[dr, "s_0^0"] & & & X_2 \arrow[dl, "(d_2^2, d_0^2)"'] \arrow[dr, "d_1^2"] & \\
\bullet & & X_1 & X_1 \times X_1 & & X_1
\end{tikzcd}$$

$d^\bullet_\bullet$:= face maps , $s^\bullet_\bullet$:= degeneracy maps

Lemma (C-, Keller, Mehta '21)

Let $X_\bullet$ be a 2-truncated simplicial set. The unit axiom holds if and only if for all $\zeta \in X_2$

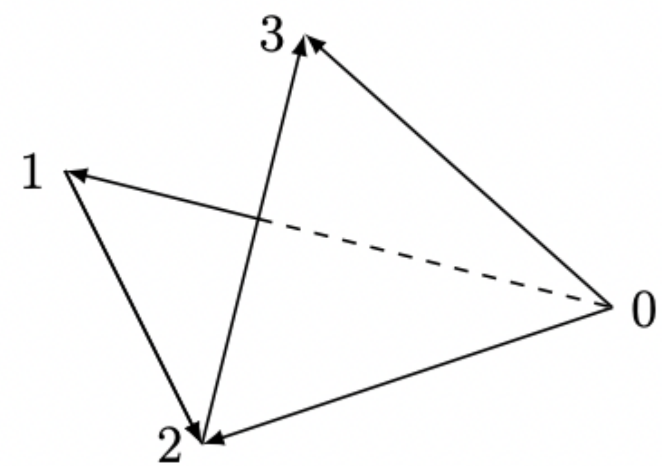
(1) *If $d_2^2\zeta \in im(s_0^0)$, then $\zeta \in im(s_0^1)$*

(2) *If $d_0^2\zeta \in im(s_0^0)$, then $\zeta \in im(s_1^1)$*

Conditions on the simplicial sets II

- (Associativity): We introduced the notion of $(i, j)-$ taco:

$$T_{ij}\mathcal{X} = \{(\zeta, \zeta') \in X_2 \times X_2 | d^2_{j-1}\zeta = d^2_i\zeta'\}.$$



A (13)-taco.

Conditions on the simplicial sets III

Let $S\mathcal{X} = \{(x_{01}, x_{12}, x_{23}, x_{03}) \in (X_1)^4$ such that

$$\begin{aligned} d_0^1 x_{01} &= d_1^1 x_{12}, & d_0^1 x_{12} &= d_1^1 x_{23}, \\ d_0^1 x_{23} &= d_0^1 x_{03}, & d_1^1 x_{03} &= d_1^1 x_{01}\}. \end{aligned}$$

Lemma (C-, Keller, Mehta '21)

Associativity holds if and only if there is a bijection $T_{02}\mathcal{X} \cong T_{13}\mathcal{X}$ that commutes with the boundary maps to $S\mathcal{X}$.

The boundary maps $\partial_{02} : T_{02}\mathcal{X} \to S\mathcal{X}$ and $\partial_{13} : T_{13}\mathcal{X} \to S\mathcal{X}$ are defined by

$$\begin{aligned} \partial_{02}(\zeta, \zeta') &= (d_2^2\zeta', d_2^2\zeta, d_0^2\zeta, d_1^2\zeta'), \\ \partial_{13}(\zeta, \zeta') &= (d_2^2\zeta', d_0^2\zeta', d_0^2\zeta, d_1^2\zeta). \end{aligned}$$

Diagrammatics

Frobenius	Simplicial sets
$\eta : \{\cdot\} \to X$	$\{\cdot\} \leftarrow E \xrightarrow{s_0^0} X$
$\mu : X \times X \to X$	$X \times X \xleftarrow{(d_2^2, d_0^2)} M \xrightarrow{d_1^2} X$
$\varepsilon : X \to \{\cdot\}$	$X \xleftarrow{\hat{\alpha} \circ s_0^0} E \to \{\cdot\}$ for some bijection $X \xrightarrow{(\hat{\alpha})^2} X$ (Nakayama automorph)

Theorem (C-, Keller, Mehta '21)

These maps come from a (truncated) simplicial set structure

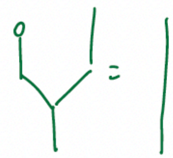
$$
\begin{tikzcd}
M \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \arrow[l, bend right] \arrow[l, bend left] \arrow[loop below, "(\hat{\alpha})^2"] \arrow[r, shift left] \arrow[r, shift right] & E
\end{tikzcd}
$$

These condition are related to the axioms of 2-Segal sets in [Dyckerhoff-Kapranov] and [Galvez-Carrillo-Kock-Tonks]. Stay tuned...

Idea of the proof

Diagram chasing:

Unitality



$\Rightarrow$

$$\begin{tikzcd} X \arrow[r, "\eta\times 1"'] \arrow[rr, bend left, "1"] & X\times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & & (E\times X)\times M \arrow[dl] \arrow[dr] & & \\ & E\times X \arrow[dl, "p_2"'] \arrow[dr, "s_0^0\times 1"] & & M \arrow[dl, "(d_2^2, d_0^2)"'] \arrow[dr, "d_1^2"] & \\ X & & X\times X & & X \end{tikzcd}
\quad \overset{\text{iso}}{\Longleftrightarrow} \quad
\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "1"] & \\ X & & X \end{tikzcd}$$

Find maps $X \xrightarrow{d_1^1} E$, $X \xrightarrow{s_0^1} M$ s.t the following diagram commutes:

$$\begin{tikzcd}[column sep=small] & & X \arrow[dl, "(d_1^1, 1)"'] \arrow[dr, "s_0^1"] \arrow[ddll, bend right, "1"'] \arrow[ddrr, bend left, "1"] & & \\ & E\times X \arrow[dl, "p_2"'] \arrow[dr, "s_0^0\times 1"] & & M \arrow[dl, "(d_2^2, d_0^2)"'] \arrow[dr, "d_1^2"] & \\ X & & X\times X & & X \end{tikzcd}$$

Similarly with the other axioms.

Final output: Simplicial set $M \substack{\leftarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \\ \leftarrow} X \substack{\leftarrow \\ \rightarrow \\ \rightarrow} E$ with a bijection $X \xrightarrow{(\hat{\alpha})^2} X$

Example

Group $G \longrightarrow$ nerve of G:

$$\begin{tikzcd} G \times G \arrow[r, shift left=2, "p_1, p_2, m"] \arrow[r] \arrow[r, shift right=2] & G \arrow[r, shift left] \arrow[r, shift right] & \{\cdot\} \arrow[l, bend right=60, "e"'] \end{tikzcd}$$

(Twisted) co-units:

$$\begin{tikzcd} & \{\cdot\} \arrow[dl, "\omega"'] \arrow[dr] & \\ G & & \{\cdot\} \end{tikzcd}$$

Theorem (C-, Keller, Mehta '22)

If G is finite and abelian, Σ_g is a closed surface with genus g:

$$Z(\Sigma_g) = \begin{cases} |G|^g & \text{if } \omega^g = \omega \\ 0 & \text{otherwise} \end{cases}$$

Work in progress I

(with Rajan Mehta and Walker Stern)

- **Symplectic groupoids** as Frobenius objects in Symp. There is a 2D TFT (Poisson sigma model) that produces a symplectic groupoid via reduced phase space.
- Higher dimensions: 3D TFT (Chern-Simons).

Thank you!