When is a surface Quasi-Ordinary?

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Introduction

- There is a well established theory of deformations of isolated singularities, but not much is known about deformation of non isolated singularities.
- Quasi-ordinary surface singularities admit very well behaved parametrizations, therefore we may ask if the theory of equisingular deformations can be generalized for them.
- The first obstacle to overcome is to answer questions of the type: given $\varphi, \psi \in \mathbb{C}\{t_1, t_2\}$ such that φ is a parametrization is a quasi-ordinary surface, for which ψ is

$$(t_1, t_2) \mapsto \varphi(t_1, t_2) + s\psi(t_1, t_2)$$

the parametrization of a quasi-ordinary surface, for $|s| << 1$?

Introduction

In order to solve these problems we must first answer a question which is the goal of this seminar:

Question

Given a parametrization

$$(t_1, t_2) \mapsto (x(t_1, t_2), y(t_1, t_2), z(t_1, t_2))$$

can we give a criteria to decide if it is the parametrization of a quasi-ordinary surface?
All this can be generalized to arbitrary hypersurfaces.

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Theorem

Let $Y = \{f(x,y) = 0\}$ *be an irreducible plane curve. Then there exists* $m \in \mathbb{N}$ *such that there is a solution* $y \in \mathbb{C}\{x^{1/m}\}$ *of* $f(x,y) = 0$ *and* Y *can be parametrized as* $Y = \{(x^m, \sum_\alpha a_\alpha x^{m\alpha})\}$.

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Definition

The series $y = \sum a_i x^{i/m}$ is called a *Puiseux expansion* for the curve with equation $f(x,y) = 0$.

Lemma

Let f be irreducible with puiseux expansion $y = \sum_\alpha a_\alpha x^\alpha$. If every α with $a_\alpha \neq 0$ is an integer, f is regular, otherwise there is a smallest $\alpha_1 = \frac{n_1}{m_1} \in \mathbb{Q} \backslash \mathbb{Z}$ with $gcd(n_1, m_1) = 1$ for which $a_{\alpha_1} \neq 0$. Assume there is an $\alpha = \frac{n_2}{m_1 m_2} \in \mathbb{Q} \backslash \mathbb{Z}$ with $\alpha_1 < \alpha, gcd(n_2, m_2) = 1$ and $a_\alpha \neq 0$, we set α_2 as the minimal α with this property. After a finite number g of iterations we get a sequence $(m_1, n_1), \dots, (m_g, n_g)$.

Example

- If $y = x^{3/2} + x^{7/2} + x^{17/4}$, the puiseux pairs are $(m_1, n_1) = (2, 3)$ and $(m_2, n_2) = (2, 17)$.

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- If $y = x^{3/2} + x^{5/3} + x^{37/2} = x^{3/2} + x^{10/2\cdot 3} + x^{111/2\cdot 3}$, then $(m_1, n_1) = (2, 3)$ and $(m_2, n_2) = (3, 10)$.

Definition

The pairs $(m_1, n_1), \ldots, (m_g, n_g)$ are called the *Puiseux pairs* of f.

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Theorem

The Puiseux pairs of f are topological invariants of $Y = \{f(x, y) = 0\}$ and determine completely the topology of Y.

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Let N be the germ at a point of a hypersurface of a complex manifold M. We say that N is the germ of a *normal crossing divisor* if there is a local system of coordinates $(x_1, \dots, x_n)$ and a positive integer k such that $x_1 \cdots x_k$ generates the defining ideal of N.

Definition

Let $(X, o) = (\{F = 0\}, o) \subset (\mathbb{C}^{d+1}, o)$ be a germ of a hypersurface with $F \in \mathbb{C}\{x_1, \ldots, x_d, y\}$. Let $p : \mathbb{C}^{d+1} \to \mathbb{C}^d$ be the projection $(x_1, \ldots, x_d, y) \mapsto (x_1, \ldots, x_d)$. We call *apparent contour* of X *relative to the projection* p to the set $\Sigma = \{f = \frac{\partial f}{\partial y} = 0\}$. We call *discriminant* of X *relative to the projection* p to the set $\Delta_y(S) = p(\Sigma)$.

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Definition

We say that X is a *quasi-ordinary hypersurface* if there is a system of local coordinates $(x_1, \ldots, x_d, y)$ such that the restriction of the projection p to X is a finite map and its discriminant is a normal crossing divisor.

Theorem

If X is quasi-ordinary relatively to the projection p, there is a positive integer k and $\zeta \in \mathbb{C}\{x_1^{1/k}, \dots, x_d^{1/k}\}$ given by

$$\zeta = \sum_{\alpha} c_\alpha x^\alpha$$

for $\alpha \in \frac{1}{k}\mathbb{N}^d$, such that $F(x_1, \dots, x_d, \zeta) = 0$.

