Future stability of models of the universe

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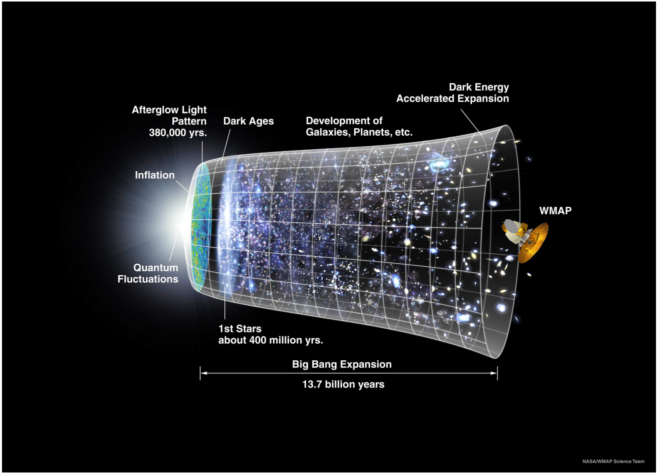
Introduction

The standard models of the universe

- satisfy the **cosmological principle** (i.e., they are spatially homogeneous and isotropic),
- are spatially flat,
- have matter content consisting of ordinary matter, dark matter and dark energy.

Current model of the universe



Current model of the universe: NASA/WMAP Science Team.

Questions

Since the universe is not exactly spatially homogeneous and isotropic, it is natural to ask: **are the standard models future stable?**

Since the assumption of spatial homogeneity and isotropy yields strong restrictions on the allowed topologies, it is also of interest to ask: **what are the restrictions on the global topology of the universe imposed by the constraint that what we observe seems to be close to a standard model?**

Matter models

Perfect fluids: Matter described by energy density ρ and pressure $\mathfrak{p}$. *Dust:* $\mathfrak{p} = 0$. *Radiation:* $\mathfrak{p} = \rho/3$.

Vlasov matter: collection of particles, where

- the particles all have unit mass,
- collisions are neglected,
- the particles follow geodesics,
- collection described statistically by a distribution function.

Einstein's equations

Einstein's equations:

$$G + \Lambda g = T,$$

where

- (M, g) is a Lorentz manifold,
- $G = \mathrm{Ric} - \frac{1}{2}Sg$ is the Einstein tensor,
- Λ is the cosmological constant,
- T is the stress energy tensor.

Vlasov matter, mathematical structures

In the Vlasov setting, the relevant mathematical structures are

- the **mass shell** P; the future directed unit timelike vectors in (M, g),
- the **distribution function** $f : P \to [0, \infty)$,
- the **stress energy tensor**

$$T_{\alpha\beta}|_{\xi} = \int_{P_{\xi}} f p_{\alpha} p_{\beta} \mu_{P_{\xi}},$$

- the **Vlasov equation**

$$\mathcal{L}f = 0.$$

The Einstein-Vlasov system

The Einstein-Vlasov system consists of the equations

$$\begin{aligned} G + \Lambda g &= T, \\ \mathcal{L} f &= 0 \end{aligned}$$

for g and f. Note that the second equation corresponds to the requirement that f be constant along timelike geodesics.

Standard models

Spatial homogeneity, isotropy and flatness imply that the metric takes the form

$$g = -dt^2 + a^2(t)\bar{g}.$$

The stress energy tensor of the Vlasov matter then takes perfect fluid form, and the energy density and pressure are given by

$$\begin{aligned}
\rho_{\mathrm{Vl}}(t) &= \int_{\mathbb{R}^3} \bar{\mathfrak{f}}\,(a(t)\bar{q})\,(1+|\bar{q}|^2)^{1/2} d\bar{q}, \\
\mathfrak{p}_{\mathrm{Vl}}(t) &= \frac{1}{3}\int_{\mathbb{R}^3} \bar{\mathfrak{f}}\,(a(t)\bar{q})\,\frac{|\bar{q}|^2}{(1+|\bar{q}|^2)^{1/2}} d\bar{q},
\end{aligned}$$

where $\bar{\mathfrak{f}}$, a function on $\mathbb{R}^3$, is the initial datum for the distribution function.

Approximating fluids

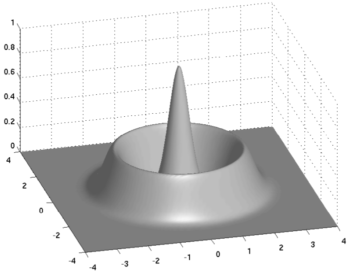


Figure: An illustration of an initial datum for the distribution function which is appropriate when approximating a standard model.

