

On the birational geometry of Fano threefold complete intersections

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Overview

- 1 What is it all about?
- 2 MMP in dimension 2
- 3 MMP in dimension $n \geq 3$
- 4 Results

Algebraic Varieties

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Let $R = K[x_1, \dots, x_{n+1}]$ where K is a field and $I \subseteq R$ an ideal.

$$X_I = \{ \mathbf{a} = (a_1, \dots, a_{n+1}) \in K^{n+1} \mid f(\mathbf{a}) = 0, \forall f \in I \}$$

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Theorem (Hilbert)

K is a Noetherian ring $\Rightarrow K[x_1, \dots, x_{n+1}]$ is a Noetherian ring.

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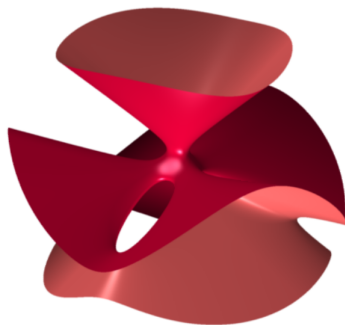
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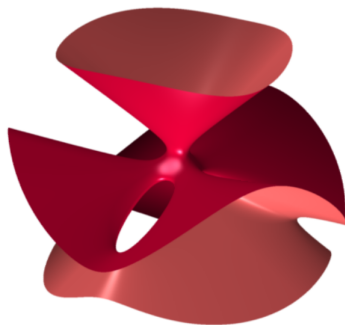
To ease notation we usually write it as $X: (f_1 = \dots = f_s = 0)$.

Example: Clebsch Cubic



$$S_{\text{Clebsch}} : (x^3 + y^3 + z^3 + t^3 + w^3 = x + y + z + t + w = 0) \quad \mathbb{P}^4$$

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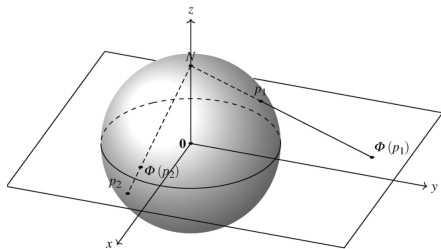
Classify Algebraic Varieties up to Birational equivalence.

Definition

We say that X and Y are **birationally equivalent** or **birational** if there is an isomorphism between open dense sets of X and Y . We write it as $X \dashrightarrow Y$.

Any birational map between *smooth projective curves* extends to a morphism.

Example

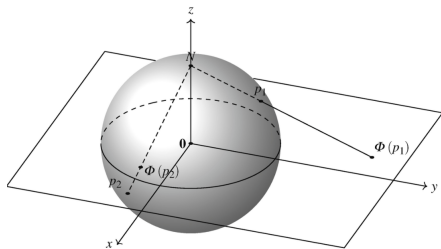


The unit sphere

$$S^n: (x_1^2 + \dots + x_{n+1}^2 = 1) \subset \mathbb{R}^{n+1}$$

projects from the north pole $\mathbf{N} = (0; \dots; 1)$ to the plane $x_{n+1} = 0$, where we use coordinates $y_1; \dots; y_n$.

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We have,

$$\Phi(x_1; \dots; x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right)$$

and

$$\Phi^{-1}(y_1; \dots; y_n) = \left(\frac{2y_1}{1 + S}, \dots, \frac{2y_n}{1 + S}, \frac{1 + S}{1 + S} \right)$$

where $S = \sum_{i=1}^n y_i^2$. We write,

$$S^n \cong \mathbb{R}^n \cup \{\infty\}$$

The Canonical Divisor

Definition

Recall that for a smooth variety X of dimension n , the **canonical bundle** is the line bundle $\omega_X = \Omega_X^n$, that is, the n th exterior power of the cotangent bundle on X . A **canonical divisor** is any divisor D for which $\omega_X = \mathcal{O}_X(D)$. We denote it by K_X .

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Example

Let $X = \mathbb{P}^1 = \mathbb{C}_z [f] / g$. Let $\omega = dz$. At 1 the local coordinate changes to $w = 1-z$ and $\omega = d(1-w) = -dw$. Then ω has a pole of order 2 at 1 . We write it as $K_X = -2 [f] / g$

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Definition

Let X be a normal projective variety with *good singularities*. We say that X is

- **Fano** if K_X is ample;
- **Calabi-Yau** if K_X is trivial;
- **Canonically Polarised** if K_X is ample.