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Definition

The function ζ given from the theorem is called a *quasi-ordinary branch.*

Lemma

Let ζ be a quasi-ordinary branch. Assume there is α such that $c_\alpha \neq 0$ and $\alpha \notin \mathbb{Z}^d$. Then there is a minimal α with this property. Set $\alpha_1 = \alpha$.
Assume that there is α such that $c_\alpha \neq 0$ and $\alpha \notin \mathbb{Z}^d + \mathbb{Z}\alpha_1$. Then there is a minimal α with this property and we set $\alpha_2 = \alpha$. We iterate this procedure. After a finite number g of iterations we get that $c_\alpha \neq 0$ implies $\alpha \in \mathbb{Z}^d + \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_g$. Moreover, $\alpha_1 < \cdots < \alpha_g$.
The $d-$tuples $\alpha_1, \ldots, \alpha_g$ are called the Puiseux exponents *of X relative to the projection p.*

Example

Assume z is a quasi-ordinary branch given by

$$z = x^{3/2}y + x^{5/2}y + x^{7/4}y^{5/4}.$$

The puiseux exponents are $(3/2, 1)$ and $(7/4, 5/4)$. Note that the Puiseux exponents are well defined even though the pairs of exponents are not totally ordered.

Theorem

The Puiseux exponents of X relative to p are topological invariants of X and determine completely the topology of the pair of germs $(X, \mathbb{C}^{d+1})$.

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Remark

We have a parametrization f on irreducible quasi-ordinary hypersurface X given by

$$x_1 = t_1^k, \ldots, x_d = t_d^k, \; y = \sum_{\alpha} c_\alpha t^{\alpha k}.$$

Proposition

Let S be a quasi-ordinary surface, then we can find an irreducible quasi-ordinary polynomial f and a root ζ of f, so that either $\zeta = 0$ or it is a quasi-ordinary branch of the form

$$\zeta = x^{\lambda} y^{\mu} H(x^{1/k}, y^{1/k})$$

with $H(0,0) \neq 0$ and $\lambda, \mu \in \frac{1}{k}\mathbb{Z}$ such that:

1. $\lambda, \mu \notin \mathbb{Z}$.
2. *If $\lambda\mu = 0$, then $\lambda + \mu > 1$.*

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Assume now we have a parametrization of a surface S given by

$$x = a(t,s),\ y = b(t,s),\ z = c(t,s) \tag{1}$$

where $a, b, c \in \mathbb{C}\{t,s\}$. Is S quasi-ordinary?
We will introduce an algorithm that answers this question.
First of all we associate to (1) the set Γ of pairs $(\alpha, \beta) \in \mathbb{N}^2$ such that there is $f \in \mathbb{C}\{x,y,z\}$ with $f(a,b,c) = t^\alpha s^\beta \cdot u$, u a unit of $\mathbb{C}\{t,s\}$.

Lemma

The set Γ *is a subsemigroup of* $(\mathbb{N}^2, +)$*, independent of the system of local coordinates* (x, y, z)*.*

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Lemma

Let S *be a quasi-ordinary surface with a parametrization in normal form. Then*

1. $(k, 0), (0, k), (\alpha, \beta) \in \Gamma$.
2. *If* $(l, 0) \in \Gamma$ *or* $(0, l) \in \Gamma$, $l \geq k$.
3. *If* $(\gamma, \delta) \in \Gamma \backslash k\mathbb{Z}^2$, $(\alpha, \beta) \leq (\gamma, \delta)$.

Lemma

Assume $x = a(t,s), y = b(t,s), z = c(t,s)$ *is the parametrization of a surface such that the associated semigroup is the semigroup of a quasi-ordinary surface. After a change of coordinates, we can assume that*

$$a(t,s) = t^k(1+A),\ b(t,s) = s^k(1+B),\ c(t,s) = t^\alpha s^\beta(1+C) \quad (2)$$

where $A, B, C \in \mathbb{C}\{t,s\}$*.*

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where $A, B, C \in \mathbb{C}\{t,s\}$.

After a change of parameter, we can transform (2) into a parametrization of the type

$$x = t^k,\ y = s^k,\ z = \sum_{\alpha,\beta} c_{\alpha,\beta} t^\alpha s^\beta.$$

We need to know beforehand if the parametrization is quasi-ordinary or not.