Induced initial data

Let (M, g, f) be a solution and Σ be a spacelike hypersurface in (M, g). Then the **initial data induced on** Σ consist of

- the Riemannian metric induced on Σ by g, say $\bar{g}$,
- the second fundamental form induced on Σ by g, say $\bar{k}$,
- the induced distribution function $\bar{f} : T\Sigma \to [0, \infty)$.

Here

$$\bar{f} = f \circ \mathrm{proj}_{\Sigma}^{-1},$$

where $\mathrm{proj}_{\Sigma} : P_{\Sigma} \to T\Sigma$ represents projection orthogonal to the normal.

Constraint equations

Due to the fact that (M, g, f) solve the Einstein-Vlasov system, $(\bar{g}, \bar{k}, \bar{f})$ have to solve the **constraint equations**:

$$\bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\mathrm{tr}\bar{k})^2 = 2\Lambda + 2\rho^{\mathrm{Vl}}, \quad (1)$$

$$\overline{\nabla}^j \bar{k}_{ji} - \overline{\nabla}_i(\mathrm{tr}\bar{k}) = -\bar{J}_i^{\mathrm{Vl}}. \quad (2)$$

Here

$$\rho^{\mathrm{Vl}}(\xi) = \int_{T_\xi \Sigma} \bar{f}(\bar{p})[1 + \bar{g}(\bar{p}, \bar{p})]^{1/2} \bar{\mu}_{\bar{g},\xi}, \quad (3)$$

$$\bar{J}^{\mathrm{Vl}}(\bar{X}) = \int_{T_\xi \Sigma} \bar{f}(\bar{p}) \bar{g}(\bar{X}, \bar{p}) \bar{\mu}_{\bar{g},\xi}. \quad (4)$$

Abstract initial data

Abstract initial data consist of

- an n-dimensional manifold Σ,
- a Riemannian metric $\bar{g}$ on Σ,
- a symmetric covariant 2-tensor field $\bar{k}$ on Σ,
- a function $\bar{f} : T\Sigma \to [0, \infty)$ (belonging to a suitable function space),

all assumed to be smooth and such that the constraint equations (1) and (2) are satisified.

Developments

Given abstract initial data, a **development** is

- a solution (M, g, f) to the Einstein-Vlasov system, and
- an embedding $i : \Sigma \to M$

such that the initial data induced on $i(\Sigma)$ by (M, g, f) correspond to the abstract initial data.

Maximal globally hyperbolic development

Definition

Given initial data, a *maximal globally hyperbolic development* of the data is a globally hyperbolic development (M, g, f), with embedding $i : \Sigma \to M$, such that if (M', g', f') is any other globally hyperbolic development of the same data, with embedding $i' : \Sigma \to M'$, then there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image such that $\psi^* g = g'$, $\psi^* f = f'$ and $\psi \circ i' = i$.

Function spaces

If Σ is a compact manifold, $\bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$ denotes the space of smooth functions $f : T\Sigma \to \mathbb{R}$ such that

$$\|\bar{f}\|_{H^{l}_{\mathrm{Vl},\mu}}$$

$$= \left(\sum_{i=1}^{j} \sum_{|\alpha|+|\beta|\leq l} \int_{\bar{\mathrm{x}}_i(U_i)\times\mathbb{R}^n} \langle\bar{\varrho}\rangle^{2\mu+2|\beta|} \bar{\chi}_i(\bar{\xi})(\partial^{\alpha}_{\bar{\xi}}\partial^{\beta}_{\bar{\varrho}}\bar{\mathrm{f}}_{\bar{\mathrm{x}}_i})^2(\bar{\xi},\bar{\varrho})d\bar{\xi}d\bar{\varrho} \right)^{1/2}$$

is finite for every $l \geq 0$, where

$$\langle\bar{\varrho}\rangle = (1+|\bar{\varrho}|^2)^{1/2}.$$

Previous results

- Stability of de Sitter space in $3+1$-dimensions, etc., Helmut Friedrich, '86, '91.
- Stability of even dimensional de Sitter spaces, Michael Anderson '05.
- Stability in the non-linear scalar field case, H.R. '08.
- Einstein-Euler with a positive cosmological constant, Igor Rodnianski and Jared Speck, *preprint* '09.

Bianchi initial data

Let

- G be a 3–dimensional Lie group,
- $5/2 < \mu \in \mathbb{R}$,
- $\bar{g}$ and $\bar{k}$ be a left invariant Riemannian metric and symmetric covariant 2–tensor field on G respectively,
- $\bar{f} \in \bar{\mathfrak{D}}^{\infty}_{\mu}(TG)$ be left invariant and non-negative.

