$U_q(\mathfrak{gl}(1|1))$ and $U(1|1)$ Chern–Simons theory

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Executive summary of Geer's talk

Reshetikhin–Turaev: a modular tensor category defines a 3d TFT

MTCs have the following properties:

1. semisimple
2. finite (finitely many simple objects up to isomorphism)
3. simple objects have non-zero dimension
4. non-degenerate (modularity condition).

BCGP, De Renzi: A relative MTC defines a 3d decorated TFT

Relative MTCs weaken ❶, ❷, ❸ and modify ❹.

Today's goals

1. New examples of relative modular tensor categories
   - Representation categories of a non-standard quantization of the complex Lie superalgebra $\mathfrak{gl}(1|1)$
   - Generic/root of unity dichotomy of quantization parameter leads to two classes of examples
2. Realization of known physical models via the associated TFT
   - Rozansky–Saleur: $U(1|1)$ Wess–Zumino–Witten theory
   - Mikhaylov, Mikhaylov–Witten: Supergroup Chern–Simons theories

Compact Chern–Simons theory I (Witten)

Three dimensional quantum gauge theory defined by

- compact simple simply connected Lie group G, the *gauge group*
- $k \in H^4(BG;\mathbb{Z}) \simeq \mathbb{Z}$, the *level*.

Formally, invariants of 3-manifolds are

$$\mathcal{Z}(M) \sim \int_{\Omega^1(M;\mathfrak{g})/C^\infty(M;G)} e^{\sqrt{-1}kCS(A)}\mathcal{D}A.$$

Invariants of coloured knots arise from Wilson operators

$$\langle K_V \rangle \sim \int_{\Omega^1(M;\mathfrak{g})/C^\infty(M;G)} Hol_K(A;V)e^{\sqrt{-1}kCS(A)}\mathcal{D}A.$$

When $M = S^3$, $G = SU(2)$ and $V = \mathbb{C}^2$, this is the Jones polynomial.

Compact Chern–Simons theory II

Key feature (Witten): Computations in Chern–Simons theory can be approached via its boundary Wess–Zumino–Witten theory.

This leads to connections with the representation theory of

- (rational) vertex operator algebras,
- loop groups,
- affine Lie algebras.

RT: 3d TFTs from modular tensor categories

Let

- $\mathfrak{g}$ be a simple complex Lie algebra
- $k \in \mathbb{Z}$ suitable integer.

The category

$$U_q(\mathfrak{g})\text{-mod}, \qquad q^k = 1,$$

is neither semisimple nor finite and has simple objects with vanishing quantum dimension and so is not modular. However,

$$\mathcal{C} = \text{semisimplified } U_q(\mathfrak{g})\text{-mod}$$

is a modular tensor category [Reshetikhin–Turaev, Andersen, ...].

Chern–Simons via Reshetikhin–Turaev theory II

The associated Reshetikhin–Turaev TFT

$$\mathcal{Z}_{\mathcal{C}} : \mathsf{Cob} \to \mathsf{Vect}$$

models Chern–Simons theory with gauge group G at level $\overline{k}$.

In particular, this defines invariants of

- closed surfaces $\mathcal{Z}_{\mathcal{C}}(\Sigma) \in \mathsf{Vect}$ and
- closed 3-manifolds $\mathcal{Z}_{\mathcal{C}}(M) \in \mathbb{C}$.

Physical perspective: $\mathcal{C}$ is the category of Wilson (line) operators of Chern–Simons theory.

