VOAs and Twisted Chern-Simons-Matter TQFTs

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Overview

Recall:

$SU(n)_k$ Chern-Simons theory

WRT

holo. BC

$\mathcal{C}$

Quantum Group $\mathcal{U}_q(\mathfrak{sl}(n))$ $\longleftrightarrow$ $\mathfrak{su}(n)_k$ WZW VOA

Kazhdan-Lusztig correspondence

no non-trivial junctions of Wilson lines

$\Updownarrow$

$\mathcal{C}$ semisimple

$\Updownarrow$

$\mathfrak{su}(n)_k$ rational

The TQFTs coming from topological twists are often non-semisimple! Non-semisimple TQFT is harder...

- partition fcns need subtle regularization
- often ∞-dim state spaces
- non-trivial algs of local operators
- logarithmic VOAs

Today:

0) Reminder of $TQFT_3$/VOA correspondence

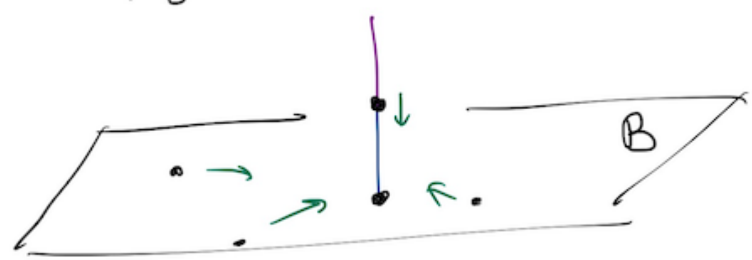
1) Consequences of $TQFT_3$/VOA

2) Examples: Chern-Simons theories based on $\mathfrak{gl}(1|1)$

3d TQFT/VOA correspondence

A choice of BC $\mathcal{B}$ yields a representation of the category of line operators in the 3d TQFT:

$$\rho_{\mathcal{B}} : \mathcal{C} \to \mathrm{Ops}_{\mathcal{B}}\text{-mod} \qquad \mathcal{L} \longmapsto \left\{ \begin{array}{c} \text{local ops at} \\ \text{junction of} \\ \mathcal{L} \text{ \& } \mathcal{B} \end{array} \right\}$$



When $\mathcal{B}$ is a "holomorphic BC", $\mathrm{Ops}_{\mathcal{B}}$ is a VOA!

Optimistically: $\rho_{\mathcal{B}}$ is fully faithful & conservative.

Expectations/dictionary:

line operator $\mathcal{L}$ $\leftrightarrow$ complex of VOA modules $\mathcal{M}$

junction $\mathcal{O} \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}')$ $\leftrightarrow$ derived VOA module map $f \in \mathrm{Hom}(\mathcal{M}, \mathcal{M}')$

state space on Σ $\mathcal{H}(\Sigma)$ $\leftrightarrow$ derived conformal blocks on Σ

Consequences

What is the utility of having a pair of a 3d TQFT $\mathcal{T}$ and a boundary VOA $\mathcal{V}$?

Obvious: gives concrete, algebraic description of $\mathcal{T}$; can offer insight into physics beyond perturbation theory.

e.g. boundary VOAs allow us to <u>solve</u> Chern-Simons theory for compact groups.

Also obvious: symmetries of $\mathcal{T}$ can provide structure to the representation theory of $\mathcal{V}$.

e.g. higher form/group symmetries will act on $\mathcal{V}$-mod

More subtle: if $\mathcal{T}$ admits multiple boundary VOAs $\mathcal{V}_1, \ldots, \mathcal{V}_n$ we get a (hopefully provable) equivalence of categories $\mathcal{V}_1\text{-mod} \simeq \cdots \simeq \mathcal{V}_n\text{-mod}$

e.g. Neuman & Dirichlet BCs in Chern-Simons $\Rightarrow$ level-rank duality

Chern-Simons for $\mathfrak{gl}(1|1)$

$\mathfrak{gl}(1|1)$: N, E bosonic $\psi_\pm$ fermionic

$[N, \psi_\pm] = \pm \psi_\pm \qquad \{\psi_+, \psi_-\} = E$

Standard bilinear form:

