Symplectic Torelli groups of rational surfaces

(Jt with Jun Li & Tian-Jun Li)

Def: (Symplectic Torelli groups)

$$G_{X,\omega} := \pi_0 \mathrm{Symp}_h(X,\omega)$$

h ↳ hom. trivial

Interesting and closely related:

* $\ker[\pi_0(\mathrm{Symp}(X,\omega)) \to \pi_0 \mathrm{Diff}(X)]$

= infinite group for many algebraic hypersurfaces.

Q1: (Seidel) Is G_X finitely generated?

Q2: (Donaldson) Is G_X generated by Lagrangian Dehn twists?

What is known:

1) For some K3's, G_X contains a subgroup $PB_m(S^2)$ pure braid group of m strands (Seidel)

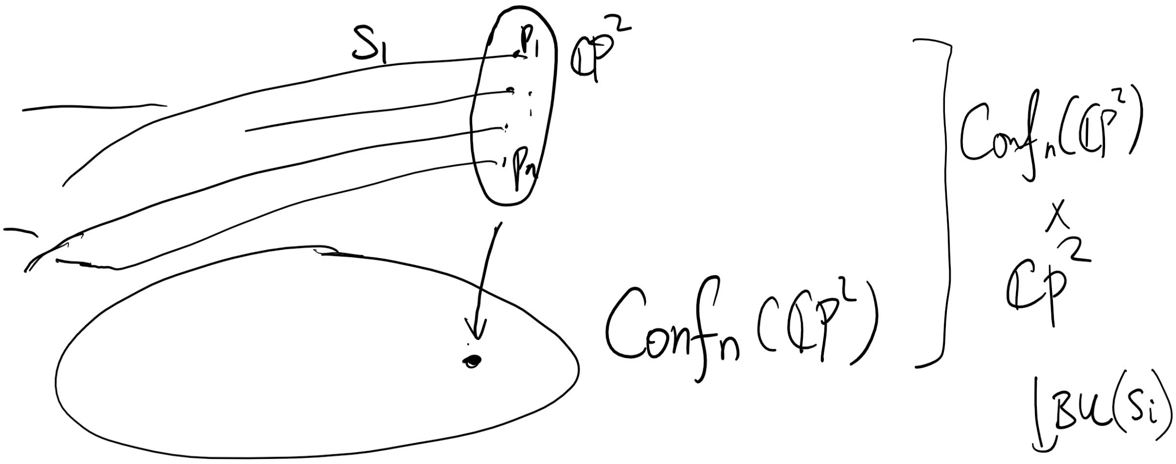
2) (Sheridan-Smith) G_X is not f.g. for some K3's.
(Hacking-Keating) Non-compact examples.

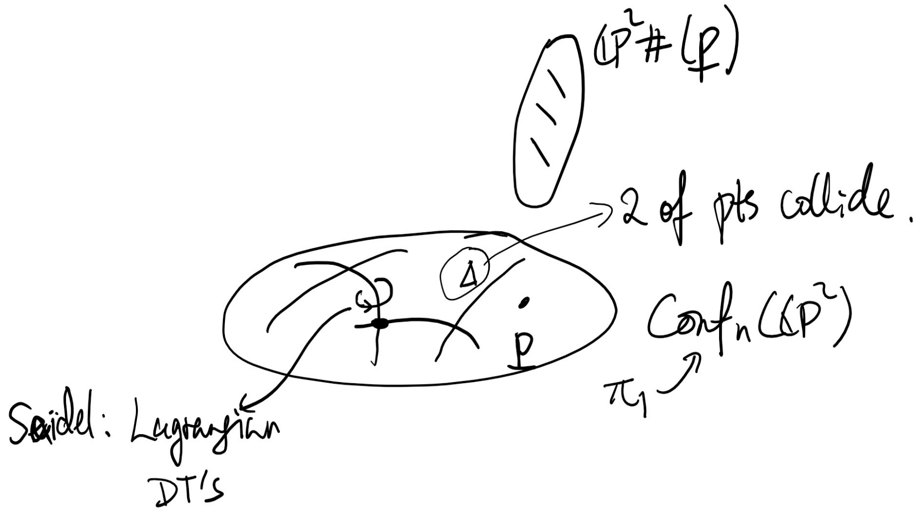
3) Rational surfaces $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, $n \le 5$, both Q1/Q2 have positive answers (Lalonde-McDuff, Abreu, Abreu-McDuff, Evans, Li-Li-Wu) etc.

4) For $\pi_k(\mathrm{Symp}(X))$, only $S^2\times S^2$, $\mathbb{CP}^2 \# n\mathbb{CP}^2$, $n\le 3$ largely known.

Motivation for Donaldson/Seidel's questions

* A partial compactification picture





Donaldson: Are these the only monodromies?

Seidel: Finite generation?