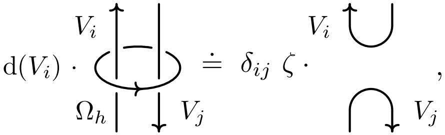
Renormalized Reshetikhin–Turaev theory I

A *relative modular category* is a ribbon category $\mathcal{C}$ with a

1. compatible abelian group grading, $\mathcal{C} = \bigoplus_{g \in \mathcal{G}} \mathcal{C}_g$,
2. monoidal action $Z \to \mathcal{C}_0$ of an abelian group Z,
3. a non-zero modified trace on the ideal of projectives

such that

- $\mathcal{C}_g$ is semisimple unless $g \in X$ for some small subset $X \subset \mathcal{G}$
- each $\mathcal{C}_g$, $g \in \mathcal{G} \backslash X$, has finitely many simples modulo Z
- non-degeneracy: there exists $\zeta \in \mathbb{C}^{\times}$ such that



$g, h \in \mathcal{G} \backslash X$
$V_i, V_j \in \mathcal{C}_g$

- $\cdots$.

Renormalized Reshetikhin–Turaev theory II

Theorem (Blanchet–Costantino–Geer–Patureau-Mirand, De Renzi)

A relative modular category $\mathcal{C}$ defines a 3d decorated TFT

$$\mathcal{Z}_{\mathcal{C}} : \mathrm{Cob}_{\mathcal{C}}^{\mathrm{ad}} \to \mathrm{Vect}^{Z\text{-gr}}.$$

In particular, $\mathcal{Z}_{\mathcal{C}}$ encodes

- invariants of decorated surfaces $(\Sigma, \omega \in H^1(\Sigma; \mathcal{G}))$,
- invariants of *admissible* 3-manifolds $(M, T, \omega \in H^1(M \setminus T; \mathcal{G}))$
  - the $\mathcal{C}$-coloured ribbon graph T has a projective colour, or
  - ω is generic: $\omega(\gamma) \in \mathcal{G} \setminus X$ for some simple closed curve $\gamma \subset M$.

Question: Is there a physical realization of $\mathcal{Z}_{\mathcal{C}}$?

TFT from QFT I

TFTs appearing in supersymmetric QFT often arise as topological twists

- Chern–Simons theory with gauge supergroup
- Rozansky–Witten theory of a holomorphic symplectic manifold (intuition: fermionic counterpart of compact Chern–Simons theory)

Resulting categories of line operators are naturally differential graded, usually non-semisimple.

> Expectation: TFTs arising from topological twists of physical QFTs are differential graded.

TFT from QFT II

If the physical QFT has global symmetry group $\mathcal{G}$, then the theory can be coupled to background flat $\mathcal{G}$-connections.

Expectation: The category of line operators decomposes as

$$\mathcal{C} = \bigoplus_{g \in \mathcal{G}} \mathcal{C}_g .$$

TFT from QFT III

Earlier results:

- QFT for unrolled quantum $\mathfrak{sl}(2)$
  - BCGP: relative MTC of representations of unrolled quantum $\mathfrak{sl}(2)$, many computations in resulting TFT
  - Creutzig–Dimofte–Garner–Geer: computations in A-type topological twist of $\mathcal{N}=4$ Chern–Simons-matter theory with gauge group $SU(2)$ match BCGP
  - Gukov–Hsin–Nakajima–Park–Pei: computations in equivariant Rozansky–Witten theory match BCGP
  - Costantino–Gukov–Putrov: $\hat{Z}$-invariants as expansions of CGP invariants
- Quantum topology of $\mathfrak{gl}(1|1)$
  - Alexander polynomial: Kauffman–Saleur, Frohman–Nicas, Kerler, Viro, ...
  - Heegaard–Floer theory: Manion–Rouquier, Manion

The unrolled quantum group $U_q^E(\mathfrak{gl}(1|1))$ I

The complex Lie superalgebra $\mathfrak{gl}(1|1) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{1|1})$ has homogeneous basis

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The defining relations are that E is central and

$$[G, X] = X, \qquad [G, Y] = -Y,$$

$$[X, X] = 0, \qquad [Y, Y] = 0,$$

$$[X, Y] = E.$$

The unrolled quantum group $U_q^E(\mathfrak{gl}(1|1))$ II

Fix $\hbar \in \mathbb{C}$ such that $q := e^{\hbar} \in \mathbb{C}^{\times} \backslash \{\pm 1\}$.

