Complete Calabi-Yau metrics asymptotic to cones

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This talk is based on joint work with Junsheng Zhang (UC Berkeley).

We are interested in studying complete (noncompact) Calabi-Yau metrics.

A Calabi-Yau metric is given by the data (X, p, Ω, ω), where X is a complex manifold of dimension n, $p \in X$, Ω is a holomorphic volume form on X, ω is a Kähler metric on X. They satisfy the equation

$$\omega^n = C\Omega \wedge \bar{\Omega}.$$

Calabi-Yau metrics have vanishing Ricci curvature. They play fundamental roles in supersymmetric string theory.

Locally write $\omega = \sqrt{-1}\partial\bar{\partial}\phi$, then the equation becomes a complex Monge-Ampére equation $\det(\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}) = e^F$.

Yau's solution to the Calabi conjecture classifies compact Calabi-Yau metrics in terms of algebro-geometric data.

Complete Calabi-Yau metrics are models for singularity formation: one can simply consider the family $(X, p, \Omega, \lambda\omega)$ as $\lambda \to 0$.

Algebraization Question: Classify complete Calabi-Yau metrics in terms of complex/algebro-geometric data.

This is related to the longstanding conjecture of Yau

Compactification conjecture

Given a complete Calabi-Yau manifold (X, p, Ω, ω), the underlying complex manifold X is biholomorphic to $M \setminus D$, where M is a compact complex manifold and D is a divisor in M.

Remark

- *Many examples of complete Calabi-Yau metrics are constructed out of a pair (M, D). The compactification conjecture is the first step toward the classification problem.*
- *Without extra hypothesis, the conjecture fails even when $n = 2$ (Anderson-Kronheimer-LeBrun 1989).*

Bishop-Gromov inequality implies that $Vol(B(p, R)) \leq CR^{2n}$ for some $C > 0$ and all $R \geq 1$. We say X satisfies Euclidean volume growth assumption (EV) if $Vol(B(p, R)) \geq cR^{2n}$ for some $c > 0$ and all $R \geq 1$.

Assuming (EV):

- when $n = 2$, X is ALE (Cheeger-Naber); These were classified by Kronheimer in the 1980s.
- when $n > 2$, the compactification conjecture is still open in general. By Cheeger-Colding theory we know X is asymptotic to metric cones $C(Y)$ at infinity. The uniqueness of asymptotic cones is not proved in general.

We say X satisfies quadratic curvature decay assumption (QC) if $|Rm| = O(r^{-2})$ as $r \to \infty$.

Assuming (EV) and (QC):

- There is a unique asymptotic Calabi-Yau cone $C(Y)$, and Yau's compactification conjecture holds(Colding-Minicozzi, Donaldson-S., G. Liu).
- There is a dual result for local singularities of Calabi-Yau metrics (i.e., a Calabi-Yau metric on $B(p, 1) \setminus \{p\}$ with $|Rm| = O(r^{-2})$ as $r \to 0$).
- Colding-Minicozzi gives a logarithmic convergence rate.

Tian-Yau construction: Let M be a Fano manifold and D be a smooth divisor such that $K_M^{-1} = \alpha[D]$ for some integer $\alpha > 1$. Denote by N the normal bundle of D. For example, $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, $D =$ diagonal $\mathbb{CP}^1$ with $\alpha = 2$, $N = \mathcal{O}(2)$.

There exists a holomorphic volume form Ω on $X = M \setminus D$ with pole of order α along D. One can ask for a Calabi-Yau metric ω on X.

Observe that D is also a Fano manifold. We assume it admits a Kähler-Einstein metric ω_D, which induces a hermitian metric h on N.

A tubular neighborhood of D can be approximated by a neighborhood of the 0-section $\mathbf{0}$ in N_D. Calabi's ansatz provides a Calabi-Yau cone metric $(\mathcal{C} = N \setminus \mathbf{0}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$, where $\omega_{\mathcal{C}} = \sqrt{-1}\partial\bar{\partial}F(|\xi|_h^2)$ for some function F.

One can graft $\omega_{\mathcal{C}}$ to X and get a Kähler metric ω_X' which satisfies $Ric(\omega_X') = O(r^{-2-\epsilon})$ for some $\epsilon > 0$.

Then one adapts Yau's analysis to this setting to solve a complete Calabi-Yau metric ω_X. By construction it is asymptotic to the cone $\mathcal{C}$ at a polynomial rate.

In the example above, one obtains a Calabi-Yau metric which is isometric to the Calabi-Eguchi-Hanson metric.

Conlon-Hein: There is a generalized Tian-Yau construction which gives rise to all complete Calabi-Yau metrics X asymptotic to a Calabi-Yau cone $\mathcal{C}$ at a polynomial rate.

A curious question is whether there exists a complete Calabi-Yau metric asymptotic to a cone at only a logarithmic rate.

For the analogous question for local singularities, it is indeed possible to find examples where the logarithmic rate is optimal.

Also, it seems possible to construct “ends" of such metrics (so are incomplete), using the strategy of Adams-Simon.

A naive idea of constructing such complete metrics is by extending the Tian-Yau construction to the case when D is not Kähler-Einstein, but is only almost Kähler-Einstein (or K-semistable in terms of algebro-geometric language).

For example, one can consider the case when D admits an equivariant degeneration to a smooth Kähler-Einstein Fano manifold. Then one can construct a Kähler metric ω'_X which is only asymptotic to $\mathcal{C}$ at a logarithmic rate, but then we only have $Ric(\omega'_X) = O(r^{-2}(\log r)^{-\delta})$, which causes difficulties in solving the equation.

Theorem (S.-Zhang)

A complete Calabi-Yau metric satisfying (EV) *and* (QC) *is always asymptotic to the Calabi-Yau cone at a polynomial rate.*

This together with Conlon-Hein yields the classification of such metrics.

We call such a result “no semi-stability at infinity".

2-step degeneration theory [Donaldson-S.]: algebraization of metric asymptotic cones.

There is a similar picture for local tangent cones at singularities.

Differences:

- In the local setting, it is conjectured by Donaldson-S. and confirmed by Li-Wang-Xu that both W and $\mathcal{C}$ are local algebro-geometric invariants of the singularity. There are examples where W is different from $\mathcal{C}$.
- In the asymptotic setting, it is known that neither W nor $\mathcal{C}$ are algebraic invariants of the underlying complex manifold X. Our result however shows that under (QC), W is always isomorphic to $\mathcal{C}$, hence “no semistability". We also conjecture this is true in general.

To prove the result, the strategy is to construct on W a Calabi-Yau metric ω_W such that

(1) it is asymptotic to $\mathcal{C}$ at infinity;

(2) the tangent cone at 0 is also $\mathcal{C}$.

Given this, by Bishop-Gromov monotonicity we know that (W, ω_W) is a metric cone, so is isomorphic to $\mathcal{C}$.

To construct ω_W, we graft the unknown metric ω on X to a Kähler metric ω'_W on W using the 2-step degeneration theory. The resulting metric satisfies that $Ric(\omega'_W) = O(r^{-2-\epsilon})$. This puts us in a good position to solve the equation to get ω_W satisfying (1). That it satisfies (2) follows from the algebraicity of local tangent cones of singularities.

The central technical difficulty lies in the grafting procedure, which involves the application of the L^2 estimates of Hörmander.