Tail bounds for detection times in dynamic hyperbolic graphs

Dieter Mitsche
(Joint work with Marcos Kiwi and Amitai Linker)

Instituto Superior Técnico, Lisboa

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Goal

Find models that exhibit characteristic properties of "real world networks/complex networks"

Example of networks:	*Power grid* *Internet* *Social networks* *Biological interaction networks* *...*
Typical properties:	*Sparse* *Heterogeneous* *Locally dense (exhibit clustering phenomena)* *Small world* *Navigable* *Scale free (with exponent between 2 and 3)* *...*

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	Locally dense (exhibit clustering phenomena)
	Small world
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	Scale free (with exponent between 2 and 3)
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Also, we want models that are susceptible to mathematical analysis!

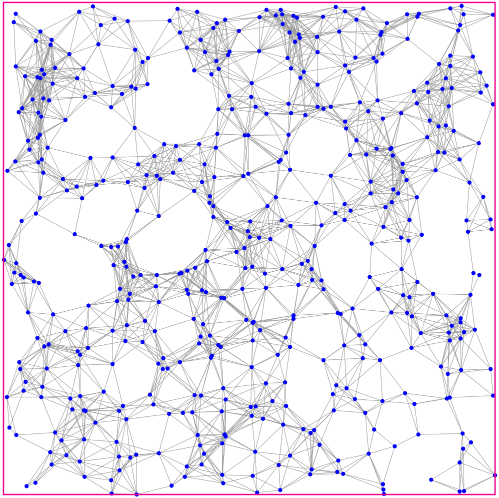
First model: random geometric graphs

Define $G = (V, E)$ as follows:

- Choose location of each $v \in V$ uniformly and independently in $[0,1]^2$ (or Poisson process with intensity n).
- Let $uv \in E$ iff Euclidean distance between u and v is at most r.

Note: No power law degree distribution, no small diameter in general ...

Example of random geometric graphs



$r = 0.09$

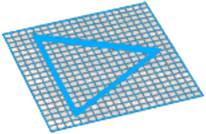
$n = 500$ points

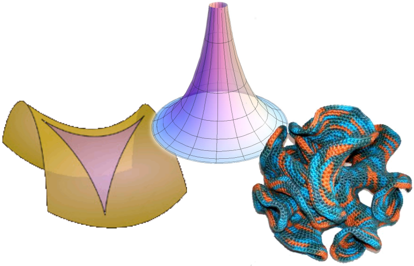
Alternative model: Random hyperbolic graphs (RHGs)

Introduced by Krioukov, Papadopoulos, Kitsak, Vahdat, Boguñá [Phys. Rev. '10]
Like random geometric graphs but where the underlying space instead of being Euclidean is Hyperbolic.

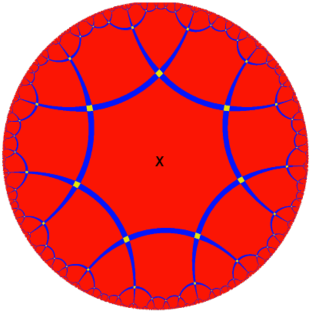
Hyperbolic plane $\mathbb{H}^2$

Euclidean plane $\mathbb{R}^2$





Poincaré disk model of $\mathbb{H}^2$



[Rendered with KaleidoTile by J. Weeks]

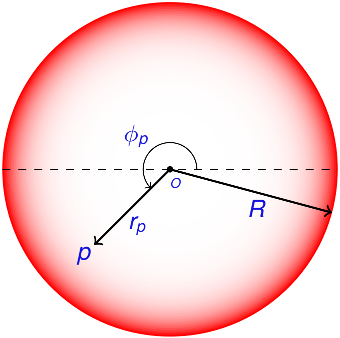
- $\mathbb{H}^2$ is represented as an open disk D.
- Blue curves are geodesics (arcs of circles perpendicularly incident to D).
- Each heptagon has the same area.
- Points in ∂D are at infinite distance from X.
- Points at (Euclidean) distance y from X are at hyperbolic distance r from X where

$$r = \ln \frac{1+y}{1-y}.$$

Space expands at exponential rate!

