2-Representations of affine type A Soergel bimodules: some observations and examples.

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January 25, 2023

Introduction

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• 2022: M., Miemietz and Vaz started to study **finitary, wide finitary and triangulated** 2-representations of Soergel bimodules of **affine type A**.

Outline

- The decategorified story
  - (Affine) symmetric group.
  - The Hecke algebra of (affine) type A.
  - Evaluation representations.
- The categorified story
  - Soergel bimodules in (affine) type A.
  - The evaluation functor.
  - Evaluation birepresentations.

The (affine) symmetric group

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Definition

- The **affine symmetric group** $\widehat{S}_d$ is the Coxeter group of type $\widehat{A}_{d-1}$, generated by the simple transpositions $s_0, s_1, \dots, s_{d-1}$ (**simple reflections**), subject to the relations

$$s_i^2 = e, \quad s_i s_j = s_j s_i \quad \text{if } |i-j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

where the indices have to be taken modulo d.

- The **(finite) symmetric group** $S_d \subset \widehat{S}_d$ is the subgroup generated by $s_1, \dots, s_{d-1}$ (where the indices are no longer modulo d).

The permutation representation

Definition

The **permutation representation** of S_d is given by

$$V := \mathbb{C}\langle x_1, \ldots, x_d \rangle,$$

where S_d acts by permuting the x_i.

- This representation is clearly faithful.

The affine permutation representation

Definition

The **affine permutation representation** of $\widehat{S}_d$ is given by

$$\widehat{V} := \mathbb{C}\langle y, x_1, \ldots, x_d \rangle,$$

where $S_d \subset \widehat{S}_d$ acts by permuting the x_i and fixing y, and s_0 fixes y and, furthermore, is determined by

$$\begin{aligned} s_0(x_d) &:= x_1 + y; \\ s_0(x_1) &:= x_d - y; \\ s_0(x_i) &:= x_i \quad (i \neq 1, n). \end{aligned}$$

- This representation is also faithful.
- Modding out by $\langle y \rangle$ yields a non-faithful representation of $\widehat{S}_d$ on V.

The (affine) Hecke algebra

Hecke algebras

Definition

The **affine Hecke algebra** $\widehat{H}_d$ is the unital associative $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-algebra generated by $t_0, t_1, \ldots, t_{d-1}$, subject to the relations

$$t_i^2 = (\mathrm{v}^{-1} - \mathrm{v})t_i + 1, \quad t_it_j = t_jt_i \text{ if } |i-j| > 1,$$
$$t_it_{i+1}t_i = t_{i+1}t_it_{i+1},$$

where the indices are to be taken modulo d.
The **(finite type) Hecke algebra** $H_d \subset \widehat{H}_d$ is the unital $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-subalgebra generated by $t_1, \ldots, t_{d-1}$.

- For $\mathrm{v} = 1$, we get $t_i^2 = 1$ again.

Hecke algebras: the standard basis

Let $W \in \{S_d, \widehat{S}_d\}$ and $H = H(W)$ the corresponding Hecke algebra.

- By **Matsumoto's theorem**, we can define

$$t_w := t_{i_1} \cdots t_{i_\ell} \in H,$$

for any $w \in W$, using any **reduced expression (rex)** $(s_{i_1}, \cdots, s_{i_\ell})$ for w.

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Theorem

As a $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-module, H is freely generated by the **standard** *basis:*

$$\{t_w \mid w \in W\}.$$

- H is a flat deformation of $\mathbb{Z}[W]$: $H|_{\mathrm{v}=1} \cong \mathbb{Z}[W]$.

Hecke algebras: the Kazhdan-Lusztig basis

Theorem (Kazhdan-Lusztig)

There is an alternative basis of H **(Kazhdan-Lusztig basis)***:*

$$\{b_w \mid w \in W\}.$$

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There is an alternative basis of H **(Kazhdan-Lusztig basis)***:*

$$\{b_w \mid w \in W\}.$$

- Define $b_u b_v = \sum_{w \in W} h_{u,v,w} b_w, \quad u, v \in W$.

