Heaviness and SH-visibility

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Hamiltonian flows

- Consider the unit sphere $S^2 \subset \mathbb{R}^3$ with its area form
$$dA = \sin\theta d\theta \wedge d\phi.$$
Total area 4π.
- A Hamiltonian isotopy on S^2 simply means an area preserving isotopy. Example: rotations along an axis.
- A Hamiltonian isotopy on $S^2 \times S^2$ means an isotopy for which the symplectic area (the sum of signed areas of two projections) of any embedded disk is preserved at all times.
- When I say symplectic manifold you can instead think about these examples.

Displaceability

- Let (M, ω) be a closed symplectic manifold.
- $K \subset M$ is called **displaceable** if at the end of some Hamiltonian isotopy ϕ we have $\phi(K) \cap K = \emptyset$
- Consider an embedded connected graph $G \subset S^2$ with finitely many smooth edges.
- When is G displaceable?
- Without the area preservation requirement, always.
- The complement of G is a disjoint union of finitely many open disks with nice exhaustions.
- Displaceability is equivalent to one of these disks having area more than 2π.
- We could prove this by hand now!

Displaceability in higher dimensions

- When is $G \times G \subset S^2 \times S^2$ displaceable?
- This seems impossible to decide by hand. Note that the product form of the subset need not persist under the isotopy.
- Lagrangian Floer theory is only of limited use because these sets are not immersed submanifolds.
- For general compact subsets of M, Entov-Polterovich introduced the symplectomorphism invariant notion of **heaviness**.
- The definition relies on Hamiltonian Floer theory (rephrased definition soon).
- **Theorem** (Entov-Polterovich 2009): If $K \subset M$ is heavy, then it is non-displaceable.
- It turns out that $G \subset S^2$ is non-displaceable if and only if it is heavy, but this is not always true.

Symplectic cohomology of compact subsets

- Let us define the Novikov field:

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lambda_i < \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = +\infty \right\}$$

- In my thesis I constructed a presheaf on compact subsets of M,

$$K \mapsto SH^*_M(K;\Lambda),$$

  of $\mathbb{Z}/2$-graded Λ-vector spaces, also using Hamiltonian Floer theory.
- **Theorem** (V. 2018): If $K \subset M$ is SH-**visible**, that is $SH_M(K;\Lambda) \neq 0$, then it is non-displaceable.
- **Conjecture** (Tonkonog-V. 2019): SH-visibility is equivalent to heaviness.
- **Theorem** (Mak-Sun-V. 2022): True.

A consequence of Mayer-Vietoris property for SH_M

- **Theorem** (V. 2018): Assume that we have an involutive system $f : M \to \mathbb{R}^N$ and a finite cover by compact subsets P_i of its image. Then, at least one of $f^{-1}(P_i)$ is SH-visible.
- As a consequence of our recent Theorem one can replace SH-visible with heavy.
- Using also the approximation properties of heaviness, in fact we can deduce that f has to have a heavy/SH-visible fiber.
- With these properties, it is an elementary exercise to prove that $G \times G \subset S^2 \times S^2$ is displaceable if and only if $G \subset S^2$ is displaceable (originally due to Entov-Polterovich).
- If direction is trivial. For only if, construct an appropriate involutive system.

Banach Λ-dga's

- Λ has a non-archimedean valuation.
- $(C, d, \cdot)$ is a $\mathbb{Z}/2$-graded Λ-chain complex with a degree 0 bilinear operation satisfying the super-Leibniz rule.
- C is equipped with a non-archimedean Λ-valuation (i.e. e^{-val} is a Λ-norm satisfying ultrametric inequality) and it is complete with respect to this norm.
- d and $\cdot$ are valuation non-decreasing (continuous enough but for our purposes this simplification is ok).
- (C, d) alone (no product) is called a Banach Λ-chain complex.
- $H^*(C)$ has an induced semi-valuation (i.e. semi-norm) and a product structure. The semi-valuation is by taking the supremum of the valuations of all representatives.

Hamiltonian Floer theory

- Given non-degenerate Hamiltonian $H : M \times S^1 \to \mathbb{R}$ and choices, we obtain a chain complex $CF^*(H;\Lambda)$ over Λ:
  1. vector space over Λ generated by the (finitely many) 1-periodic orbits of X_H (mod 2 degree by the Lefschetz sign)
  2. self-map d by counting Floer solutions with weights $T^{topE(u)}$
- Valuation coming from declaring the defining basis orthonormal make it into a Banach Λ-chain complex.
- Different choices lead to valuation preserving homotopy equivalent complexes.
- Can define chain maps $CF^*(H,\Lambda) \to CF^*(H',\Lambda)$ with a lower bound on the operator valuation given by

$$\int_{S^1} \min_x (H'(t,x) - H(t,x))dt.$$

- These continuation maps give canonical isomorphisms on homology

PSS maps and spectral invariants

- There are also canonical PSS maps

$$C^*(M;\mathbb{Z})\otimes\Lambda\to CF^*(H;\Lambda)$$

- These are chain maps with operator valuation bounded below by $\int_{S^1}\min_x H(t,x)dt$
- Spectral invariant $c(H)$ is defined as the valuation of $PSS(1)$, where 1 is the unit in $H^*(M;\Lambda)$. Note that the definition involves a max-min process.
- The definition can be extended
  1. to any continuous $H : M\times S^1\to\mathbb{R}$ by C^0-approximations (commonly used)
  2. to any lower semi-continuous $H : M\to\mathbb{R}\cup\{+\infty\}$ by monotone approximations (less used, upper ok too).

Heaviness

- The characteristic function of a compact $K \subset M$:

$$\chi_K(x) = \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{otherwise} \end{cases}$$

  is lower semi-continuous.
- It is easy to see that $c(\chi_K) \geq 0$, which is all we need below. Using product structures, we can prove more.
- **Proposition**: $c(\chi_K)$ is either 0 or ∞.
- **Definition**: K is called heavy, if it is 0.
- The going up of the functions outside of K is trying to increase the spectral invariant but K (being heavy) keeps it down.

Product structure on symplectic cohomology

- One should think of $SH^*_M(K;\Lambda)$ as $HF^*(\chi_K;\Lambda)$. The definition also involves a monotone approximation and an appropriate direct limit in the category of Banach Λ-chain complexes.
- Because $\chi_K + \chi_K = \chi_K$, we expect a pair-of-pants type product structure on $SH^*_M(K;\Lambda)$.
- **Theorem** (Tonkonog-V. 2019, Abouzaid-Groman-V. 2022): There exists a Banach Λ-dga $\mathcal{A}_K$ and a chain map $PSS_K : C^*(M;\mathbb{Z}) \otimes \Lambda \to \mathcal{A}_K$ such that
  1. $\mathcal{A}_K$ is a chain level model for $SH^*_M(K;\Lambda)$.
  2. $val_{\mathcal{A}_K}(PSS_K(a)) = c(a;\chi_K)$.
  3. $PSS_K(1)$ is a unit.

The proof of equivalence

- If K is not heavy, we get that $PSS_K(1)$ has a positive valuation representative a in $\mathcal{A}_K$.
- We construct a primitive of a using a telescoping trick, since $a \cdot (a \cdot (\ldots \cdot (a \cdot a) \ldots)$ all represent $PSS_K(1)$ and have valuations going to infinity. This only uses that $PSS_K(1)$ is an idempotent.
- Hence, $PSS_K(1) = 0$ and by unitality we get that $SH^*_M(K; \Lambda) = 0$.
- Conversely, if $PSS_K(1)$ is zero, then $c(1; \chi_K) = \infty$.
- Thank you for listening!