Continuous analogue of regular trees

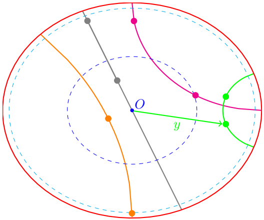
Native representation of $\mathbb{H}^2$



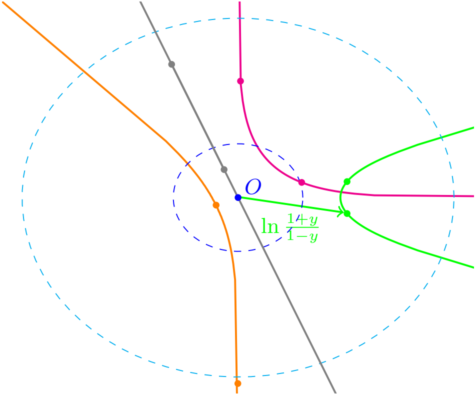
$B_O(R)$: Ball of radius R centered at origin O with perimeter $2\pi \sinh R = \Theta(e^R)$.

- $\mathbb{H}^2$ is represented as $\mathbb{R}^2$.
- A point p is represented in polar coordinates.
- r_p is the hyperbolic distance between p and O

Poincaré vs native representation of $\mathbb{H}^2$



Poincaré model



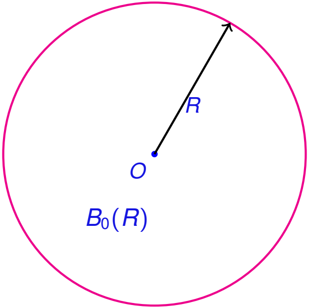
Native representation

Formal definition of RHG model: $G_{\alpha,\nu}(n)$

(Gugelmann, Panagiotou, Peter [ICALP'12])

Model parameters:
$\alpha, \nu \in \mathbb{R}_+, n \in \mathbb{N}$

Set $R := 2 \ln \frac{n}{\nu}$.



Choose an n-node graph $G = (V, E)$ as follows (or Poisson model with intensity n):

- Each $v \in V$ uniformly and independently in $B_O(R)$.
- $uv \in E$ iff $u \in B_v(R)$.

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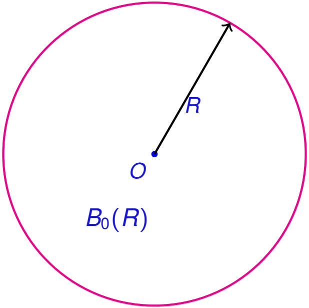


Choose an n-node graph $G = (V, E)$ as follows (or Poisson model with intensity n):

- Each $v \in V$ so $\phi_v \sim \mathrm{Unif}[0, 2\pi)$ independent of r_v with density:

$$f(r) := \frac{\alpha}{C_{\alpha,R}} \sinh(\alpha r) \approx \alpha e^{-\alpha(R-r)} \qquad \text{if } 0 \le r < R \text{ and } 0 \text{ otherwise.}$$

  (Here, $C_{\alpha,R}$ is a normalizing constant).

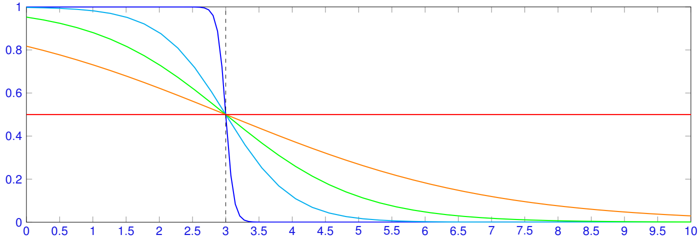
- $uv \in E$ iff $u \in B_v(R)$.

Soft version

Incorporates a temperature T and a *probability of connecting* u and v:

$$p(d) := \frac{1}{1 + e^{\frac{1}{2T}(d-R)}}$$

where $d := d_{\mathbb{H}^2}(u, v)$ is the (hyperbolic) distance between $u, v \in \mathbb{H}^2$.