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Example

Let $X = \mathbb{P}^d$. Then $K_X = -(d+1)H$, where $H \subset \mathbb{P}^d$ is a hyperplane.

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Example

Let $C \subset \mathbb{P}^2$ be a smooth projective curve. Then,

$$K_C = (K_{\mathbb{P}^2} + C)|_C = (-3L + dL)|_C = (d-3)L|_C:$$

Taking degrees,

$$\deg(K_C) = 2g(C) - 2 = -3L \cdot C + C^2 = -3d + d^2:$$

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C is Fano	$(-)$	$g(C) = 0$	$(-)$	$d < 3$
C is Calabi-Yau	(0)	$g(C) = 1$	$(-)$	$d = 3$
C is Canonically Polarised	$(+)$	$g(C) \geq 2$	$(-)$	$d > 3$

The Canonical Divisor

Example

Let $X := X_{d_1, \dots, d_n} \subset \mathbb{P}^d$ be a smooth complete intersection of multidegree (d_1, \dots, d_n) . Then, $K_X = (d - 1 + \sum d_i)H|_X$, where H is a generic hyperplane section of \mathbb{P}^d not containing X and

X is Fano	$(-)$	$d + 1$	$\sum d_i > 0$
X is Calabi-Yau	(0)	$d + 1$	$\sum d_i = 0$
X is Canonically Polarised	$(+)$	$d + 1$	$\sum d_i < 0$

The Three Mosqueteers

Let W be a smooth projective variety. The goal of the Minimal Model Program (MMP) is to find a good representative of the birational class of W .

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Each smooth projective variety is birational to a projective variety with *good singularities* Y such that either

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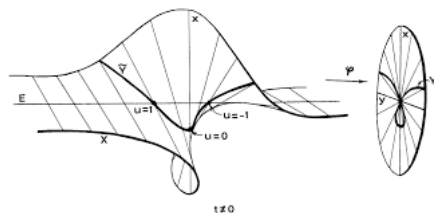
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Theorem (Birkar-Cascini-Hacon-McKernan, '10)

Let W be a smooth projective variety which is uniruled. Then W is birational to a Fano fibration.

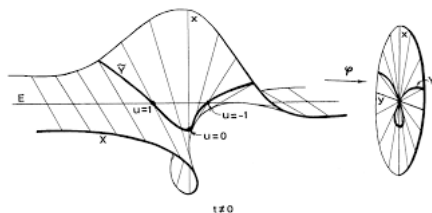
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$$\begin{aligned} \varphi : \mathbb{P}^1 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2; \quad E := \varphi^{-1}(0) \\ \mathbb{P}^1 \times \mathbb{C}^2 \cong \mathbb{C}^2 \cup E \cong \mathbb{C}^2 \cup \mathbb{P}^1 \\ E \cong \mathbb{P}^1; \quad E^2 = -1 \end{aligned}$$

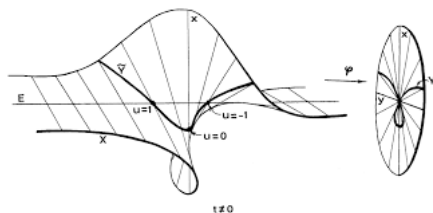
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- **Weak Factorisation Theorem** (Abramovich, Karu, Matsuki, Włodarczyk, 1999): Any birational map between two smooth complex projective varieties can be decomposed into finitely many blow-ups or blow-downs of smooth subvarieties.

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- **Weak Factorisation Theorem** (Abramovich, Karu, Matsuki, Włodarczyk, 1999): Any birational map between two smooth complex projective varieties can be decomposed into finitely many blow-ups or blow-downs of smooth subvarieties.
- **Resolution of Singularities** (Hironaka, 1964): Every variety is birational to a *smooth* projective variety.

Example

The blowup map is the main source of birational but non-isomorphic projective surfaces.

Example

Consider the smooth cubic surface

$$S: (x^3 + y^3 + z^3 + t^3 = 0) \quad \mathbb{P}^3:$$

It is well known that $S = Bl_{p_1, \dots, p_6} \mathbb{P}^2$. Hence, $S \dashrightarrow \mathbb{P}^2$.