Theorem (Kazhdan–Lusztig)

The $h_{u,v,w}$ belong to $\mathbb{N}[\mathrm{v}, \mathrm{v}^{-1}]$. **(Positive integrality)**

Kazhdan-Lusztig basis: examples

- The change-of-basis matrix is **unitriangular**, e.g., for all i:

$$b_i := b_{s_i} = t_i + \mathtt{v},$$

Kazhdan-Lusztig basis: examples

- The change-of-basis matrix is **unitriangular**, e.g., for all i:

$$b_i := b_{s_i} = t_i + \mathrm{v},$$

- For $i \neq j$, we have

$$\begin{aligned} b_i^2 &= (\mathrm{v} + \mathrm{v}^{-1}) b_i \\ b_i b_j &= b_{ij} \\ b_i b_{i+1} b_i &= b_{i(i+1)i} + b_i. \end{aligned}$$

Evaluation representations

The evaluation map

Definition

The *evaluation map* $\mathrm{ev}\colon \widehat{H}_d \to H_d$ is the homomorphism of $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-algebras determined by

$$\begin{aligned} \mathrm{ev}(t_i) &:= t_i, \quad \text{for} \quad 1 \leq i \leq d-1, \\ \mathrm{ev}_a(t_0) &:= t_\rho t_1 t_\rho^{-1}, \end{aligned}$$

where $\rho = s_{d-1} \cdots s_1$.

In terms of the Kazhdan–Lusztig generators, we have

$$\begin{aligned} \mathrm{ev}_a(b_i) &= b_i, \quad \text{for} \quad 1 \leq i \leq d-1, \\ \mathrm{ev}_a(b_0) &= t_\rho b_1 t_\rho^{-1} \end{aligned}$$

Evaluation representations: definition

Definition

The **evaluation representations** of $\widehat{H}_d$ are the pull-backs of the irreducible representations of H_d through the evaluation map.

- By construction, evaluation representations are finite-dimensional and irreducible.

Evaluation representations: example

Lemma

Take $M_d := \mathrm{span}\{m_1, \ldots, m_{d-1}\}$ *over* $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$.

- *The following defines an* H_d*-action on* M_d*:*

$$b_i m_j = \begin{cases} [2] m_i, & \text{if } j = i; \\ m_i, & \text{if } j = i \pm 1; \\ 0, & \text{else}, \end{cases}$$

for $i, j = 1, \ldots, d-1$. *Here* $[2] := \mathrm{v} + \mathrm{v}^{-1}$.

- $M_d^{\mathbb{C}(\mathrm{v})} := M_d \otimes_{\mathbb{Z}[\mathrm{v},\mathrm{v}^{-1}]} \mathbb{C}(\mathrm{v})$ *is irreducible.*

Evaluation representations: example

Definition

Let M_d^{ev} be the evaluation representation of $\widehat{H}_d$ obtained by pulling back M_d through $\mathrm{ev}\colon \widehat{H}_d \to H_d$.

- In the next slides, we are going to show that M_d^{ev} an also be obtained as the irreducible quotient of a Graham-Lehrer cell module.

Graham-Lehrer cell modules: example

Definition (Graham-Lehrer cell module)

Let

$$\widehat{M}_d := \mathrm{Span}_{\mathbb{Z}[\mathrm{v},\mathrm{v}^{-1}]}\{m_i \mid i = 0, \ldots, d-1\},$$

where the indices of the m_i have to be taken modulo d by convention, and define an action of $\widehat{H}_d$ by

$$b_i m_j = \begin{cases} [2]m_i, & \text{if } j \equiv i \bmod d; \\ (-\mathrm{v})^d m_1, & \text{if } i-1 \equiv 0 \equiv j \bmod d; \\ (-\mathrm{v})^{-d} m_0, & \text{if } i \equiv 0 \equiv j-1 \bmod d; \\ m_j, & \text{if } i \equiv j \pm 1 \bmod d, \text{ but none of the above}; \\ 0, & \text{else.} \end{cases}$$

