On the birational geometry of Fano threefold complete intersections

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Overview

1. What is it all about?
2. MMP in dimension 2
3. MMP in dimension $n \geq 3$
4. Results

Algebraic Varieties

Definition

Classical **Algebraic Geometry** is the study of geometric structures defined by polynomials equations.

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A is a Noetherian ring $\implies A[x_1, \ldots, x_{n+1}]$ is a Noetherian ring.

Hence,

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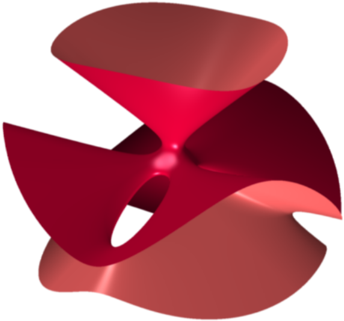
To ease notation we usually write it as $X\colon (f_1 = \dots = f_s = 0)$.

Example: Clebsch Cubic



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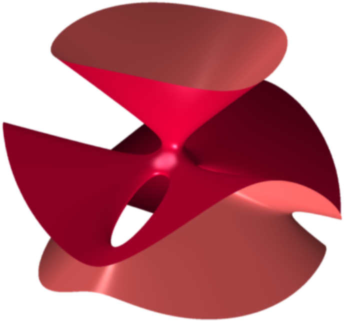

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$$\mathrm{Aut}(S_{\text{Clebsch}}) = S_5$$

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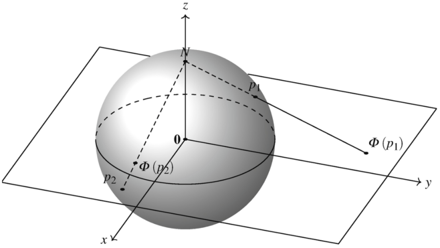
Any birational map between *smooth projective curves* extends to a morphism.

Example



The unit sphere

$$\mathbb{S}^n : (x_1^2 + \cdots + x_{n+1}^2 = 1) \subset \mathbb{R}^{n+1}$$

projects from the north pole $\mathbf{N} = (0, \ldots, 1)$ to the plane $x_{n+1} = 0$, where we use coordinates $y_1, \ldots, y_n$.

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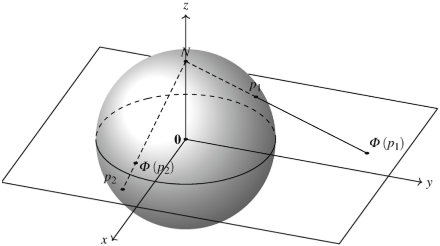

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We have,

$$\Phi(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right)$$

and

$$\Phi^{-1}(y_1, \ldots, y_n) = \left(\frac{2y_1}{1+S}, \ldots, \frac{2y_n}{1+S}, \frac{-1+S}{1+S} \right)$$

where $S = \sum y_i^2$. We write,

$$\mathbb{S}^n \dashrightarrow \mathbb{R}^n.$$

The Canonical Divisor

Definition

Recall that for a smooth variety X of dimension n, the **canonical bundle** is the line bundle $\omega_X = \Omega_X^n$, that is, the nth exterior power of the cotangent bundle on X. A **canonical divisor** is any divisor D for which $\omega_X = \mathcal{O}_X(D)$. We denote it by K_X.

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Example

Let $X = \mathbb{P}^1 = \mathbb{C}_z \cup \{\infty\}$. Let $\omega = dz$. At ∞ the local coordinate changes to $w = 1/z$ and $\omega = d(1/w) = -1/w^2 dw$. Then ω has a pole of order 2 at ∞. We write it as $K_X = -2 \cdot \{\infty\}$

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Definition

Let X be a normal projective variety with *good singularities*. We say that X is

- **Fano** if $-K_X$ is ample;
- **Calabi-Yau** if $-K_X$ is trivial:
- **Canonically Polarised** if K_X is ample.