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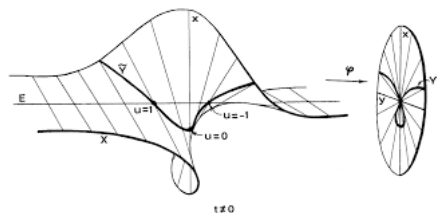
It is well known that $S = Bl_{p_1, \dots, p_6} \mathbb{P}^2$. Hence, $S \simeq \mathbb{P}^2$. However, S and \mathbb{P}^2 are *not* isomorphic since S contains disjoint lines but any two lines in \mathbb{P}^2 intersect in a point.

This leads to the idea of minimal model:

Question

Is there a simpler representative in a birational equivalence class of a surface?

Castelnuovo's Contraction Criterion



$$\varphi : B_0 \mathbb{C}^2 \rightarrow \mathbb{C}^2; \quad E := \varphi^{-1}(0)$$

$$B_0 \mathbb{C}^2 \cong \mathbb{C}^2 \setminus \{0\}$$

$$E \cong \mathbb{P}^1; \quad E^2 = -1:$$

Theorem (Castelnuovo Contraction Criterion, XIX)

Let S be a smooth projective surface and $E \cong \mathbb{P}^1$ with $E^2 = -1$ an irreducible curve in S . Then, there exists a smooth surface S^0 and a contraction morphism $\varphi : S \rightarrow S^0$ such that $\varphi : S \setminus E \rightarrow S^0 \setminus \{0\}$ is an isomorphism and $\varphi(E) = 0$.

A graph theoretic viewpoint

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A graph theoretic viewpoint

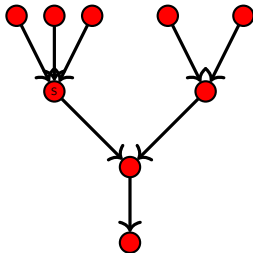
Let G be a directed graph such that

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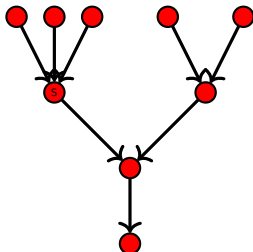
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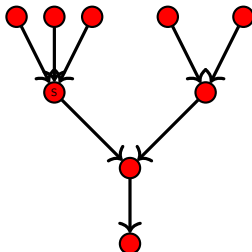


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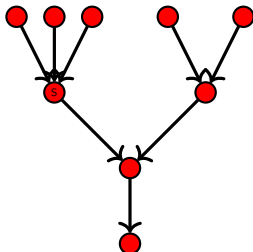


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- 3 G has an end-point.

Minimal Model Program for Surfaces

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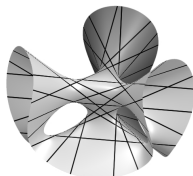
Let S be the smooth cubic surface

$$S: (x^3 + y^3 + z^3 + t^3 = 0) \subset \mathbb{P}^3.$$

Then S has 27 lines, all of which are (-1) -curves. Applying the steps of the MMP for surfaces, we contract 6 curves to get the birational morphism

$$\sigma: S \rightarrow S_1 \rightarrow \dots \rightarrow S_6 \rightarrow \mathbb{P}^2.$$

Since \mathbb{P}^2 has no (-1) -curves, we are done.



Minimal Model Program for Surfaces

Theorem (MMP for Surfaces)

Let S be a smooth projective surface. Then, the graph G containing S has an end-point S^0 such that either

- 1 $S^0 = \mathbb{P}^2$ or $S^0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $S^0 = C$;
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- 2 K_{S^0} is nef.

Remark

The first case happens when S is a rational or ruled surface and, in this case, there are infinitely many end points. For instance, if we consider the connected component of rational surfaces, \mathbb{P}^2 is an end-point but so is any Hirzebruch Surface

$$F_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

for $n \neq 1$.



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Minimal Model Program in Higher Dimension

Let X be a smooth projective variety of dimension $n = 3$.

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To extend the MMP to higher dimensions, one needs to extend the category we work with to allow for mild singularities.

Singularities

Definition

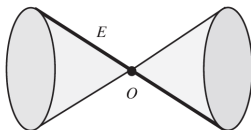
A prime divisor D on a normal variety X is \mathbb{Q} -Cartier if there is a non-zero multiple m such that mD is Cartier. If every divisor on X is \mathbb{Q} -Cartier then X is called \mathbb{Q} -factorial.

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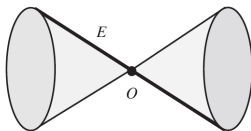


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Example

The cone $(xy - uv = 0) \subset \mathbb{C}^4$ is not \mathbb{Q} -factorial. On the other hand, $(xy + zw + z^3 + w^3 = 0) \subset \mathbb{C}^4$ is.