Graham-Lehrer cell modules: example

Lemma

$\widehat{M}_d$ has a rank-one subrepresentation, generated by (recall $m_d := m_0$)

$$n_d := \sum_{k=1}^{d} (-q)^{-k} m_k,$$

and there is a natural isomorphism of $\widehat{H}_d$-representations

$$\begin{array}{rcll} \widehat{M}_d/\langle n_d \rangle & \xrightarrow{\cong} & M_d^{\mathrm{ev}} & \\ m_i & \mapsto & m_i & \qquad i = 1, \ldots, d-1. \end{array}$$

(Affine) Soergel Bimodules

Polynomial algebras

Definition

Define two polynomial algebras

$$R := \mathbb{C}[x_1, \ldots, x_d] \subset \widehat{R} := \mathbb{C}[y, x_1, \ldots, x_d].$$

We define a $\mathbb{Z}$-grading on $\widehat{R}$ (and R, of course) by

$$\mathrm{deg}(x_i) = \mathrm{det}(y) = 2$$

and the $\widehat{S}_d$-action on $\widehat{V}$ extends to an $\widehat{S}_d$-action on $\widehat{R}$ by degree-preserving algebra-automorphisms, which restricts to an S_d-action on R, of course.

The subalgebra of s_i-invariant polynomials

Definition

For any $i = 0, 1, \ldots, d-1$, define

$$\widehat{R}^{s_i} := \left\{ f \in \widehat{R} \mid s_i(f) = f \right\}.$$

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Concretely,

Lemma

For $i = 1, \ldots, d-1$, we have

$$\widehat{R}^{s_i} = \mathbb{C}[y, x_1, \ldots, x_i + x_{i+1}, x_i x_{i+1}, \ldots, x_d],$$

and, for $i = 0$, we have

$$\widehat{R}^{s_0} = \mathbb{C}[y, x_1 + x_d, x_1(x_d - y), x_2, \ldots, x_{d-1}].$$

Note that $(x_1 + y)x_d = x_1(x_d - y) + y(x_1 + x_d)$.

The subalgebra of s_i-invariant polynomials

Lemma

For any $i = 0, \ldots, d-1$, there is a degree-preserving isomorphism of graded R^{s_i}-modules

$$R \cong R^{s_i} \oplus R^{s_i}\langle -2 \rangle.$$

Proof: The isomorphism is obtained by splitting any $f \in \widehat{R}$ into its s_i-symmetric part and its s_i-antisymmetric part. Concretely, for any $i = 1, \ldots, d-1$,

$$f = \frac{1}{2}(f + s_i(f)) + \frac{1}{2}\left(\frac{f - s_i(f)}{x_i - x_{i+1}}\right)(x_i - x_{i+1}),$$

and, for $i = 0$,

$$f = \frac{1}{2}(f + s_0(f)) + \frac{1}{2}\left(\frac{f - s_0(f)}{x_d - x_1 - y}\right)(x_d - x_1 - y).$$

Bott-Samelson bimodules

Definition

For every $i = 0, \ldots, d-1$, define the graded $\widehat{R}$-$\widehat{R}$ bimodule

$$\widehat{B}_i = \widehat{B}_{s_i} := \widehat{R} \otimes_{\widehat{R}^{s_i}} \widehat{R}\langle 1 \rangle.$$