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Example

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Let $C \subset \mathbb{P}^2$ be a smooth projective curve. Then,

$$K_C = (K_{\mathbb{P}^2} + C)|_C = (-3L + dL)|_C = (d-3)L|_C.$$

Taking degrees,

$$\deg(K_C) = 2g(C) - 2 = -3L \cdot C + C^2 = -3d + d^2.$$

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$$\begin{array}{lcccc} C \text{ is Fano} & \iff & g(C) = 0 & \iff & d < 3 \\ C \text{ is Calabi-Yau} & \iff & g(C) = 1 & \iff & d = 3 \\ C \text{ is Canonically Polarised} & \iff & g(C) \geq 2 & \iff & d > 3 \end{array}$$

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Let $X := X_{d_1,\dots,d_s} \subset \mathbb{P}^d$ be a smooth complete intersection of multidegree $(d_1, \dots, d_n)$. Then, $K_X = (-d - 1 + \sum d_i)H|_X$, where H is a generic hyperplane section of $\mathbb{P}^d$ not containing X and

$$\begin{array}{lcl} X \text{ is Fano} & \iff & d + 1 - \sum d_i > 0 \\ X \text{ is Calabi-Yau} & \iff & d + 1 - \sum d_i = 0 \\ X \text{ is Canonically Polarised} & \iff & d + 1 - \sum d_i < 0 \end{array}$$

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Each smooth projective variety is birational to a projective variety with *good singularities* Y such that either

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Theorem (Birkar-Cascini-Hacon-McKernan, '10)

Let W be a smooth projective variety which is uniruled. Then W is birational to a Fano fibration.

The Blowup

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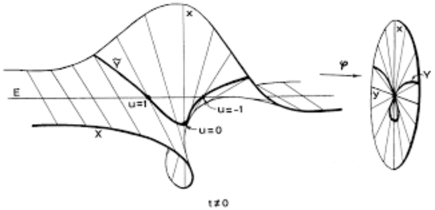

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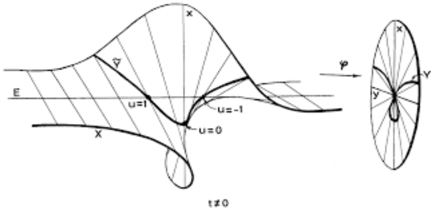

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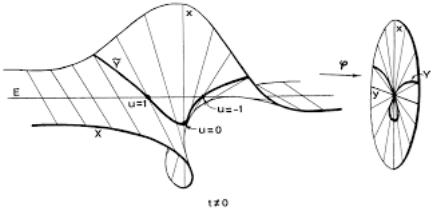

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- **Resolution of Singularities** (Hironaka, 1964): Every variety is birational to a *smooth* projective variety.

Example

The blowup map is the main source of birational but non-isomorphic projective surfaces.

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Consider the smooth cubic surface

$$S\colon (x^3 + y^3 + z^3 + t^3 = 0) \subset \mathbb{P}^3.$$

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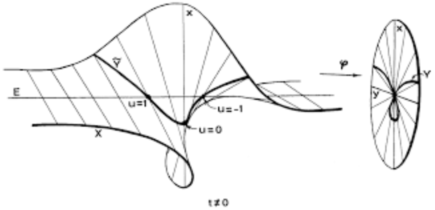
This leads to the idea of minimal model:

Question

Is there a simpler representative in a birational equivalence class of a surface?

Castelnuovo's Contraction Criterion



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Theorem (Castelnuovo Contraction Criterion, XIX)

Let S be a smooth projective surface and $E \simeq \mathbb{P}^1$ with $E^2 = -1$ an irreducible curve in S. Then, there exists a smooth surface S' and a contraction morphism $\varphi\colon S \to S'$ such that $\varphi\colon S \setminus E \to S' \setminus 0$ is an isomorphism and $\varphi(E) = 0$.

