Generalisations To Infinity In Finitary 2-Representation Theory

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Categorification

- Basic ethos: create a ‘higher level’ n-category that encodes a structure of interest.
- Use the extra machinery of the n-category to derive new information about the lower level structure.
- First known example: [Kho00], categorified Jones polynomials (invariants in knot theory) as the Euler characteristics of complexes of modules (i.e. elements of some category).
- Problem: higher level structures are more complicated and harder to study.
- Solution: Representation Theory.

2-Representation Theory

- This leads to '2-Representation Theory' - that is, the representation theory of 2-categories.
- Various authors have approached this in different ways:
  - Etingof–Ostrik: 2-representations of tensor categories.
  - Khovanov–Lauda and Rouquier: 2-representations in Lie theory.
  - Mazorchuk–Miemietz: finitary 2-representation theory.

Talk Structure

- I will first introduce finitary 2-categories and their 2-representations.
- A particular focus on ‘external vs. internal’ results.
- Second part of the talk will introduce wide finitary 2-categories.
- Include results regarding internal 2-representations in this setup.

Finitary Categories

- Basic idea: A finitary (2-)category is a (2-)category with a significant degree of additive/linear structure.
- Let $\mathbb{k}$ be an algebraically closed field.
- Always working with strict 2-categories in this talk.
- Mostly drawn from initial papers by Mazorchuk–Miemietz.

Definition

A *finitary category* is an additive $\mathbb{k}$-linear idempotent complete category with:

- finitely many isomorphism classes of indecomposable objects;
- finite dimensional hom-spaces.

- We denote the 2-category of finitary categories, additive $\mathbb{k}$-linear functors and natural transformations by $\mathfrak{A}_{\mathbb{k}}^{f}$.

Basic Example

Consider the category $\mathrm{Rep}_{\Bbbk}^{\mathrm{fd}}(G)$ of finite dimensional representations of a finite group G over $\Bbbk$.

- A module category over the group algebra, so additive and $\Bbbk$-linear.
- Schur's Lemma shows that the category is idempotent complete.
- Up to isomorphism, set of irreducible (i.e. indecomposable) representations in bijection with conjugacy classes of G, and so finitely many isoclasses of indecomposable objects.
- Hom-spaces between finite dimensional vector spaces are finite dimensional.

Finitary 2-Categories

Definition

A *finitary 2-category* is a 2-category $\mathscr{C}$ with finitely many objects such that:

- For any two objects $\mathtt{i}, \mathtt{j}$ of $\mathscr{C}$, $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is a finitary category.
- Horizontal composition is biadditive and $\Bbbk$-linear.
- For each object $\mathtt{i}$, the identity 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable.

Internal Adjunctions

- Often desire extra structure for finitary 2-categories.

Definition

A finitary 2-category $\mathscr{C}$ is a *quasi-fiat* 2-category if it has internal adjoints. More formally, for each 1-morphism $F : \mathtt{i} \to \mathtt{j}$ of $\mathscr{C}$ there is a 1-morphism $F^* : \mathtt{j} \to \mathtt{i}$ along with 2-morphisms $\epsilon : FF^* \to \mathbb{1}_{\mathtt{j}}$ and $\eta : \mathbb{1}_{\mathtt{i}} \to F^*F$ which obey certain axioms.

- ϵ and η follow the standard axioms for an adjunction.

Definition

A quasi-fiat 2-category $\mathscr{C}$ is *fiat* if $F^{**} \cong F$ for any 1-morphism F.

2-Representations

- What are our analogues of vector spaces in classical representation theory?
- Answer: Finitary or abelian 2-representations.
- Let $\mathbf{Ab}_{\Bbbk}$ denote the 2-category of $\Bbbk$-linear abelian categories, $\Bbbk$-linear additive functors and natural transformations.

Definition

Let $\mathscr{C}$ be a finitary 2-category.