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For any word $\underline{w} = (s_{i_1}, \dots, s_{i_r})$ in $\{s_0, \dots, s_{d-1}\}$, the **Bott-Samelson bimodule** $\widehat{BS}(\underline{w})$ is defined as

$$\begin{aligned} \widehat{BS}(\underline{w}) &:= \widehat{B}_{i_1} \otimes_{\widehat{R}} \cdots \otimes_{\widehat{R}} \widehat{B}_{i_r} \\ &\cong \widehat{R} \otimes_{\widehat{R}^{s_{i_1}}} \widehat{R} \otimes_{\widehat{R}^{s_{i_2}}} \cdots \otimes_{\widehat{R}^{s_{i_r}}} \widehat{R}\langle r \rangle. \end{aligned}$$

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- $\widehat{B}_i$ is an indecomposable $\widehat{R}$-$\widehat{R}$-bimodule, because it is generated by $1 \otimes 1$ and $\widehat{R}$ is positively graded.
- $\widehat{BS}(\underline{w})$ need not be indecomposable, e.g. next two slides.

Decomposition: examples

- Recall that

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$$\widehat{B}_i \otimes_{\widehat{R}} \widehat{B}_i \quad = \quad (\widehat{R} \otimes_{\widehat{R}^{s_i}} \widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_{\widehat{R}^{s_i}} \widehat{R})\langle 2\rangle$$

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\end{aligned}$$

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- Let $\widehat{R}^{s_i, s_{i+1}} := \{f \in \widehat{R} \mid s_i(f) = f = s_{i+1}(f)\}$. Then there is an isomorphism of graded $\widehat{R}$-$\widehat{R}$-bimodules

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- We omit the proof, which is a bit tricky, but note that $\widehat{R} \otimes_{\widehat{R}^{s_i, s_{i+1}}} \widehat{R}\langle 3 \rangle$ is indecomposable, as it's generated by $1 \otimes 1$.

Soergel bimodules

Definition (Soergel)

Let $\widehat{\mathcal{S}}_d$ be the additive closure in $\widehat{R}\text{-}\mathrm{mod}_{\mathrm{gr}}^{\mathrm{fg}}\text{-}\widehat{R}$ (only degree-preserving bimodule maps!) of the full, additive, graded, monoidal subcategory generated by $\widehat{B}_i\langle t\rangle$, for $i = 0, \ldots, d-1$ and $t \in \mathbb{Z}$.

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- Let $X \in \widehat{R}\text{-}\mathrm{mod}_{\mathrm{gr}}^{\mathrm{fg}}\text{-}\widehat{R}$. Then

$$X \in \widehat{\mathcal{S}}_d \;\Leftrightarrow\; X \subseteq^{\oplus} \bigoplus_{\underline{w}} \widehat{BS}(\underline{w})\langle t_{\underline{w}}\rangle$$

for some words $\underline{w}$ in $\{s_0, \ldots, s_{d-1}\}$ and shifts $t_w \in \mathbb{Z}$.

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Let $\widehat{\mathcal{S}}_d$ be the additive closure in $\widehat{R}\text{-}\mathrm{mod}^{\mathrm{fg}}_{\mathrm{gr}}\text{-}\widehat{R}$ (only degree-preserving bimodule maps!) of the full, additive, graded, monoidal subcategory generated by $\widehat{B}_i\langle t\rangle$, for $i = 0, \ldots, d-1$ and $t \in \mathbb{Z}$.

- Let $X \in \widehat{R}\text{-}\mathrm{mod}^{\mathrm{fg}}_{\mathrm{gr}}\text{-}\widehat{R}$. Then

$$X \in \widehat{\mathcal{S}}_d \;\Leftrightarrow\; X \subseteq^{\oplus} \bigoplus_{\underline{w}} \widehat{BS}(\underline{w})\langle t_{\underline{w}}\rangle$$

for some words $\underline{w}$ in $\{s_0, \ldots, s_{d-1}\}$ and shifts $t_w \in \mathbb{Z}$.
- $\widehat{\mathcal{S}}_d$ is **linear** and **additive** but **not abelian**, e.g. the kernel of

$$\widehat{B}_i\langle -1\rangle = \widehat{R} \otimes_{\widehat{R}^{s_i}} \widehat{R} \xrightarrow{a\otimes b \mapsto ab} \widehat{R}$$

is isomorphic to $\widehat{R}$ as a right $\widehat{R}$-module but the left $\widehat{R}$-action is twisted by s_i, so it does not belong to $\widehat{\mathcal{S}}_d$.