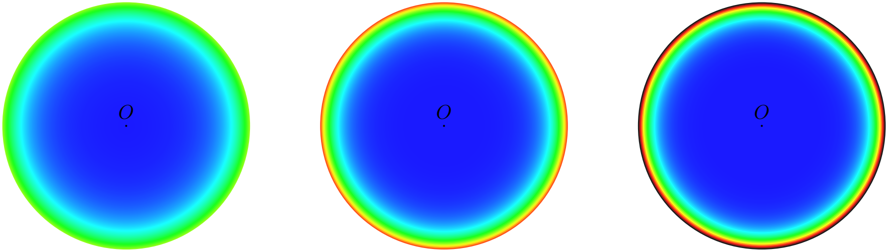


$+\infty \approx T > T > T > T > T \approx 0$

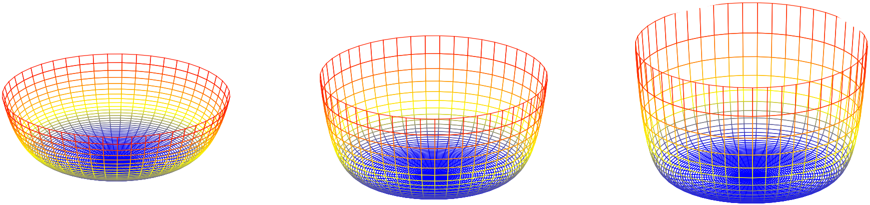
$R = 3.0.$

Pdf of (r_v, ϕ_v) and its heat plot

(Colder colors correspond to smaller density)



$\alpha = \frac{1}{2}$ $\alpha = \frac{3}{4}$ $\alpha = 1$



Calculating distances

Hyperbolic distance from v to origin O, ... easy! Just r_v.

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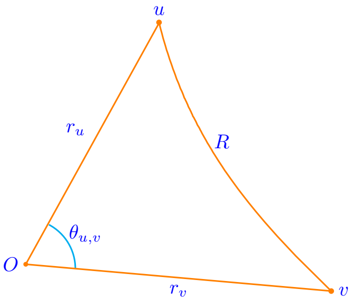
In general, use hyperbolic law of cosines

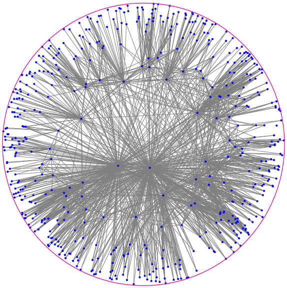
$$\cosh(d) = \cosh(r_u)\cosh(r_v) - \sinh(r_u)\sinh(r_v)\cos(\phi_{u,v}).$$



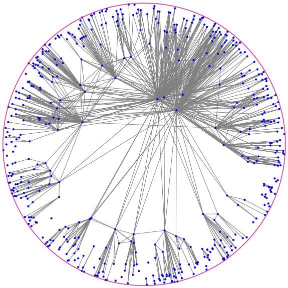
Lemma: $\phi_{u,v} \le \theta_R(r_u, r_v) \iff d_{\mathbb{H}^2}(u,v) \le R$.

Examples of RHGs

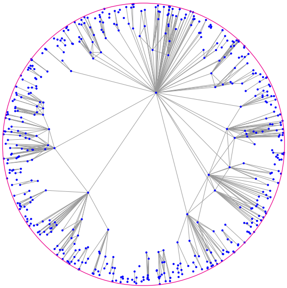
($\nu = 1$ fixed, $n = 500$)



$\alpha = 0.60$



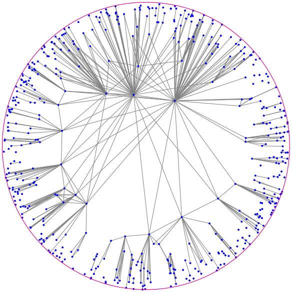
$\alpha = 0.75$



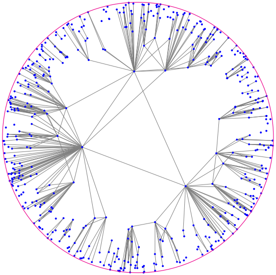
$\alpha = 0.90$

Examples of RHGs

($\alpha = \frac{3}{4}$ fixed, $n = 500$)