- A *finitary* 2-representation of $\mathscr{C}$ is a strict 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$.
- A *abelian* 2-representation of $\mathscr{C}$ is a strict 2-functor $\mathbf{M} : \mathscr{C} \to \mathbf{Ab}_{\Bbbk}$.

2-Representations in Detail

- In more detail:
- Let $\mathscr{C}$ be a finitary 2-category. A *finitary* 2-representation $\mathbf{M}$ of $\mathscr{C}$ consists of the following:
  - For each object $\mathtt{i}$ of $\mathscr{C}$, a finitary category $\mathbf{M}(\mathtt{i})$.
  - For each 1-morphism $F : \mathtt{i} \to \mathtt{j}$ of $\mathscr{C}$, a $\mathbb{k}$-linear additive functor $\mathbf{M}(F) : \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{j})$.
  - For each 2-morphism $\alpha : F \to G$ of $\mathscr{C}$, a natural transformation $\mathbf{M}(\alpha) : \mathbf{M}(F) \to \mathbf{M}(G)$.
- Two 2-representations $\mathbf{M}$ and $\mathbf{N}$ are *equivalent* if there exists a 2-natural transformation from $\mathbf{M}$ to $\mathbf{N}$ that induces an equivalence of categories for every $\mathtt{i}$.

Basic Example: Principal 2-Representations

- The most straightforward example of a finitary 2-representation is the principal 2-representation $\mathbf{P}_{\mathtt{i}}$ for some object $\mathtt{i}$ of $\mathscr{C}$:
  - $\mathbf{P}(\mathtt{j}) = \mathscr{C}(\mathtt{i}, \mathtt{j})$;
  - For $F \in \mathscr{C}(\mathtt{j}, \mathtt{k})$, $\mathbf{P}(F) = F \circ - : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
  - For a 2-morphism $\alpha : F \to G$ and a 1-morphism $H \in \mathbf{P}(\mathtt{j}) = \mathscr{C}(\mathtt{i}, \mathtt{j})$,

$$\mathbf{P}(\alpha)_G = \alpha \circ_H \mathrm{id}_H : FH \to GH.$$

Simple Transitive 2-Representations

- In modular representation theory, simple modules (a.k.a. irreducible representations) play an important role.
- Equivalent concept is *simple transitive 2-representations*.
- *Simple:* the (finitary) 2-representation has no proper $\mathscr{C}$-stable ideals (analogous to simple rings, simple modules).
- *Transitive:* for any $X \in \mathbf{M}(\mathtt{i})$, $Y \in \mathbf{M}(\mathtt{j})$, there exists some 1-morphism $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ such that Y is a direct summand of $\mathbf{M}(F)(X)$.
- Slogan: Any object in a transitive 2-representation generates the whole 2-representation (up to equivalence) under the action of $\mathscr{C}$.
- Legitimate analogue of simple modules, e.g. there exists a weak Jordan-Hölder Theorem.

Internal Vs. External

- In representation theory, we often want to reduce ‘external’ problems to ‘internal’ ones.
- Classical example: in characteristic 0, for a group G, there are a lot of vector spaces to try constructing irreducible representations on.
- But representations are determined up to isomorphism by their characters, which are in bijection with conjugacy classes of G.
- Reduced ‘external’ problem of classifying representations to ‘internal’(ish) problem of finding a known number of class functions on G.
- A lot of powerful theorems and concepts in 2-representation theory do similar things.
- I will detail two examples: cell 2-representations, and comodule 2-representations.

Cells in 2-Categories

- Based on Green's cells in semigroups from 1951.
- Given (isomorphism classes of) indecomposable 1-morphisms F and G of a finitary 2-category $\mathscr{C}$, say $F \leq_{\mathscr{L}} G$ (resp. $F \leq_{\mathscr{R}} G$, $F \leq_{\mathscr{J}} G$) if there exists some 1-morphism H with G a direct summand of HF (FH, HFK resp.).
- Equivalence classes of these pre-orders are $\mathscr{L}$*-cells* (resp. $\mathscr{R}$*-cells*, $\mathscr{J}$*-cells*).
- Useful fact: Given an $\mathscr{L}$-cell $\mathcal{L}$, every $X \in \mathcal{L}$ has the same source object.