Indecomposable Soergel bimodules

Theorem (Soergel)

$\widehat{\mathcal{S}}_d$ *is* **Krull-Schmidt**. *For every* $w \in \widehat{S}_d$, *there is an* **indecomposable** *bimodule* $\widehat{B}_w \in \widehat{\mathcal{S}}_d$, **unique** *up to degree-preserving isomorphism, such that*

(1) $\widehat{B}_w \subseteq^{\oplus} \widehat{BS}(\underline{w})$ *with multiplicity one, for any rex* $\underline{w}$ *of* w*;*

(2) $\widehat{B}_w\langle t\rangle \not\subseteq^{\oplus} \widehat{BS}(\underline{u})$ *for any* $t \in \mathbb{Z}$, $u \prec w$ *and rex* $\underline{u}$ *of* u.

(3) Every indecomposable Soergel bimodule is isomorphic to $\widehat{B}_w\langle t\rangle$, *for some* $w \in W$ *and* $t \in \mathbb{Z}$.

Decategorification: split Grothendieck group

Let $\mathcal{A}$ be a Krull-Schmidt category.

Definition

The **split Grothendieck group** $[\mathcal{A}]_\oplus$ is the abelian group generated by the isoclasses $[X]$ of the objects $X \in \mathcal{A}$ modulo the relations

$$[X \oplus Y] = [X] + [Y]$$

for all $X, Y \in \mathcal{A}$.

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Lemma

$[\mathcal{A}]_{\oplus}$ *is the free abelian group generated by the isoclasses of the indecomposable objects in* $\mathcal{A}$*.*

Decategorification: split Grothendieck ring

Definition

If $\mathcal{A}$ is a monoidal Krull-Schmidt category, then $[\mathcal{A}]_{\oplus}$ is a $\mathbb{Z}$-algebra with product defined by

$$[X][Y] := [X \otimes Y]$$

for all $X, Y \in \mathcal{A}$.

The categorification theorem

Since the objects of $\widehat{\mathcal{S}}_d$ can be shifted, the split Grothendieck algebra $[\widehat{\mathcal{S}}_d]_\oplus$ is an algebra over $\mathbb{Z}[\mathrm{v},\mathrm{v}^{-1}]$:

$$\mathrm{v}^t[X] := [X\langle t\rangle], \quad X \in \widehat{\mathcal{S}}_d, t \in \mathbb{Z}.$$

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Theorem (Soergel, Fiebig)

The $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-linear map given by

$$b_w \mapsto [B_w]$$

defines an algebra isomorphism between $\widehat{H}_d$ and $[\widehat{\mathcal{S}}_d]_\oplus$.

Finite type Soergel bimodules

• Similarly, one can define the monoidal category $\mathcal{S}_d$ of Soergel bimodules of finite type A_{d-1}, which are defined over the ring $R = \mathbb{C}[x_1, \ldots, x_{d-1}]$.

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• $\mathcal{S}_d$ categorifies the Hecke algebra H_d, such that the indecomposables B_w correspond to the Kazhdan-Lusztig basis elements b_w, for $w \in S_d$.

The evaluation functor

Rouquier complexes

- Recall $b_i = t_i + v$. Therefore $t_i = b_i - v$.