Definition

The *unrolled quantum group* $U_q^E(\mathfrak{gl}(1|1))$ is the superalgebra generated by $E, G, K^{\pm 1}$ and X, Y such that $E, K^{\pm 1}$ are central and

$$KK^{-1} = K^{-1}K = 1,$$

$$[G, X] = X, \qquad [G, Y] = -Y,$$

$$X^2 = Y^2 = 0,$$

$$XY + YX = \frac{K - K^{-1}}{q - q^{-1}}.$$

There is a natural Hopf structure on $U_q^E(\mathfrak{gl}(1|1))$.

Integral weight modules I

A $U_q^E(\mathfrak{gl}(1|1))$-module is called *integral weight* if

- E and G are simultaneously diagonalizable,
- G has integral weights and
- $K = q^E$ as operators.

The category $\mathcal{D}^{q,\mathrm{int}}$ of integral weight modules is rigid monoidal.

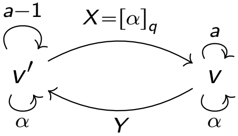
One dimensional simples: $(n, b, \bar{p}) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$

$$\epsilon\left(\frac{n\pi\sqrt{-1}}{\hbar}, b\right)_{\bar{p}} =$$

Integral weight modules II

Quantum Kac modules: $(\alpha, a, \bar{p}) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{Z}_2$

$V(\alpha, a)_{\bar{p}} =$



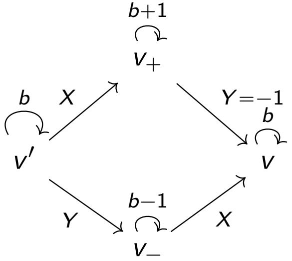
Then

- $V(\alpha, a)_{\bar{p}}$ is simple $\Leftrightarrow [\alpha]_q \neq 0 \Leftrightarrow \alpha \notin \frac{\pi\sqrt{-1}}{\hbar}\mathbb{Z}$.
- If $\alpha = \frac{n\pi\sqrt{-1}}{\hbar}$, then $V(\alpha, a)_{\bar{p}}$ is reducible indecomposable.

Integral weight modules III

Projective indecomposables: $(n, b, \bar{p}) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$

$P(\frac{n\pi\sqrt{-1}}{\hbar}, b)_{\bar{p}} =$



, $E = \frac{n\pi\sqrt{-1}}{\hbar}$.

Relative modular structures I

Theorem (Geer-Y.)

$\mathcal{D}^{q,\text{int}}$ admits two classes of relative modular structures

- q is arbitrary
- q is a primitive r^{th} root of unity (say, odd)
  - $\mathcal{G} = \mathbb{C}/\mathbb{Z}$ via E-weights
  - $X = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$
  - $Z = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \ni (n, a, \bar{p}) \mapsto \epsilon(\frac{n\pi\sqrt{-1}}{\hbar}, a)_{\bar{p}}$.

Ingredients of proof:

- Prove generic semisimplicity via injectivity of simple Kac modules
- Use Viro's explicit R-matrix as braiding
- Verify generic coherence of candidate twist, then complete
- Explicit description of projective indecomposables to establish existence of a modified trace

Relative modular structures II

Theorem (Geer-Y.)

If $q^r = 1$, then $\mathcal{D}^{q,\mathrm{int}}$ admits a relative modular structure with

- $\mathcal{G} = \mathbb{C}/\mathbb{Z}$ via E-weights
- $X = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$
- $Z = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \ni (n, a, \bar{p}) \mapsto \epsilon(\frac{n\pi\sqrt{-1}}{\hbar}, a)_{\bar{p}}$.

Relation to supergroup Chern–Simons theories

Proposal (Geer-Y.)