Cell 2-Representations

- We will define a specific type of simple transitive 2-representation that categorify cell modules.
- Let $\mathscr{C}$ be a finitary 2-category and let $\mathcal{L}$ be an left cell of $\mathscr{C}$ with domain $\mathtt{i}$. We define a 2-representation $\mathbf{N}_{\mathcal{L}}$ of $\mathscr{C}$ as follows:
  - $\mathbf{N}_{\mathcal{L}}(\mathtt{j})$ is the full subcategory of $\mathscr{C}(\mathtt{i},\mathtt{j})$ generated by $\mathrm{add}\{FX | X \in \mathcal{L}, X : \mathtt{i} \to \mathtt{k}, F \in \mathscr{C}(\mathtt{k},\mathtt{j})\}$.
  - The action of 1- and 2-morphisms is the same as in the principal 2-representation $\mathbf{P}_{\mathtt{i}}$.

Proposition (Mazorchuk, Miemietz '16)

$\mathbf{N}_{\mathcal{L}}$ *has a unique simple transitive quotient 2-representation* $\mathbf{C}_{\mathcal{L}}$*, called the cell 2-representation associated to* $\mathcal{L}$*.*

The First Big Theorem

- Cell 2-representations are 'internal' structures - entirely defined by information from $\mathscr{C}$.
- When can we use them to classify 'external' 2-representations?
- One example is *strongly regular* fiat 2-categories, which are fiat 2-categories where the cells form a particularly pleasant structure.

Theorem (Mazorchuk, Miemietz '16)

Any simple transitive 2-representation of a strongly regular fiat 2-category is equivalent to a cell 2-representation.

Coalgebra 1-Morphisms

- Given a fiat 2-category $\mathscr{C}$ with transitive 2-representation $\mathbf{M}$, we use the notation $\mathscr{M} = \coprod_{\mathtt{j}\in\mathscr{C}} \mathbf{M}(\mathtt{j})$ and $\mathscr{C}_{\mathtt{i}} = \coprod_{\mathtt{j}\in\mathscr{C}} \mathscr{C}(\mathtt{i},\mathtt{j})$.
- Let $X \in \mathbf{M}(\mathtt{i})$. There is an 'evaluation at X' functor $\mathrm{ev}_X : \mathscr{C}_{\mathtt{i}} \to \mathscr{M}$, given by $\mathrm{ev}_X(F) = \mathbf{M}(F)X$, $\mathrm{ev}_X(\alpha) = \mathbf{M}(\alpha)_X$.
- We would like this functor to have a left adjoint. To do this, we need a larger 'enveloping' 2-category for $\mathscr{C}$.

Abelianisation I

Definition

Let $\mathscr{B}$ be an additive category. We define its *injective Freyd abelianisation* $\underline{\mathscr{B}}$ as follows:

- Objects of $\underline{\mathscr{B}}$ are morphisms of $\mathscr{B}$.
- Morphisms are commutative diagrams
$$\begin{CD} X @>f>> Y \\ @VgVV @VVhV \\ X' @>f'>> Y' \end{CD}$$
modulo 'homotopy' - i.e. modulo those diagrams where there exists a morphism $q : Y \to X'$ such that $g = qf$.

Abelianisation I

Theorem (Freyd '66)

If $\mathscr{B}$ has weak kernels, then $\underline{\mathscr{B}}$ is an abelian category, and any additive functor $F : \mathscr{B} \to \mathscr{D}$, where $\mathscr{D}$ is an abelian category, extends uniquely to a left exact functor $\underline{F} : \underline{\mathscr{B}} \to \mathscr{D}$. In addition, $\mathscr{B}$ embeds into $\underline{\mathscr{B}}$ as the full subcategory of injective objects.