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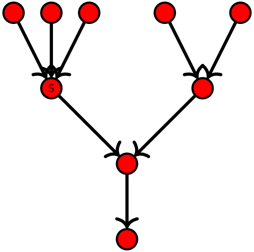

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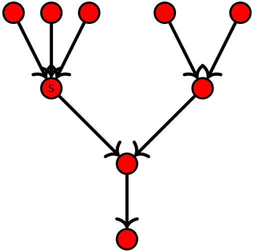

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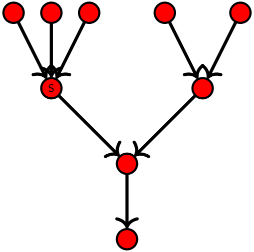

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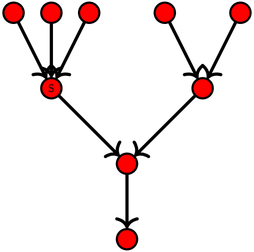

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3. G has an end-point.

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Example

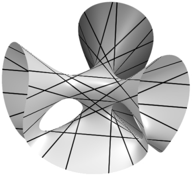
Let S be the smooth cubic surface

$$S \colon (x^3 + y^3 + z^3 + t^3 = 0) \subset \mathbb{P}^3.$$

Then S has 27 lines, all of which are (-1)-curves. Applying the steps of the MMP for surfaces, we contract 6 curves to get the birational morphism

$$\varphi \colon S \to S_1 \to \cdots \to S_6 \simeq \mathbb{P}^2.$$

Since $\mathbb{P}^2$ has no (-1)-curves, we are done.



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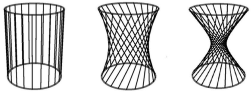
Remark

The first case happens when S is a rational or ruled surface and, in this case, there are infinitely many end points. For instance, if we consider the connected component of rational surfaces, $\mathbb{P}^2$ is an end-point but so is any Hirzebruch Surface

$$\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$$

for $n \neq 1$.



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To extend the MMP to higher dimensions, one needs to extend the category we work with to allow for mild singularities.

Singularities

Definition

A prime divisor D on a normal variety X is $\mathbb{Q}$-**Cartier** if there is a non-zero multiple m such that mD is Cartier. If every divisor on X is $\mathbb{Q}$-Cartier then X is called $\mathbb{Q}$-**factorial**.

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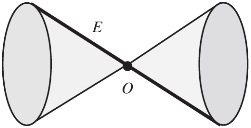

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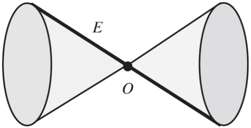

Example

The cone $(xy - uv = 0) \subset \mathbb{C}^4$ is not $\mathbb{Q}$-factorial. On the other hand, $(xy + zw + z^3 + w^3 = 0) \subset \mathbb{C}^4$ is.

Singularities

Definition

A normal $\mathbb{Q}$-factorial variety X has **terminal singularities** if for any resolution $\varphi : Y \to X$ we have,

$$K_Y - \varphi^* K_X = \sum a_i E_i, \quad a_i > 0$$

where E_i are all the exceptional divisors of the resolution. It has **canonical singularities** if $a_i \geq 0$.

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- Let $X = \mathbb{P}(1,1,2)$. Then a resolution of X is a blowup of the vertex, $\varphi\colon \mathbb{F}_2 \to X$ and is crepant, i.e.,

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- Let $X = \mathbb{P}(1,1,1,2)$. Then a resolution of X is a blowup of the vertex, $\varphi \colon T \to X$ and it satisfies
$$K_T = \varphi^* K_X + \frac{1}{2} E.$$

Singularities

Definition

Let $F \in \mathbb{C}\{x_1, x_2, x_4, x_4\}$ be a convergent power series around 0. Then $(F = 0)$ is a **compound du Val Singularity** (or cDV) if F is of the form

$$h(x_1, x_2, x_3) + x_4 g(x_1, x_2, x_3, x_4) = 0$$

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Theorem (Reid, '83)