$(N,E) = (E,N) = 1 = (\psi_-, \psi_+) = -(\psi_+, \psi_-)$

$$S = \frac{1}{2\pi}\int A_E\, d A_N + Z^-(d Z^+ + A_N Z^+)$$

Discrete data: choice of global form for bosonic gauge group $\Longleftrightarrow$ cocharacter lattice $\Lambda \subset \mathbb{C}^2$

Requirements:
1) Λ integral for $(-,-)$
2) weight of Z^+, Z^- belong to dual lattice

Rank 1: $\Lambda = \langle \nu N + \frac{\xi}{\nu} E \rangle \qquad \nu \in \mathbb{Z}_{>0},\ \xi \in \frac{1}{2}\mathbb{Z}$

$\xi = 0$: "B-twist of 3d $\mathcal{N}=4$ $U(1)$ gauge theory w/ hyper of charge ν"

Rank 2: $\Lambda = \langle \overset{v_1}{\nu N + \frac{\xi}{\nu} E}, \overset{v_2}{\frac{K}{\nu} E} \rangle \qquad K, \nu \in \mathbb{Z}_{>0},\ \xi \in \frac{1}{2}\mathbb{Z}/K\mathbb{Z}$

$\nu = 1,\ K = 2\xi = k$: "$U(1|1)_k$ Chern-Simons theory"

Boundary VOAs

Perturbative answer: $V(\mathfrak{gl}(1|1))$

$$N\,N \sim \frac{1}{(z-w)^2} \qquad N\,E \sim \frac{1}{(z-w)^2}$$
$$N\,\psi_\pm \sim \frac{\pm 1}{z-w}\psi_\pm \qquad \psi_+\psi_- \sim \frac{1}{(z-w)^2} + \frac{1}{z-w}E$$

Indep. of discrete data

Non-perturbative corrections: boundary monopoles

For these $\mathfrak{gl}(1|1)$ theories, expect full boundary VOA to be realized as

$$\mathcal{V} = \bigoplus_{\lambda\in\Lambda} \sigma_\lambda\Big(V(\mathfrak{gl}(1|1))\Big)$$

∞-order simple current extension of $V(\mathfrak{gl}(1|1))$

Where σ_λ is the spectral flow automorphism assoc. to the cocharacter λ. For $\Lambda_{(k,\nu,\xi)} = \langle \nu N + \frac{\xi}{\nu}E, \frac{k}{\nu}E\rangle$

$$\sigma_1 : N \mapsto N - \frac{\xi/\nu + \nu}{z}, \quad E \mapsto E - \frac{\nu}{z}, \quad \psi_\pm \mapsto z^{\mp\nu}\psi_\pm \qquad \lambda = v_1$$

$$\sigma_2 : N \mapsto N - \frac{k/\nu}{z}, \quad E \mapsto E, \quad \psi_\pm \mapsto \psi_\pm \qquad \lambda = v_2$$

For rank 1 case, only need to use σ_1.

Boundary VOAs cont.

The rank 1 examples were studied by Creutzig-Ridout,

$$\Lambda = \langle \nu N + \tfrac{\varepsilon}{\nu} E \rangle \rightsquigarrow \mathcal{W}_{\frac{\varepsilon}{\nu}+\frac{1-\nu}{2},\nu}$$

Thm: (Creutzig-Ridout)

$$\mathcal{W}_{0,1} \simeq \underline{\beta\gamma} \otimes bc \quad \text{cf. A-twist of free hyper}$$

This is a boundary VOA for the B-twist of $\mathcal{N}=4$ $U(1)$ gauge theory with a charge 1 hyper!

The rank 2 VOAs can be thought of as simple current extensions of these VOAs. Some examples of these VOAs admit descriptions in terms of familiar VOAs:

Thm: (G-Niu)

$$V_{(k,1,\frac{k}{2})} \simeq \left(\begin{array}{c}\text{symplectic}\\ \text{fermions}\end{array} \otimes \begin{array}{c}\text{rank 2}\\ \text{selfdual}\\ \text{Lorentzian}\\ \text{lattice VOA}\end{array} \right)^{\mathbb{Z}_k}$$

Category of Line Operators

What about line operators? Our proposal for these theories:

$$KL := \begin{matrix}\text{finite-length, grading-restricted} \\ \text{generalized } V(\mathfrak{gl}(1|1)) \text{ modules}\end{matrix} \rightsquigarrow \mathcal{C} = \mathrm{Rep}^{\circ}_{KL} \mathcal{V}$$

