Counting periodic geodesics and improving Weyl's Law for predominant sets of metrics

Joint with J.Galkowski

Structure of the talk

1. What is a predominant set?
2. Counting periodic geodesics
3. Counting Laplace eigenvalues
4. Volume of near periodic trajectories
5. Reduction to the Poincaré map

1 What is a predominant set?

Note. We will work with $\mathscr{G} = \{\text{Riemannian metrics over a manifold } M\}$.

Definition

Let $\mathscr{G}$ be an open subset of Banach space.

We say that $G \subset \mathscr{G}$ is predominant if

- for each $g \in \mathscr{G}$ there is a submanifold $\mathcal{L}_g \subset \mathscr{G}$ through g, with Borel measure μ_g:

$$G \cap \mathcal{L}_g \text{ has } \textbf{full} \text{ measure.}$$

- $g \mapsto \mathcal{L}_g$ is C^1.
- $\mu_g(U_g) > 0$ for every open nbhd of g.

Properties.

- predominant sets are dense
- intersection of predominant sets are predominant
- in finite dimensions, predominant sets have full measure

2 Counting periodic geodesics

M compact, no boundary.

$$\mathfrak{c}(T,g) := \#\{\gamma \text{ a primitive, periodic, geodesic for } g \text{ of length } \le T\}.$$

- $\mathfrak{c}(T,g) \sim ce^{hT}$ for g with negative curvature [Margulis '69]
- $\mathfrak{c}(T,g) \sim ce^{hT}$ for g with Anosov flow [Bowen '72]
- $\mathfrak{c}(T,g) \sim ce^{hT}$ for many g with non-positive curvature [Knieper '98]
- $\mathfrak{c}(T,g_f) \ge f(T)$ on S^2 with g_f close to g_{S^2} and any f [Burns-Paternain'95]

- $\mathfrak{c}(T,g) < \infty$ for a Baire generic g [Abraham '70, Anosov '82]
- $\mathfrak{c}(T,g) \to \infty$ for a Baire generic g [Hingston '84]
- $\mathfrak{c}(T,g) \ge ce^{cT}$ for open and dense set of g [Contreras '10]

Theorem (C–Galkowski '22)

Let M be a C^ν manifold with $\nu \ge 5$. Then, there is $\Omega_\nu > 0$ and for all $\Omega > \Omega_\nu$ there is a predominant set of C^ν-metrics G_Ω s.t. for every $g \in G_\Omega$ there is $C > 0$:

$$\mathfrak{c}(T,g) \le Ce^{CT^\Omega}.$$

❸ Counting Laplace eigenvalues

(M^n, g) compact, no boundary. Eigenvalues of $-\Delta_g$: $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$

$$\#\{j : \lambda_j \leq \lambda\} = \frac{\mathrm{vol}_{\mathbb{R}^n}(B_1)\mathrm{vol}_g(M)}{(2\pi)^n}\lambda^n + E(\lambda, g)$$

- $E(\lambda, g) = O(\lambda^{n-1})$ always [Levitan '52, Avakumovic '56]
- $E(\lambda, g) = o(\lambda^{n-1})$ for g aperiodic [Duistermaat-Guillemin '75]
- $E(\lambda, g) = O(\frac{\lambda^{n-1}}{\log \lambda})$ for g with no conjugate points [Berard '77+Bonthoneau '17]
- $E(\lambda, g) = o(\lambda^{n-1})$ for g Baire generic [Duistermaat-Guillemin '75+Anosov'82]

Theorem (C–Galkowski '22)

Let M be a C^ν manifold with $\nu \geq \nu_0$. Then, there is $\Omega_\nu > 0$ such and for all $\Omega > \Omega_\nu$ there is a predominant set of C^ν-metrics G_Ω such that for every $g \in G_\Omega$

$$E(\lambda, g) = O\left(\frac{\lambda^{n-1}}{(\log \lambda)^{1/\Omega}}\right).$$

4 Volume of near periodic trajectories

Theorem (C–Galkowski '22)