Suppose F is equivariant with respect to the action given by $\boldsymbol{\mu}_r$. Then, every terminal 3-fold singularity over $\mathbb{C}$ is isomorphic to

$$(F(x_1, x_2, x_3, x_4) = 0)/\boldsymbol{\mu}_r.$$

The Atiyah Flop

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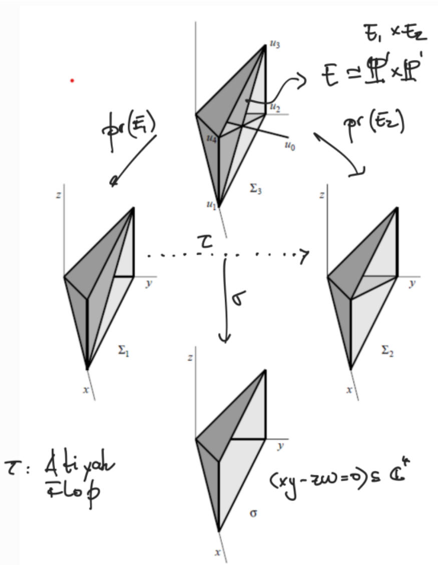

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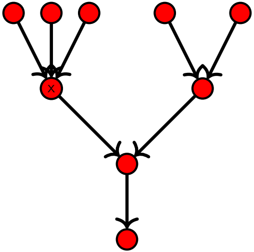

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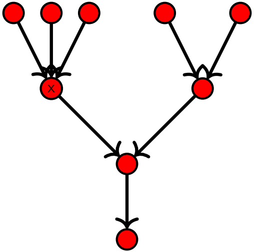

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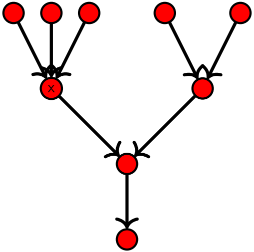

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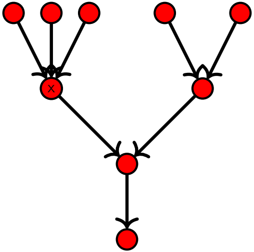

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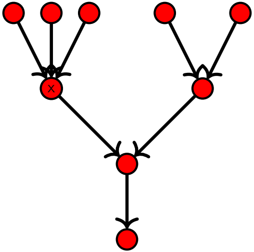

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3. Does G have an end-point?

Minimal Model Program in dimension 3

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Question

The first case happens if X is a *uniruled* variety. If G has more than one end-point, then how are these related?

Birational Rigidity

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Theorem (Iskovskikh-Manin, '71 - Corti, '95)

A smooth quartic threefold $X_4 \subset \mathbb{P}^4$ is birationally rigid.

In particular, X_4 is *non*-rational.

Birational Rigidity

Theorem (Corti-Mella, '04)

Let $X_4 \subset \mathbb{P}^4$ *be a quartic threefold with a singularity* $\mathbf{p} \in X_4$ *analytically equivalent to* $(xy + z^3 + t^3 = 0)$*, but otherwise general. Then, the only Fano fibration birational but non-biregular to* X_4 *is a quasismooth complete intersection* $Y_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$*.*

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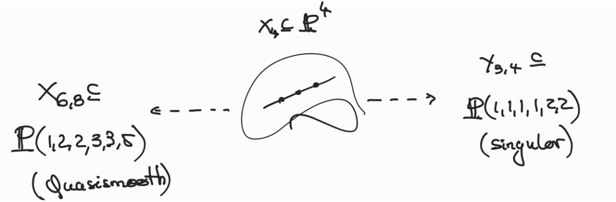
In particular, X_4 is bi-rigid and *non*-rational even though it is not birationally rigid.

Theorem (DG, '22)

Let $X_4 \subset \mathbb{P}^4$ be a quartic threefold with three cA_2 singularities along a line $L \subset X_4$, but otherwise general. Then we have birational maps



Birational Rigidity

Let X be polarised by an ample divisor A for which $-K_X = \iota_X A$. Then we consider the multsection ring

$$R(X, A) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mA)).$$

A (minimal) choice of generators for $R(X, A)$ determines an embedding into some weighted projective space

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Theorem ((Cheltsov-Park '14), (Abban-Cheltsov-Park '20), (Okada - '14-21), (DG - '22))