$\nu = 0.50$



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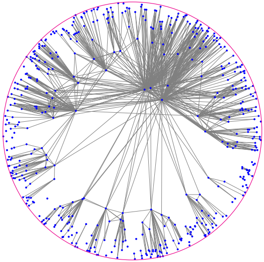
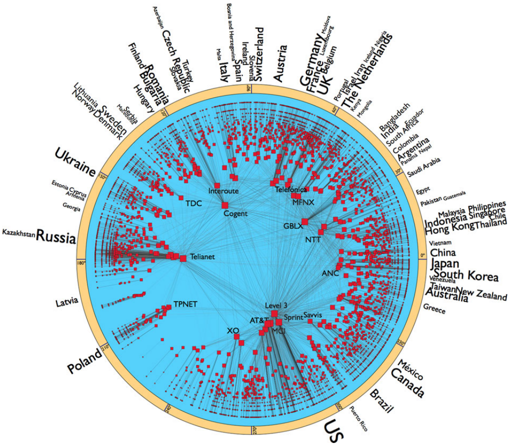


$\nu = 1.00$

What drew attention...

Mapping of Internet's Autonomous Systems (ASs, 2009)



[From Boguña, Papadopoulus, Krioukov (Nat. Comm. '10)]

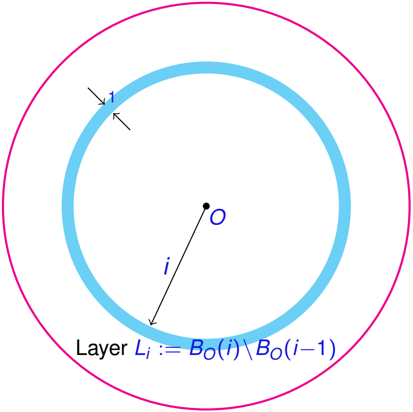
Data set:

- $23,752$ ASs
- $58,416$ links
- Average degree 4.92

"Maximum Likelihood" fit:

- $\alpha = 0.55$
- $R = 27$
- Temperature $T = 0.69$

Analysis of RHGs - vertices per layer
(measure centered balls)



Calculations yield[GPP'12]

$$\mu(L_i) \cong \frac{\mu(B_O(i))}{1-e^{-\alpha}}.$$
$$\mu(B_O(i)) \cong e^{-\alpha(R-i)}.$$

Most vertices close to the boundary of $B_O(i)$

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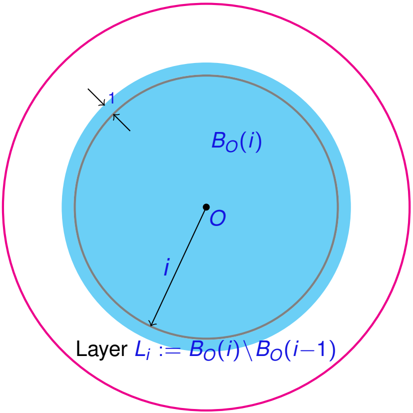

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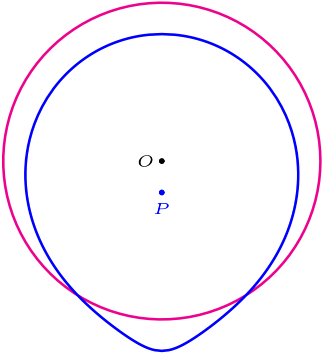
Vertex degrees
(measure of non-centered balls)



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$$\mu(B_P(R)) = C_\alpha e^{-\frac{r_P}{2}}(1 + o(e^{-(\alpha-\frac{1}{2})r_P})).$$

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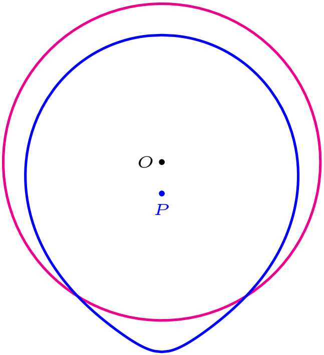