There is a predominant set of C^ν-metrics G_Ω such that for every $g \in G_\Omega$ there is $C > 0$:

$$\mathfrak{c}(T, g) \le C e^{CT^\Omega}.$$

- [C–Galkowski('22)] For a predominant set of metrics g,

$$\mathrm{vol}\Big(\rho : \ \exists t \in [t_0, T] \ \text{ s.t. } \ d(\rho, \varphi_t^g(\rho)) \le \varepsilon\Big) \le C\varepsilon^{2n-2} e^{BT^\Omega}$$

Theorem (C–Galkowski '22)

There is a predominant set of C^ν-metrics G_Ω such that for every $g \in G_\Omega$

$$E(\lambda, g) = O\Big(\lambda^{n-1}/(\log \lambda)^{1/\Omega}\Big).$$

- Definition. (M, g) is said to be $\mathbf{T}(R)$-aperiodic if

$$\mathrm{vol}\Big(\rho : \ \exists t \in [t_0, \mathbf{T}(R)] \ \text{ s.t. } \ d(\rho, \varphi_t^g(B(\rho, R))) \le R\Big) \le \frac{C}{\mathbf{T}(R)} \qquad R \to 0^+$$

- [C–Galkowski('20)] If (M, g) is $\mathbf{T}(R)$-aperiodic, then

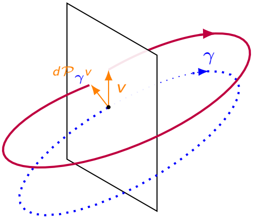
$$E(\lambda, g) = O(\lambda^{n-1}/\mathbf{T}(\tfrac{1}{\lambda})).$$

- [C–Galkowski('22)] Let $\mathbf{T}(R) = (\log R^{-1})^{1/\Omega}$. Then

  (M, g) is $\mathbf{T}(R)$-aperiodic for a predominant set of metrics g.

5 Reduction to the Poincaré map

The Poincare map, $\mathcal{P}_\gamma$, associated to $\gamma \in \mathcal{C}(T, g)$



$$\mathcal{C}(T, g) = \{\gamma : \text{ periodic geodesic for } g,\ \text{length}(\gamma) \in [T, 2T]\}$$

Theorem (C–Galkowski '22)

For all $\nu \geq 5$ and $\Omega > \Omega_\nu$ there is a predominant set of metrics G_Ω such that for all $g \in G_\Omega$ there is $C > 0$ s.t. for all T,

$$\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^\Omega}, \qquad \gamma \in \mathcal{C}(T, g).$$

6 Perturbing away periodicity

Goal: given g_0 find g_∞ nearby s.t. $\underbrace{\|(I - d\mathcal{P}_\gamma)^{-1}\| \leq Ce^{CT^\Omega}}_{(*)}$ for all $\gamma \in \mathcal{C}(T, g_\infty)$

Hypothesis: $\exists g_\ell$ s.t. $(*)$ holds for all $\gamma \in \mathcal{C}(2^j, g_\ell)$ and $j \leq \ell$

Want to show: $\exists g_{\ell+1}$ s.t. $(*)$ holds for all $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$ and $j \leq \ell + 1$

Strategy: build $g_{\ell+1}$ s.t.

- $(*)$ holds for $\gamma \in \mathcal{C}(2^j, g_{\ell+1})$ and $j \leq \ell$
- $(*)$ holds for γ primitive in $\mathcal{C}(2^{\ell+1}, g_{\ell+1})$

Not enough!

– If $\gamma \in \mathcal{C}(2^{\ell+1}, g_{\ell+1})$, we may have $\gamma = \beta^k$ with β primitive. Then, $\mathcal{P}_\gamma = (\mathcal{P}_\beta)^k$.

– If $d\mathcal{P}_\beta$ has eigenvalue $\lambda = e^{i2\pi m/k}$, then there is v such that

$$0 = (I - (d\mathcal{P}_\beta)^k)v = (I - d\mathcal{P}_\gamma)v$$

Solution: when building $g_{\ell+1}$, also make sure that $d\mathcal{P}_\gamma$ is at a safe distance from

$$\mathcal{M}_{K_\ell} = \{\text{symplectic matrices with an eigenvalue } e^{2\pi i p/k},\ 1 \leq k \leq K_\ell\}$$

Thank you!