Let $X \hookrightarrow \mathbb{P}$ be a terminal $\mathbb{Q}$-factorial complete intersection Fano threefold with at most cyclic quotient singularities. Then $\operatorname{codim}_{\mathbb{P}} X \leq 3$, and

- *If $\operatorname{codim}_{\mathbb{P}} X = 1$, then X is birationally rigid iff X is one of 95 families.*
- *If $\operatorname{codim}_{\mathbb{P}} X = 2$, then X is birationally rigid iff X is one of 19 families.*
- *If $\operatorname{codim}_{\mathbb{P}} X = 3$, then X is is the complete intersection of three quadrics and is not birationally rigid.*

Cones and Birational Geometry

To a smooth projective variety one can associate cones of (equivalence classes of) divisors in

$$N^1(X) = Div(X)/\equiv.$$

We have the inclusions

$$\mathrm{Amp}(X) \subset \mathrm{Nef}(X) \subset \overline{\mathrm{Mov}}(X) \subset \overline{\mathrm{Eff}}(X)$$

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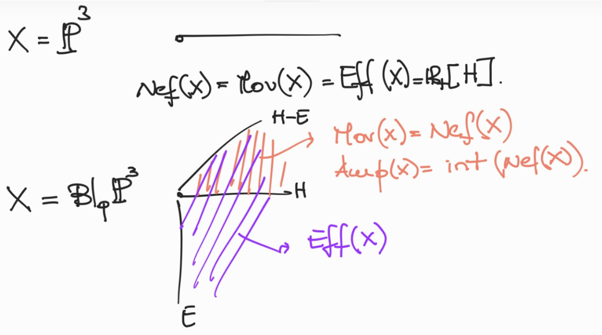


Cones and Birational Geometry

Mori: The contractions of a smooth projective variety are controlled by (the dual of) its Nef Cone.

Definition (Mori Dream Space)

Let X be a normal projective $\mathbb{Q}$-factorial variety. We say X is a **Mori Dream Space** if

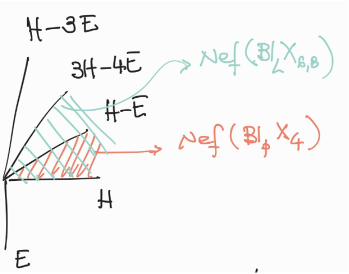
- $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$
- $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles.
- There is a finite collection of SQMs $f_i \colon X \dashrightarrow X_i$ such that each X_i satisfies the above and $\mathrm{Mov}(X) = \bigcup_i f_i^*(\mathrm{Nef}(X_i))$.

Theorem (Hu-Keel, '00)

MMP holds for any Mori Dream Space. Moreover, the chambers $f_i^*(\mathrm{Nef}(X_i))$ and their faces give a fan supported in $\mathrm{Mov}(X)$ and the cones in the fan are in one-to-one correspondence with contractions.

Cones and Birational Geometry

Example



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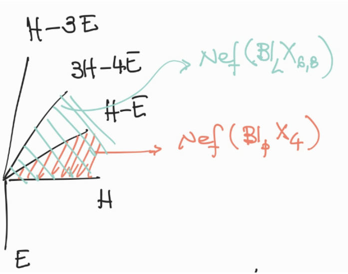

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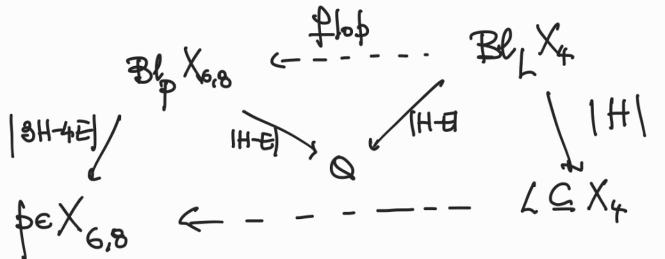

The Sarkisov Program

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Theorem (Corti, '95 and Hacon-McKernan, '13)

Let X_1 and X_2 be birational Fano fibrations with normal $\mathbb{Q}$-factorial terminal singularities. Then there is a finite sequence of Sarkisov links connecting them.