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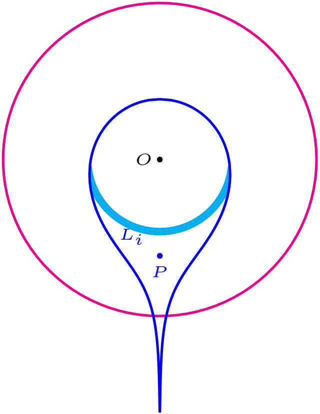
Thus,

$$\deg(P) = \begin{cases} O(\ln n) \text{ (no concentration)}, & \\ \quad \text{if } r_P \geq R - 2\ln R + O(1), & \\ \Theta(ne^{-\frac{r_P}{2}}) \text{ (concentrated)}, & \\ \quad \text{otherwise.} & \end{cases}$$

Location of neighbors of a vertex



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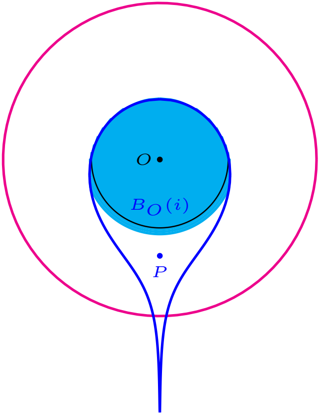

Calculations yield

$$\mu(B_P(R) \cap L_i) = \Theta(e^{-(\alpha-\frac{1}{2})(R-i)} e^{-\frac{1}{2}(R-r_P)})$$
$$= (1 - e^{-(\alpha-\frac{1}{2})})(1 + o(1))\mu(B_P(R) \cap B_O(i-1)).$$

As a function of i grows like $e^{-\alpha i}$.

So, P has:

- more neighbors towards $\partial B_O(R)$
- const. fraction of neighbors "near" $\partial B_O(R)$

Consequences/known results

- Power law degree distribution with exponent $2\alpha + 1 \in (2, 3)$.
- Average degree constant
- $\Theta(n)$ isolated vertices
- There is a giant component $\Theta(n)$ [BFM, EJC'15; FM, AAP'17]
- and much more ... !

The dynamic RHG model

The dynamic RHG model

- Vertices move, maintaining spatial distribution invariant

Our choice: Every vertex moves independently in $B_O(R)$ according to a diffusion process with angular and radial component

$$\boxed{\Delta_h := \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\alpha}{2}\frac{1}{\tanh(\alpha r)}\frac{\partial}{\partial r} + \frac{1}{2}\sigma_\theta^2(r)\frac{\partial^2}{\partial \theta^2}}$$

with **reflecting boundary** at $\partial B_O(R)$.

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with **reflecting boundary** at $\partial B_O(R)$. For a fixed time t, the edge $u_t v_t$ is there iff $d(u_t, v_t) \leq R$.

Choice of generator

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with $\beta > 0$ being a new parameter

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- For larger α, stronger drift towards $\partial B_O(R)$
- For larger β, less angular speed
- Far from the origin we use

$$\Delta_h :\approx \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\alpha}{2}\frac{\partial}{\partial r} + 2e^{-2\beta r}\frac{\partial^2}{\partial \theta^2}$$

Inspiration/related work

Peres, Sinclair, Sousi, Stauffer (2012) consider mobile geometric graphs in $\mathbb{R}^d$ with Brownian motion for each vertex.

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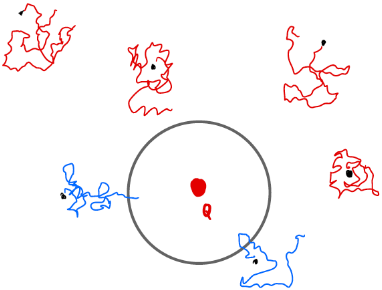
T_{det} in mobile random geometric graphs