- For our purposes, can extend the definition of abelianisation to finitary 2-categories and 2-representations.

Lemma (Mackaay,Mazorchuk,Miemietz,Tubbenhauer '16)

The left exact functor $\underline{\mathrm{ev}_X} : \underline{\mathscr{C}_{\mathtt{i}}} \to \underline{\mathscr{M}}$ has a left adjoint $[X, -] : \underline{\mathscr{M}} \to \underline{\mathscr{C}_{\mathtt{i}}}$.

The Second Big Theorem

Lemma (MMMT)

$[X, X]$ has the structure of a coalgebra 1-morphism (i.e. it has counit and comultiplication 2-morphisms).

- The category of comodule 1-morphisms over $[X, X]$, $\mathbf{comod}_{\underline{\mathscr{C}}}([X, X])$ can be given a natural structure of an abelian $\mathscr{C}$ 2-representation.
- Let $\mathbf{inj}_{\underline{\mathscr{C}}}([X, X])$ denote the sub-2-representation of $\mathbf{comod}_{\underline{\mathscr{C}}}([X, X])$ generated by its injective objects.

Theorem (MMMT '16)

There is an equivalence of 2-representations of $\mathscr{C}$ between $\underline{\mathbf{M}}$ and $\mathbf{comod}_{\underline{\mathscr{C}}}([X, X])$, which restricts to an equivalence of 2-representations between $\mathbf{M}$ and $\mathbf{inj}_{\underline{\mathscr{C}}}([X, X])$.

Finiteness Conditions Revisited

- Let's recall the finiteness conditions in the definition of a finitary 2-category:
  - Finitely many objects;
  - Finitely many isomorphism classes of indecomposable 1-morphisms;
  - Finite dimensional hom-spaces of 2-morphisms
- How can we relax these restrictions?

Wide Finitary Categories

Definition

A category $\mathscr{C}$ is *wide finitary* if it is an additive $\mathbb{k}$-linear Krull-Schmidt category with countably many isomorphism classes of indecomposable objects and where the morphism sets are $\mathbb{k}$-vector spaces of countable dimension. Define the 2-category $\mathfrak{A}_{\mathbb{k}}^{wf}$ to have as objects wide finitary categories, as 1-morphisms $\mathbb{k}$-linear additive functors, and as 2-morphisms natural transformations.

- Why Krull-Schmidt?
  - Retains any 1-morphism being a finite sum of indecomposable 1-morphisms.
  - A lot of theory (e.g. being able to define cell 2-representations) heavily uses endomorphism rings of indecomposable 1-morphisms being local.

Basic Example

- Consider $\mathrm{Rep}_{\Bbbk}^{\mathrm{fd}}(\mathfrak{sl}_2)$, the category of finite dimensional representations of $\mathfrak{sl}_2$. It is a standard result that there is a unique indecomposable representation of dimension n for each $n \in \mathbb{Z}^+$.
- Consequently, there are infinitely many isomorphism classes of indecomposable objects.
- However, the hom-spaces retain a sufficiently pleasant structure that the category is at least wide finitary.

Locally Wide Finitary 2-Categories

Definition

A 2-category $\mathscr{C}$ is *locally wide finitary* if:

- $\mathscr{C}$ has countably many objects.
- For any objects $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, $\mathscr{C}(\mathtt{i}, \mathtt{j}) \in \mathfrak{A}_{\Bbbk}^{wf}$.
- Horizontal composition is biadditive and $\Bbbk$-linear.
- For each object $\mathtt{i} \in \mathscr{C}$, the identity 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable.

Definition

Let $\mathscr{C}$ be a locally wide finitary 2-category. A *wide finitary* 2-representation of $\mathscr{C}$ is a strict 2-functor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{\mathsf{wf}}$.