Singularities

Definition

A normal \mathbb{Q} -factorial variety X has **terminal singularities** if for any resolution $\sigma : Y \rightarrow X$ we have,

$$K_Y - \sigma^* K_X = \sum a_i E_i; \quad a_i > 0$$

where E_i are all the exceptional divisors of the resolution. It has **canonical singularities** if $a_i \geq 0$.

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Example

- Let $X = \mathbb{P}(1;1;2)$. Then a resolution of X is a blowup of the vertex, $\sigma : F_2 \rightarrow X$ and is crepant, i.e.,

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$$K_T = \nu^* K_X + \frac{1}{2}E$$

Singularities

Definition

Let $F \in \mathbb{C}[x_1; x_2; x_3; x_4]$ be a convergent power series around 0. Then $(F = 0)$ is a **compound du Val Singularity** (or cDV) if F is of the form

$$h(x_1; x_2; x_3) + x_4 g(x_1; x_2; x_3; x_4) = 0$$

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Let μ_r be the cyclic group of r th roots of unity. Define the action of μ_r on \mathbb{C}^4 with coordinates $x_1; x_2; x_3; x_4$ by

$$\mu_r \cdot (x_1; x_2; x_3; x_4) = (\zeta^r x_1; \zeta^{2r} x_2; \zeta^{3r} x_3; \zeta^{4r} x_4)$$

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where $h = 0$ defines a canonical surface singularity.

Let μ_r be the cyclic group of r th roots of unity. Define the action of μ_r on \mathbb{C}^4 with coordinates $x_1; x_2; x_3; x_4$ by

$$\mu_r \cdot \mathbb{C}^4 \rightarrow \mathbb{C}^4 \\ (\lambda; (x_1; x_2; x_3; x_4)) \mapsto (\lambda^{-1} x_1; \lambda^{-2} x_2; \lambda^{-3} x_3; \lambda^{-4} x_4)$$

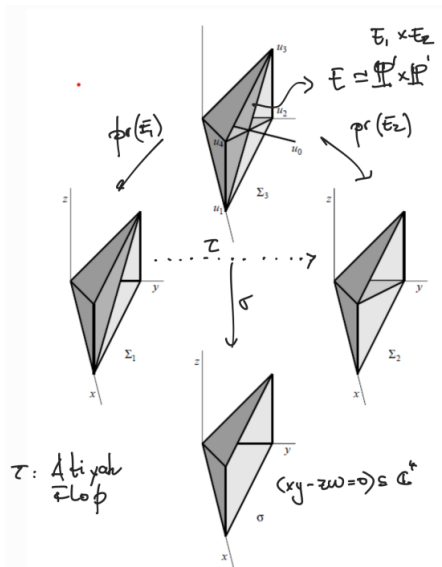
Theorem (Reid, '83)

Suppose F is equivariant with respect to the action given by μ_r . Then, every terminal 3-fold singularity over \mathbb{C} is isomorphic to

$$(F(x_1; x_2; x_3; x_4) = 0)_{\mu_r}$$

The Atiyah Flop

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A graph theoretic viewpoint

Let G be a directed graph such that

- A vertex is a **normal \mathbb{Q} -factorial projective variety of dimension at least three;**

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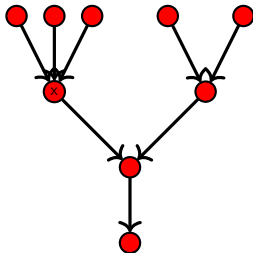
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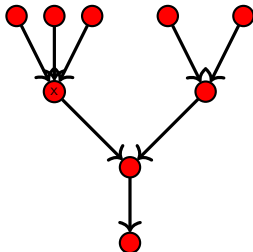
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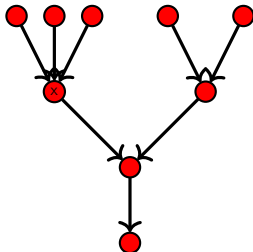


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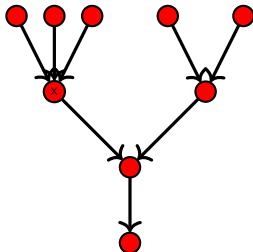


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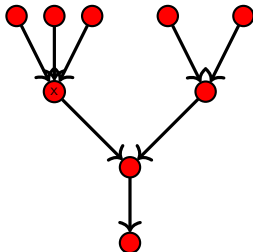


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- 3 Does G have an end-point?