Idea: Vertices detecting by time t define thinned Poisson process with point measure

$$\mathbb{P}_{x_0}(T_{det} \le t)dx_0$$



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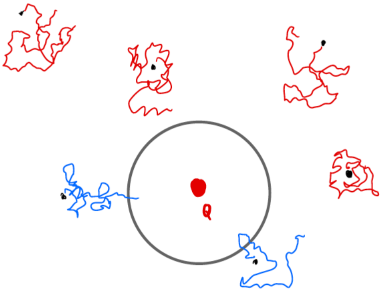
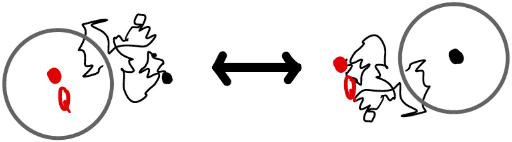


and thus

$$\mathbb{P}(T_{det} > t) = \exp\left(-\int_{\mathbb{R}^d} \mathbb{P}_{x_0}(T_{det} \leq t)dx_0\right)$$

T_{det} in mobile random geometric graphs

x_0 detects Q with the same probability as Q detects x_0



Thus

$$\int_{\mathbb{R}^d} \mathbb{P}_{x_0}(T_{det} \leq t)dx_0 = \mathbb{E}\left(\mathrm{vol}\, W(t)\right) = \begin{cases} \Theta(\sqrt{t}) & \text{if } d = 1 \\ \Theta(\frac{t}{\log t}) & \text{if } d = 2 \\ \Theta(t) & \text{if } d \geq 3 \end{cases}$$

with $W(t)$ the Wiener sausage at time t

Our result: tail bounds of detection time for RHGs

Theorem (Kiwi, Linker, M. '22+:)

Let $\alpha \in (\frac{1}{2}, 1]$, $\beta > 0$, $t := t(n)$, and assume that particles move according to the generator Δ_h. Then, the following hold:

1. *For $\beta \leq \frac{1}{2}$, if $t = \Omega((e^{\beta R}/n)^2) \cap O(1)$, then $\mathbb{P}(T_{det} \geq t) = \exp\left(-\Theta(ne^{-\beta R}\sqrt{t})\right)$.*
2. *For $\beta \leq \frac{1}{2}$ and $t = \Omega(1)$ the tail exponent depends on the relation between α and 2β as follows:*
   1. *For $\alpha < 2\beta$, if $t = O(e^{\alpha R})$, then $\mathbb{P}(T_{det} \geq t) = \exp\left(-\Theta(ne^{-\beta R}t^{\frac{\beta}{\alpha}})\right)$.*
   2. *For $\alpha = 2\beta$, if $t = O(e^{\alpha R}/(\alpha R))$, then $\mathbb{P}(T_{det} \geq t) = \exp\left(-\Theta(ne^{-\beta R}\sqrt{t \log t})\right)$.*
   3. *For $\alpha > 2\beta$, if $t = O(e^{2\beta R})$, then $\mathbb{P}(T_{det} \geq t) = \exp\left(-\Theta(ne^{-\beta R}\sqrt{t})\right)$.*
3. *For $\beta > \frac{1}{2}$, if $t = \Omega(1) \cap O(e^{\alpha R})$, then $\mathbb{P}(T_{det} \geq t) = \exp\left(-\Theta(t^{\frac{1}{2\alpha}})\right)$.*

(lower bounds on t correspond to expectation, upper bounds on t are s.t. for larger t same probability to have empty graph)

Angular movement only

To understand the proof, consider first **angular** movement only or **radial** movement only

$$\Delta_{ang} \; := \; \frac{1}{2}\frac{1}{\sinh^2(\beta r)}\frac{\partial^2}{\partial\theta^2} \approx 2e^{-2\beta r}\frac{\partial^2}{\partial\theta^2}$$

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In t time units a vertex in layer r moves (in expectation) $\Theta(\sqrt{t}e^{-\beta r})$ radians.

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Define then

$$\mathcal{D}_t = \{x_0 \in B_O(R), |\theta_0| \leq \sqrt{t}e^{-\beta r}\}$$

the set of points detecting Q by time t

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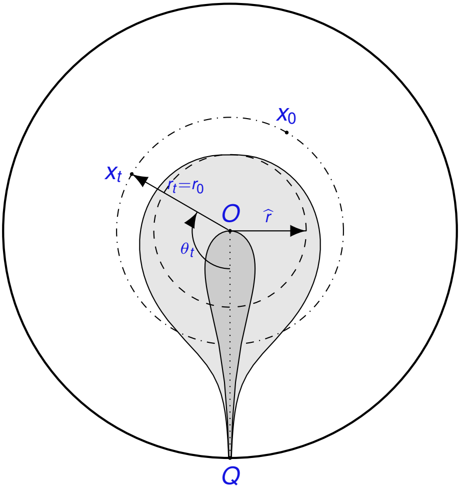
$$\mathcal{D}_t = \{x_0 \in B_O(R), |\theta_0| \leq \sqrt{t}e^{-\beta r}\} \supseteq B_O(\tfrac{1}{2\beta}\log t)$$