Things Get Complicated

- Certain concepts do generalise to this setting - e.g. locally wide (quasi-)fiat 2-categories, (simple) transitive 2-representations, cells, cell 2-representations, ideals of 2-categories and 2-representations.
- However, there are a lot of concepts from (locally) finitary 2-representation theory that break when naïvely generalised.
- The remainder of this talk will focus on fixing these generalisation problems with regards to the comodule 2-representations.

Breaking Freyd Abelianisation

- Reminder: for injective Freyd abelianisation to produce an abelian (2-)category, we need weak kernels.
- Problem: in general, the hom-categories of locally wide finitary 2-categories do not have weak kernels.
- Need to find a more general abelianisation process.

Adelman Abelianisation

- Solution given by Adelman in a 1973 paper.

Definition

Let $\mathscr{C}$ be an additive category. The *Adelman abelianisation* $\widehat{\mathscr{C}}$ is a category with:

- Objects are pairs of morphisms $X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_3$.
- Morphisms are commutative diagrams

$$\begin{array}{ccccc} X_2 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_3 \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_3} \\ Y_2 & \xrightarrow{g_1} & Y_1 & \xrightarrow{g_2} & Y_3 \end{array}$$

modulo 'homotopy' - i.e. modulo those diagrams where there exist morphisms $q_1 : X_1 \to Y_2$ and $q_2 : X_3 \to Y_1$ such that $g_1 q_1 + q_2 f_2 = h_1$.

Adelman Abelianisation

Theorem (Adelman '73)

If $\mathscr{C}$ is an additive category, then $\widehat{\mathscr{C}}$ is an abelian category. Any additive functor $F : \mathscr{C} \to \mathscr{D}$, where $\mathscr{D}$ is an abelian category, extends uniquely to an exact functor $\widehat{F} : \widehat{\mathscr{C}} \to \mathscr{D}$.

- We can extend the definition of abelianisation to locally wide finitary 2-categories and 2-representations.
- However, it turns out we need to do more.

The Problem

- Reminder: in finitary case, we construct the coalgebra 1-morphism using the left adjoint of the (abelian) evaluation functor $\underline{\mathrm{ev}_X} : \underline{\mathscr{C}_{\mathtt{i}}} \to \underline{\mathscr{M}}$.
- But in the wide finitary case, we have no guarantee that $\widehat{\mathrm{ev}_X} : \widehat{\mathscr{C}_{\mathtt{i}}} \to \widehat{\mathscr{M}}$ has such an adjoint.
- There is a solution: pro-(2-)categories.

Pro-2-Categories

- Pro-categories (and their dual, ind-categories) were first introduced by Grothendieck and Verdier in the depths of SGA (specifically [GV72]).
- Roughly, the pro-category $\mathrm{Pro}(\mathscr{C})$ of a category $\mathscr{C}$ is the free completion of $\mathscr{C}$ under cofiltered limits.
- Can (carefully) generalise the definition to 2-categories (taking pro-categories of the hom-categories).
- Important result from SGA:

Proposition (Grothendieck, Verdier '72)

A functor $F : \mathscr{C} \to \mathscr{D}$ is right exact if and only if $\mathrm{Pro}(F) : \mathrm{Pro}(\mathscr{C}) \to \mathrm{Pro}(\mathscr{D})$ has a left adjoint.

The Final Big Result

- It follows that $\mathrm{Pro}(\widehat{\mathrm{ev}_X}) : \mathrm{Pro}(\widehat{\mathscr{C}_{\mathtt{i}}}) \to \mathrm{Pro}(\widehat{\mathscr{M}})$ has a left adjoint, which we denote $[X, -]$.
- The image of $\mathscr{M}$ under $[X, -]$ has the structure of a 2-representation of $\mathscr{C}$, which we notate as $[X, \mathbf{M}]$.

Theorem (M '22)

There is an equivalence of 2-representations of $\mathscr{C}$ between $\mathbf{M}$ and $[X, \mathbf{M}]$.

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