Then $(G, \bar{g}, \bar{k}, \bar{f})$ are referred to as **Bianchi initial data** for the Einstein–Vlasov system, assuming they satisfy the constraints.

Future stability of spatially locally homogeneous solutions

Let $(G, \bar{g}_{\rm bg}, \bar{k}_{\rm bg}, \bar{f}_{\rm bg})$ be Bianchi initial data for the Einstein–Vlasov system, where

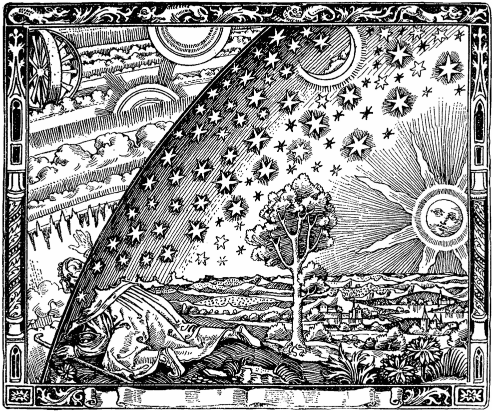
- the universal covering group of G is not isomorphic to $\mathrm{SU}(2)$,
- $\mathrm{tr}\,\bar{k}_{\rm bg} = \bar{g}^{ij}_{\rm bg}\bar{k}_{{\rm bg},ij} > 0$.

Assume that there is a cocompact subgroup Γ of the isometry group of the initial data. Let Σ be the compact quotient. Then there is an $\epsilon > 0$ such that if $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ are initial data satisfying

$$\|\bar{g} - \bar{g}_{\rm bg}\|_{H^5} + \|\bar{k} - \bar{k}_{\rm bg}\|_{H^4} + \|\bar{f} - \bar{f}_{\rm bg}\|_{H^4_{{\rm Vl},\mu}} \leq \epsilon,$$

then the maximal Cauchy development of $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ is future causally geodesically complete.

What is the shape of the universe?



Minkowski space; non-silent causality

Let $\gamma(t) = (t, 0, 0, 0)$. Then γ is an observer in Minkowski space. How much of the $t = 0$ hypersurface does γ see?

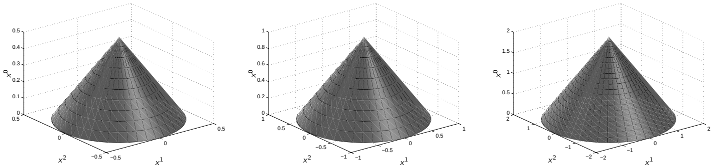


Figure: The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$, $t = 1$ and $t = 2$.

de Sitter space; silent causality

Consider the metric

$$g = -dt^2 + e^{2t}\bar{g}.$$

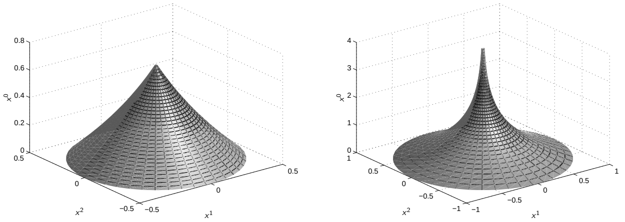


Figure: The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$ and for all t.

Question, topology

Assume that

- the observational data indicate that, to our past, the universe is well approximated by one of the standard models,
- interpreting the data in this model, we only have information concerning the universe for $t \geq t_0$,
- there is a big bang,
- analogous statements apply to all observers in the universe (with the same t_0).

The question is then: what conclusions are we allowed to draw concerning the global spatial topology of the universe?

Ingredients

Assume we are given

- a standard model, characterised by an existence interval I, a scale factor a etc.,
- a $t_0 \in I$, which represents the time to the future of which we wish the approximation to be valid,
- an $l \in \mathbb{N}$, specifying the norm with respect to which we measure proximity to the standard model,
- an $\epsilon > 0$, characterising the size of the distance,
- a closed 3-manifold Σ.

Construction

There is a solution (M, g, f) with the following properties:

- (M, g, f) is a maximal Cauchy development,
- (M, g) is future causally geodesically complete,
- there is a Cauchy hypersurface, say $\bar{S}$, in (M, g), diffeomorphic to Σ,
- given an observer γ in (M, g), there is a neighbourhood, say U, of

  $$J^-(\gamma) \cap J^+(\bar{S})$$

  such that the solution in U is ϵ-close to the standard model in a solid cylinder of the form $[t_0, \infty) \times \bar{B}_R(0)$,
- all timelike geodesics in (M, g) are past incomplete,
- the solution is stable with these properties.

Further reading...