the set of points detecting Q by time t

the detecting set $\mathcal{D}_t$



Since in $\mathcal{D}_t$ the detection probability is $\Omega(1)$, we have

$$\int_{\mathcal{D}_t} \mathbb{P}_{x_0}(T_{det} > t) d\mu(x_0) \; = \; \Omega(\mu(\mathcal{D}_t))$$

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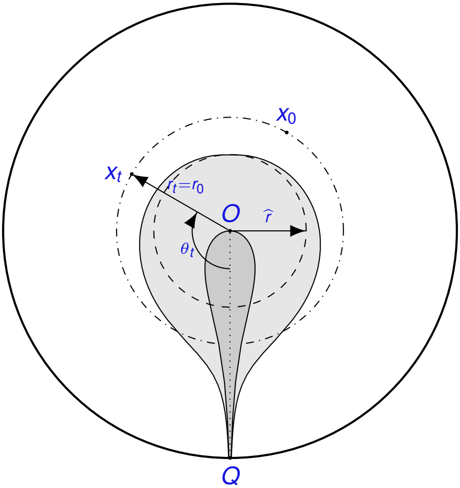

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Radial movement only

$$\Delta_{rad} := \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\alpha}{2}\frac{1}{\tanh(\alpha r)}\frac{\partial}{\partial r} \approx \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\alpha}{2}\frac{\partial}{\partial r}$$

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In t time units a vertex reaches (in expectation) radius $R - \frac{1}{\alpha}\log t$. At $\partial B_Q(R)$ this corresponds to an angle $\frac{1}{n}t^{\frac{1}{2\alpha}}$

Radial movement only

Defining a new set

$$\mathcal{D}_t = \{x_0 \in B_O(R), |\theta_0| \leq \frac{1}{n} t^{\frac{1}{2\alpha}}\}$$

of points that detect Q by time t we have

$$\int_{\mathcal{D}_t} \mathbb{P}_{x_0}(T_{det} > t) d\mu(x_0) \;=\; \Omega(\mu(\mathcal{D}_t)) = \Omega(t^{\frac{1}{2\alpha}})$$

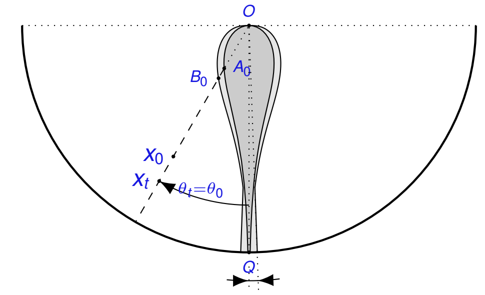
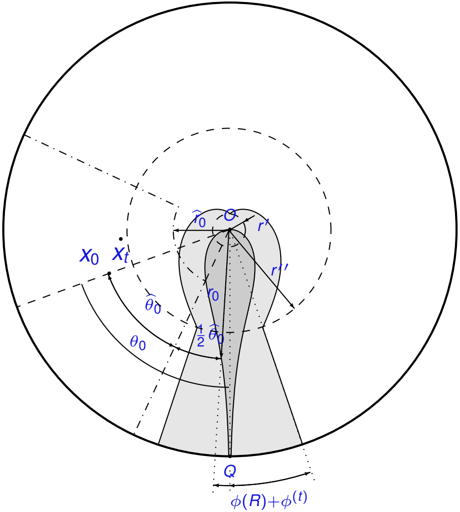


Figure: The set $\mathcal{D}_t$

Back to mixed movement - where are detecting points?