"generalized $\mathcal{V}$-modules that belong to KL"

Creutzig-Kanade-McRae & Creutzig-McRae-Yang:

If $\mathcal{V} = \bigoplus_{\gamma \in \Gamma} \mathcal{V}_\gamma$ is a simple current extension of $\mathcal{V}_0$,

and $\mathcal{D} = \mathcal{V}_0$-mod, then

restrict to modules that have trivial monodromy w/ $\gamma \in \Gamma$

$$\mathrm{Rep}^{\circ}_{\mathcal{D}} \mathcal{V} \simeq \mathcal{D}^{\circ}/\Gamma$$

identify modules that differ by the action of Γ

Moreover, $\mathrm{Rep}^{\circ}_{\mathcal{D}} \mathcal{V}$ inherits a BTC structure from $\mathcal{D}$.

Prop: (G-Niu)

$M \in KL$ has trivial monodromy w/ $\mathcal{V}_{(\kappa,\nu,\xi)}$ iff $\nu N_0 + \frac{\xi}{\nu} E_0$ & $\frac{\kappa}{\nu} E_0$ act semisimply w/ integer e-values.

Thm: (G-Niu)

$$\mathcal{C}_{(\kappa,\nu,\xi)} \simeq KL^{\nu N + \frac{\xi}{\nu} E,\, \frac{\kappa}{\nu} E} \Big/ \Lambda_{(\kappa,\nu,\xi)}$$

Connections to Q-groups

Using the above description of $\mathcal{C}_{(\kappa,\nu,\xi)}$, we can show an explicit equivalence of abelian categories:

Thm: (G-Nin)

$$\mathcal{C}_{(\kappa,\nu,\xi)} \simeq A_{(\kappa,\nu,\xi)}\text{-mod}$$

Where $A_{(\kappa,\nu,\xi)}$ is generated by $K_1, K_2, \Psi_\pm$ subject to

$$K_1^\kappa = K_2^{2\xi} \qquad K_2^\kappa = 1$$

$$K_1 \Psi_\pm = q^{\pm 2\nu} \Psi_\pm K_1 \qquad \{\Psi_+, \Psi_-\} = K_2^\nu - 1$$

for $q = \exp\left(\frac{i\pi}{\kappa}\right)$.

Better: use the q-group analog of a simple current extension due to Creutzig-Rupert, dubbed "uprolling." Roughly, we consider a module $\mathcal{M}_{(\kappa,\nu,\xi)} = \bigoplus_{\lambda \in \Lambda} A_\lambda$; this module is a commutative superalgebra object in $\overline{U}^E(\mathfrak{gl}(1|1))$. Uprolling is a way to describe modules for $\mathcal{M}_{(\kappa,\nu,\xi)}$ in $\overline{U}^E(\mathfrak{gl}(1|1))$-mod in terms of modules for a subquotient of $\overline{U}^E(\mathfrak{gl}(1|1))$, namely $A_{(\kappa,\nu,\xi)}$.

Connections to Q-groups cont.

Note:

The unrolling process involves:

1) restricting to modules w/ trivial monodromy ⇝ quotient
2) identifying modules that differ by fusion w/ A_λ ⇝ sub

Moreover, the BTC structure on $\overline{U}^E(\mathfrak{gl}(1|1))$ induces a unique BTC structure on modules for the unrolled q-group.

Remark:

In order to turn this into a 3d TQFT, one often needs more structure than the BTC structure on $\mathcal{A}_{(\kappa,\nu,\xi)}$-mod. For example, the Costantino-Geer-Patureau-Mirand TQFT works with $\overline{U}^E(\mathfrak{gl}(1|1))$-mod; the modules $\{A_\lambda\}$ used in unrolling go into the choice of free realization necessary to define the TQFT.

Odds & Ends

- Deforming VOA by background flat connections
  ⇝ extend to non-local modules "non-genuine line operators"
  cf. central quotients of $\mathfrak{g}$-groups

- Different choices of Λ can be related by gauging 0-form & 1-form symmetries
  orbifold   S.C.E.

- State spaces & line operators from geometric quantization?

- The $\mathfrak{gl}(1|1)$ C.S. theories described here can be identified with B-twisted 3d $\mathcal{N}=4$ theories. What do their A-twists look like? What about more general 3d $\mathcal{N}=4$ (abelian) Chern-Simons-matter theories?

Thank you!