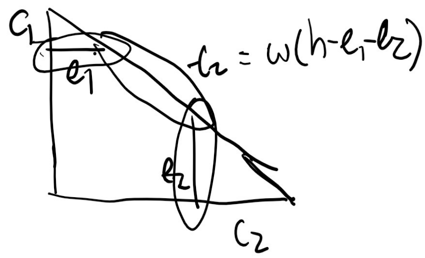
A general approach to $\pi_k(\mathrm{Symp}(X))$

Let $X = \underbrace{D}_{\text{divisor}} \sqcup \underbrace{\mathring{X}}_{\text{complement.}}$

$\underbrace{\mathrm{Symp}(X, D)}_{\substack{\text{in good cases} \\ \text{not far from contractible.}}} \longrightarrow \mathrm{Symp}(X) \longrightarrow \underbrace{\mathcal{D}}_{\text{Space of } D}$

D chosen to be collection of exceptional divisors.

Ex: $(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2, \omega)$



- If $c_1 = c_2 = \frac{1}{3}$, can verify that none of $e_1, e_2, h-e_1-e_2$ can bubble.

$$\mathcal{D} \sim \mathcal{J}_{\omega_{\frac{1}{3},\frac{1}{3}}} \sim *.$$

- $$\mathrm{Symp}(X,D) = \underbrace{\mathrm{Symp}_c(X \setminus D)}_{\overset{?}{*}} \ltimes \underbrace{\mathcal{G}}_{\text{some gauge group}}$$

In contrast If $\{c_1, c_2\} \neq \frac{1}{3}$, e.g. $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{10}$

$$e_1 = (2e_1 - h) + (h - e_1)$$

Upshot: $\omega_{\frac{2}{3},\frac{1}{10}}$ tames different a.c.s

$$\mathcal{D} \sim \boxed{\mathcal{J}^0_{\omega_{\frac{2}{3},\frac{1}{10}}}} \sim ?$$

$$\pi_0, \pi_1 \Rightarrow \pi_0(\mathrm{Symp}(X, \omega_{\frac{2}{3},\frac{1}{10}}))$$

Even worse: $n \geq 10$, Infinite chambers of a.c.s.

Unclear what should $\mathcal{J}^0$ be.

Nagata conjecture: $p_1 \cdots, p_k \in \mathbb{CP}^2$, $k \geq 10$, C irred. ← generic position.

then $\deg C > \frac{1}{\sqrt{k}} \sum_{i=1}^{k} m_i$ ↪ multiplicity at p_i.

$\Rightarrow \nexists$ curve class $ah - \sum_{i=1}^{k} m_i e_i$ if $a < \frac{1}{\sqrt{k}} \sum_{i=1}^{k} m_i$

Statement of results:

* Symplectic Log-CY (X, D) is a pair where $D \subset X$ normal, and $[D] = -K_X$

$\Leftrightarrow$ $\underline{-K_X \cdot [\omega] > 0}$, X is called "positive".

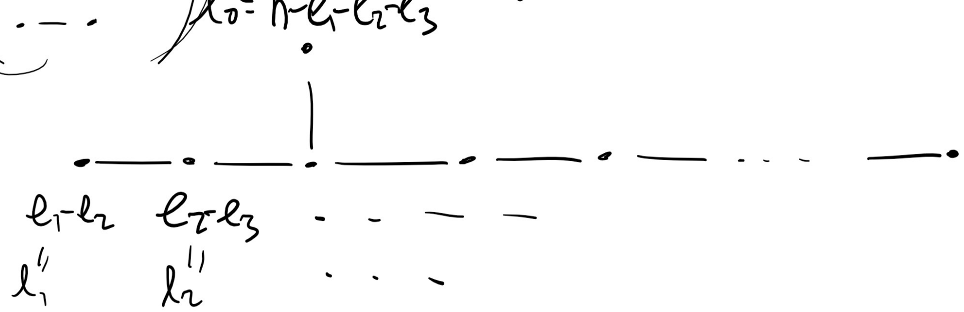
Thm ([Li, Li, W.])

(1) G_X is generated by Lagrangian DT when X positive.

(2) ω can be associated to a root system of (product) ADE-type. when ω = type A: $G_X \simeq Id$

($\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$, when) type D_n: $G_X \sim PB_{n-1}(S^2)$.

root system:



Evaluate $\omega(l_i)$, take the subsystem where $\omega(l_{k_i}) = 0$. $\rightarrow$ rep. by Lag. spheres.

$\Rightarrow$ $\Gamma(X, \omega)$ given by $\{A \in H_2(X, \mathbb{Z}) : A$ rep. by Lag. spheres$\}$

is product of A/D/E root systems.

$\Rightarrow$ types of $\underline{\omega}$.

Corollary: $\exists$ Hamiltonian $\mathbb{Z}/2$-action that cannot extend to S^1-actions.

A very rough sketch of the proof:

Why positivity of X?

Li–Zhang: If $[\omega]$ is positive then

$$[\omega] = \sum_{i \in I} a_i E_i, \quad a_i > 0$$

E_i exceptional.

Inflation (Lalonde–McDuff)

Given ω and ω', relate $\mathrm{Symp}(X,\omega)$ & $\mathrm{Symp}(X,\omega')$

If ω is positive, define $\mathcal{J}^0_\omega = \{$all involved E_i embedded$\}$

$$\mathrm{Conf}_n(\mathbb{CP}^2) \leftarrow \boxed{\mathcal{J}^0_{\omega'}} \longleftrightarrow \mathcal{J}^0_{\omega}$$

(ω′ does not need to be positive).

More refined analysis on $\mathcal{J}^2_\omega$.