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- If $\alpha > 2\beta$ main contribution from the boundary
- If $\alpha < 2\beta$ main contribution from smallest radius reached
- If $\alpha = 2\beta$ **all layers contribute**

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$$\int_{\mathcal{D}_t} \mathbb{P}_{x_0}(T_{det} > t)d\mu(x_0) \;=\; \Omega(\mu(\mathcal{D}_t)) = \Omega(n^{1-2\beta}\sqrt{t})$$

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Then

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Proof idea for the case $\alpha = 2\beta$

Ingredients:

- (easy) $\mathbb{E}_\mu\left(\int_0^t \mathbf{1}_{[0,k]}(r_s)ds\right) \approx te^{-\alpha(R-k)}$
- Second moment method: show $\mathbb{P}_\mu\left(\int_0^t \mathbf{1}_{[0,k]}(r_s)ds > \gamma te^{-\alpha(R-k)}\right) \geq \eta$ by coupling with discrete integer-valued process (typically not too close to the origin in the beginning, and typically many jumps)

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Note: If $\beta \ge \frac{1}{2}$ we may detect only by radial movement

$$\mathcal{D}_t = \{x_0 \in B_O(R), |\theta_0| \le t^{\frac{1}{2\alpha}} n^{-1}\}$$
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- We control the probability to exit from above by radial movement
- How to control exit from the sides of the box?

Control of $\overline{\mathcal{D}}_t$

$$\Delta_h \;:\approx\; \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\alpha}{2}\frac{\partial}{\partial r} + 2e^{-2\beta r}\frac{\partial^2}{\partial \theta^2}$$

- Conditional under radial movement, the angular movement follows a Brownian motion $B_{I(t)}$ where

$$I(t) \approx \int_0^t e^{-2\beta r_s} ds$$

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- known fact (Dufresne): if X_u is Brownian motion with drift $\alpha/2$,

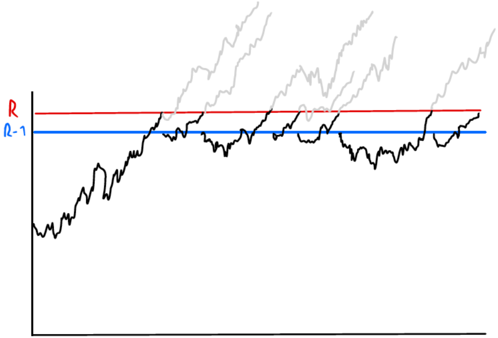
$$\int_0^\infty e^{-2\beta X_u} du \;=\; We^{-2\beta X_0}$$

where W follows an inverse Gamma distribution

$$f_W(x) = \frac{(2\beta^2)^{\frac{\alpha}{2\beta}}}{\Gamma(\frac{\alpha}{2\beta})} x^{-\frac{\alpha}{2\beta}-1} e^{-\frac{2\beta^2}{x}}$$

How to bound $I(t)$?

- Split the radial trajectory into excursions



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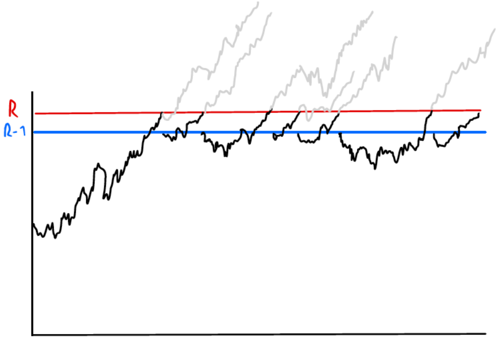

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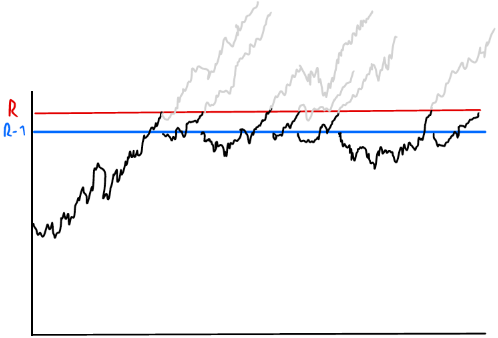

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- We control the sum by large deviation for heavy tails

Thank you!