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- Let $K^b(\mathcal{A})$ be the homotopy category of bounded complexes in a monoidal Krull-Schmidt category $\mathcal{A}$, which inherits a monoidal structure from $\mathcal{A}$.
- Let $\underline{A_0} \to A_1 \to \cdots \to A_n \in K^b(\mathcal{A})$. Then the **Euler characteristic** in the **triangulated Grothendieck group** $[\mathcal{A}]_\triangle$ is defined as

$$[\underline{A_0} \to A_1 \to \cdots \to A_n] = [A_0] - [A_1] + \cdots + (-1)^n [A_n].$$

Definition

For any $i = 1, \ldots, d-1$, define the complex

$$T_i \colon \underline{B_i} = \underline{R \otimes_{R^{s_i}} R\langle 1 \rangle} \xrightarrow{a \otimes b \mapsto ab} R\langle 1 \rangle.$$

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$$T_i \colon \underline{B_i} = \underline{R \otimes_{R^{s_i}} R\langle 1\rangle} \xrightarrow{a\otimes b \mapsto ab} R\langle 1\rangle.$$

Lemma

The T_i are invertible in $K^b(\mathcal{S}_d)$:

$$T_i^{-1} \colon R\langle -1\rangle \to \underline{B_i} = \underline{R \otimes_{R^{s_i}} R\langle 1\rangle},$$

with differential given by

$$1 \mapsto (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1}).$$

Rouquier complexes

Theorem (Rouquier)

The T_i satisfy the braid relations of type A_{d-1} in $K^b(\mathcal{S}_d)$:

$$\begin{aligned} T_i T_j &\cong T_j T_i \qquad \text{if} \quad |i-j|>1 \\ T_i T_{i+1} T_i &\cong T_{i+1} T_i T_{i+1}. \end{aligned}$$

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By Matsumoto's theorem, the following is well-defined:

Definition

Let $w \in S_d$. Choose any rex $(s_{i_1}, \ldots, s_{i_\ell})$ for w and define

$$T_w := T_{i_1} \cdots T_{i_\ell} \in K^b(\mathcal{S}_d).$$

The T_w categorify the standard basis elements t_w in H_d.

The evaluation functor

Theorem (M–Miemietz-Vaz)

There is a linear monoidal functor $\mathcal{E}v \colon \widehat{\mathcal{S}}_d \to K^b(\mathcal{S}_d)$ *which on objects is given by*

$$\begin{array}{rcll} \widehat{B}_i & \mapsto & B_i & \text{for} \quad i = 1, \ldots, d-1 \\ \widehat{B}_0 & \mapsto & T_\rho B_1 T_\rho^{-1}, & \end{array}$$

where $\rho = s_{d-1} \cdots s_1$.

- To prove this theorem, one needs to define $\mathcal{E}v$ on morphisms as well, which requires an extension of the usual **Soergel calculus** and 40 pages of diagrammatic calculations.

Evaluation birepresentations

Linear additive birepresentations of linear additive monoidal categories

Definition

Let $\mathcal{A}$ be a linear additive Krull-Schmidt monoidal category. A **linear additive birepresentation M** of $\mathcal{A}$ is a linear additive Krull-Schmidt category $\mathcal{M}$ together with a linear monoidal functor

$$\mathbf{M} \colon \mathcal{A} \to \mathrm{END}(\mathcal{M}).$$

By definition, the **rank** of **M** is equal to the rank of $[\mathcal{M}]_{\oplus}$, which is an linear representation of $[\mathcal{A}]_{\oplus}$.

A **strict** finitary birepresentation of a **strict** finitary monoidal category is called a 2-**representation**.

Recall the zigzag algebra

- $A := \mathrm{ZZ}(A_{d-1})$ (**Zigzag algebra of type** A_{d-1}).

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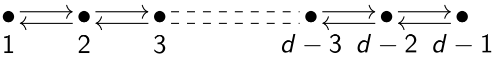

$$i|(i{+}1)|i = i|(i{-}1)|i, \quad i|(i{+}1)|(i{+}2) = 0 = (i{+}2)|(i{+}1)|i.$$

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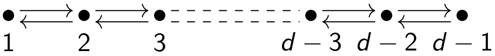

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  - Let e_i be the idempotent corresponding to vertex i and $\ell_i := i|(i+1)|i = i|(i-1)|i$ the loop on the same vertex.