Minimal Model Program in dimension 3

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Question

The first case happens if X is a *uniruled* variety. If G has more than one end-point, then how are these related?

Birational Rigidity

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Theorem (Iskovskikh-Manin, '71 - Corti, '95)

A smooth quartic threefold $X_4 \subset \mathbb{P}^4$ is birationally rigid.

In particular, X_4 is *non-rational*.

Birational Rigidity

Theorem (Corti-Mella, '04)

Let $X_4 \subset \mathbb{P}^4$ be a quartic threefold with a singularity $\mathbf{p} \in X_4$ analytically equivalent to $(xy + z^3 + t^3 = 0)$, but otherwise general. Then, the only Fano fibration birational but non-biregular to X_4 is a quasismooth complete intersection $Y_{3,4} \subset \mathbb{P}(1;1;1;1;2;2)$.

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In particular, X_4 is bi-rigid and non-rational even though it is not birationally rigid.

Theorem (DG, '22)

Let $X_4 \subset \mathbb{P}^4$ be a quartic threefold with three cA_2 singularities along a line $L \subset X_4$, but otherwise general. Then we have birational maps

$$\begin{array}{ccc}
 X_{6,8} \subseteq \mathbb{P}(1,2,2,3,3,5) & \xleftrightarrow{\text{---}} & X_4 \subset \mathbb{P}^4 \\
 \text{(Quasismooth)} & & \text{(Singular)} \\
 & & \xleftrightarrow{\text{---}} & Y_{3,4} \subseteq \mathbb{P}(1,1,1,1,2,2) \\
 & & & \text{(Singular)}
 \end{array}$$

Birational Rigidity

Let X be polarised by an ample divisor A for which $K_X = -c_1(X)A$. Then we consider the multisection ring

$$R(X; A) = \bigoplus_{m=0}^{\infty} H^0(X; \mathcal{O}_X(mA)):$$

A (minimal) choice of generators for $R(X; A)$ determines an embedding into some weighted projective space

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Theorem ((Cheltsov-Park '14), (Abban-Cheltsov-Park '20), (Okada - '14-21), (DG - '22))

Let $X \hookrightarrow \mathbb{P}$ be a terminal \mathbb{Q} -factorial complete intersection Fano threefold with at most cyclic quotient singularities. Then $\text{codim}_{\mathbb{P}} X = 3$, and

- *If $\text{codim}_{\mathbb{P}} X = 1$, then X is birationally rigid iff X is one of 95 families.*
- *If $\text{codim}_{\mathbb{P}} X = 2$, then X is birationally rigid iff X is one of 19 families.*
- *If $\text{codim}_{\mathbb{P}} X = 3$, then X is the complete intersection of three quadrics and is not birationally rigid.*

Cones and Birational Geometry

To a smooth projective variety one can associate cones of (equivalence classes of) divisors in

$$N^1(X) = \text{Div}(X) = :$$

We have the inclusions

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Example

Cones and Birational Geometry

Mori: The contractions of a smooth projective variety are controlled by (the dual of) its Nef Cone.

Definition (Mori Dream Space)

Let X be a normal projective \mathbb{Q} -factorial variety. We say X is a **Mori Dream Space** if

- $\text{Pic}(X)_{\mathbb{Q}} = N^1(X)$
- $\text{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles.
- There is a finite collection of SQMs $f_i: X \dashrightarrow X_i$ such that each X_i satisfies the above and $\text{Mov}(X) = \sum_i f_i^*(\text{Nef}(X_i))$.

Theorem (Hu-Keel, '00)

MMP holds for any Mori Dream Space. Moreover, the chambers $f_i^(\text{Nef}(X_i))$ and their faces give a fan supported in $\text{Mov}(X)$ and the cones in the fan are in one-to-one correspondence with contractions.*

Cones and Birational Geometry

Example

Let X_4 be the quartic threefold containing a line L and 3 cA_2 singular points on it. Let H be (the pull-back of) a hyperplane section and E the exceptional divisor resulting from blowing up L . Then,

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The Sarkisov Program

Question

How are end products of applying MMP to uniruled varieties related?

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Theorem (Corti, '95 and Hacon-McKernan, '13)

Let X_1 and X_2 be birational Fano fibrations with normal \mathbb{Q} -factorial terminal singularities. Then there is a finite sequence of Sarkisov links connecting them.