The 3d TFT associated to $\mathcal{D}^{q,\mathrm{int}}$ is a homological truncation of

1. $\mathfrak{psl}(1|1)$ Chern–Simons if q is arbitrary
   - Rozansky–Witten theory of $T^\vee\mathbb{C}$
   - B-twist of a 3d $\mathcal{N}=4$ free hypermultiplet
2. $U(1|1)$ Chern–Simons theory at level r if $q^r=1$
   - $U(1)\times U(1)$-equivariant Rozansky–Witten theory of $T^\vee\mathbb{C}$.

Evidence

Evidence for the proposal by comparison with physics literature:

- Rozansky–Saleur: $GL(1|1)$ Wess–Zumino–Witten theory and assumed Chern–Simons/WZW correspondence
- Mikhaylov, Mikhaylov–Witten: Supergroup Chern–Simons theory via geometric quantization and brane constructions
- Kapustin–Saulina: $U(1) \times U(1)$-equivariant Rozansky–Witten theory of $T^{\vee}\mathbb{C}$
- Aghaei–Gainutdinov–Pawelkiewicz–Schomerus: Combinatorial quantization in genus one via the small quantum group of $\mathfrak{gl}(1|1)$

Evidence I: Global symmetries

- $\mathbb{C}^{\times} \simeq \mathcal{G}$ acts as symmetries of $\mathfrak{psl}(1|1)$ and $\mathfrak{gl}(1|1)$, e.g.

$$\mathfrak{gl}(1|1)_{-1} = \mathbb{C} \cdot Y, \quad \mathfrak{gl}(1|1)_0 = \mathbb{C} \cdot G \oplus \mathbb{C} \cdot E, \quad \mathfrak{gl}(1|1)_{+1} = \mathbb{C} \cdot X$$

- $U(1|1)$ Chern–Simons theory admits Wilson operators labelled by $U(1|1)$ representations
- $\mathfrak{psl}(1|1)$ Chern–Simons theory admits
  - Wilson operators labelled by $\mathfrak{pgl}(1|1)$ representations
  - monodromy operators

Henceforth: $q^r = 1$, r odd.

Evidence II: Verlinde formula

Theorem (Geer-Y.)

Let Σ_g be a generic surface of genus $g \geqslant 1$. Then

$$\mathcal{Z}(\Sigma_g \times S^1_{\bar{\beta}}) = (-1)^{g+1} r^{2g-1} \sum_{i=0}^{r-1} (q^{\bar{\beta}+i} - q^{-\bar{\beta}-i})^{2g-2}.$$

Generating function of graded dimensions:

$$\dim_{(t_1,t_2,s)} \mathcal{Z}(\Sigma_g) = \sum_{(n,n',\bar{p}) \in Z} (-1)^{\bar{p}} \dim_{\mathbb{C}} \mathcal{Z}_{(n,n',\bar{p})}(\Sigma_g) t_1^n t_2^{n'} s^{\bar{p}}.$$

Corollary (Verlinde formula)

$$\mathcal{Z}(\Sigma_g \times S^1_{\bar{\beta}}) = \dim_{(1,q^{-2r\bar{\beta}},1)} \mathcal{Z}(\Sigma_g).$$

Evidence III: Dimensions of state spaces

Theorem (Geer-Y.)

Let Σ_g be a generic surface of genus $g \geqslant 1$. Then

$$\mathcal{Z}(\Sigma_g) = \bigoplus_{k \in [-(g-1), g-1] \cap r\mathbb{Z}} \mathcal{Z}_{(0,k,\overline{k})}(\Sigma_g)$$

with

$$\dim_{\mathbb{C}} \mathcal{Z}_{(0,k,\overline{k})}(\Sigma_g) = r^{2g} \binom{2g-2}{g-1-|k|}.$$

Evidence IV: Mapping class group actions

Theorem (Geer-Y.)

Let Σ_1 be a *non-generic surface* of genus one. Then

$$\mathcal{Z}(\Sigma_g) \simeq \mathcal{Z}_0(\Sigma_g) \simeq \mathbb{C}^{r^2+1}$$

and the mapping class group action is such that Dehn twists act with infinite order.

Thank you!