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  - Let e_i be the idempotent corresponding to vertex i and $\ell_i := i|(i+1)|i = i|(i-1)|i$ the loop on the same vertex.
  - Since

$$\ell_i^2 = 0,$$

we have $e_iAe_i \cong D = \mathbb{C}[x]/\langle x^2 \rangle$ (dual numbers).

Subregular evaluation birepresentation of $\widehat{\mathcal{S}}_d$

Theorem

There is a linear additive birepresentation $\mathbf{M}_d$ *of* $\mathcal{S}_d$ *which on objects is given by*

$$\begin{aligned} \mathbf{M}_d \colon \mathcal{S}_d &\to \mathrm{END}(A\text{-proj}_{\mathrm{gr}}) \\ B_i &\mapsto Ae_i \otimes e_i A\langle 1\rangle \otimes_A -, \qquad i = 1, \ldots, d-1. \end{aligned}$$

- $[\mathbf{M}_d]_\oplus \cong M_d$ as modules over $[\mathcal{S}_d]_\oplus \cong H_d$.

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Definition

Let $\mathbf{M}_d^{\mathcal{E}v}$ be the **triangulated** birepresentation of $\widehat{\mathcal{S}}_d$ obtained by pulling back $K^b(\mathbf{M}_d)$ through $\mathcal{E}v \colon \widehat{\mathcal{S}}_d \to K^b(\mathcal{S}_d)$

- $[\mathbf{M}_d^{\mathcal{E}v}]_\triangle \cong M_d^{\mathrm{ev}}$ as modules over $[\widehat{\mathcal{S}}_d]_\oplus \cong [K^b(\widehat{\mathcal{S}}_d)]_\triangle \cong \widehat{H}_d$.

The Graham-Lehrer cell birepresentation of $\widehat{\mathcal{S}}_d$

Let $\widehat{A}_d := ZZ(\widehat{A}_{d-1})$ be the (signed) zigzag algebra of type $\widehat{A}_{d-1}$.

Theorem

There is a **linear additive** *birepresentation* $\widehat{\mathbf{M}}_d$ *of* $\widehat{\mathcal{S}}_d$ *which on objects is given by*

$$\begin{aligned} \widehat{\mathbf{M}}_d : \widehat{\mathcal{S}}_d &\to \mathrm{END}(\widehat{A}\text{-proj}_{\mathrm{gr}}) \\ B_i &\mapsto \widehat{A}e_i \otimes e_i\widehat{A}\langle 1\rangle \otimes_{\widehat{A}} -, \qquad i = 0, \ldots, d-1. \end{aligned}$$

- $[\widehat{\mathbf{M}}_d]_\oplus \cong \widehat{M}_d$ as modules over $[\widehat{\mathcal{S}}_d]_\oplus \cong \widehat{H}_d$.

The categorified projection

Theorem

There is a **linear** *morphism of linear additive birepresentations of* $\widehat{\mathcal{S}}_d$

$$\Phi\colon \widehat{\mathbf{M}}_d \to K^b(\mathbf{M}_d)$$

which on objects is given by

$$\begin{aligned} \widehat{A}e_i &\mapsto Ae_i, \qquad i = 1, \ldots, d-1 \\ \widehat{A}e_0 &\mapsto [Ae_{d-1}\langle 1\rangle \to Ae_{d-2}\langle 2\rangle \to \cdots \to Ae_1\langle d\rangle] \end{aligned}$$

Moreover, Φ *extends to a morphism of triangulated birepresentations*

$$\widehat{\Phi}\colon K^b(\widehat{\mathbf{M}}_d) \to K^b(\mathbf{M}_d),$$

which is essentially surjective, faithful and **epimorphic** *(not full!).*

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• Is it possible to categorify parabolic induction in the triangulated setting?

• What about other affine Weyl types?

The End