Definition

Assume $D \in \{A, B, C\}$, $D = \sum_{i,j} \lambda_{i,j} t^i s^j$. We say that D is *well behaved* if there is an integer g such that the pairs recursively defined by

$$n_{D,p} = \inf\{(i,j) : \ \lambda_{i,j} \neq 0, (i,j) \notin k\mathbb{Z}^2 + (\alpha, \beta)\mathbb{Z} + \sum_{q<p} \mathbb{Z} n_{D,q}\}$$

are well defined for $1 \leq p \leq g$ and

$$\{(i,j) : \ \lambda_{i,j} \neq 0\} \subset k\mathbb{Z}^2 + (\alpha, \beta)\mathbb{Z} + \sum_{n_{D,q} \leq (i,j)} \mathbb{Z} n_{D,q}.$$

Example

$$\begin{cases} x = t^4(1 + ts + 2ts^2) \\ y = s^4(1 + 5s^2) \\ z = t^3s^2(1 + t^2s^2 + t^5s^4 + t^3s^5) \end{cases}.$$

Although the pairs of exponents are not totally ordered, the parametrization is well behaved.

Example

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Although the pairs of exponents are not totally ordered, the parametrization is well behaved.

Lemma

If A, B or C are not well behaved, the parametrization does not parametrize a quasi-ordinary surface.

Definition

If A, B, C are well behaved and there are not $p, q, r \in \mathbb{N}$ such that

$$n_{C,r} = n_{A,p} = n_{B,q}, \; c_{n_{C,r}} = \frac{\alpha}{k} a_{n_{A,p}} + \frac{\beta}{k} b_{n_{B,q}},$$

we say that the parametrization is *non resonant.*

"Non resonant" means that we can predict from the data $n_{A,i}, n_{B,j}, n_{C,l}$ which are the Puiseux exponents of the surface.

Conjecture

There is a change of coordinates after which the parametrization is non resonant or not well behaved.

Example

Consider the surface parametrized by

$$\begin{cases} x &= t^4(1+2t+ts-\frac{1}{2}t^2s) \\ y &= s^4 \\ z &= t^6s^4(1+2t+\frac{3}{2}ts-t^2s) \end{cases}$$

It is resonant because $n_{A,2}=(1,1)=n_{C,2}$ and $1=a_{n_{A,2}}=\frac{\alpha}{k}c_{n_{C,2}}=\frac{3}{2}$.

Example

After changing the coordinates by $t = u(1 - \frac{1}{4}us)$, we have

$$\begin{cases} x = u^4(1 + 2u - 3u^2s - \frac{7}{8}u^2s^2 + \dots) \\ y = s^4 \\ z = u^6s^4(1 + 2u - \frac{9}{2}u^2s - \frac{27}{16}u^2s^2 + \dots) \end{cases}$$

It is again resonant because $n_{A,2} = (2,1) = n_{C,2}$ and $-3 = a_{n_{A,2}} = \frac{\alpha}{k}c_{n_{C,2}} = -\frac{9}{2}$.

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It is again resonant because $n_{A,2} = (2,1) = n_{C,2}$ and $-3 = a_{n_{A,2}} = \frac{\alpha}{k}c_{n_{C,2}} = -\frac{9}{2}$.We change again the coordinates, this time by $u = v(1 + \frac{3}{4}v^2s)$, then

$$\begin{cases} x = v^4(1 + 2v + \frac{15}{2}v^3s - \frac{7}{8}v^2s^2 + \dots) \\ y = s^4 \\ z = v^6s^4(1 + 2v + \frac{21}{2}v^3s - \frac{27}{16}v^2s^2 + \dots) \end{cases}$$

which is clearly not well behaved.

Theorem

A surface S is quasi-ordinary if and only if after a well determined change of coordinates, it is well behaved, non resonant and the set $\{n_i\}_{1\leq i\leq g}$ is well defined, where $n_1 = (\alpha, \beta)$ and for $p \geq 2$

$$n_p = \min\{n_{A,i} + (\alpha,\beta), n_{B,j} + (\alpha,\beta), n_{C,l} + (\alpha,\beta) \text{ not in } k\mathbb{Z}^2 + \sum_{q\leq p-1} n_q\mathbb{Z}\}.$$

In this case the Puiseux exponents of S are $n_1/k, \ldots, n_g/k$.

Example

Consider

$$\begin{cases} x &= t^4(1+t^4s^6) \\ y &= s^4 \\ z &= t^6s^4(1+ts) \end{cases}$$

the surface is well behaved and non resonant, and $n_1 = (6,4), \; n_2 = (7,5)$ is well defined, therefore it is quasi-ordinary with Puiseux pairs $(\frac{3}{2}, 1), (\frac{7}{4}, \frac{5}{4})$.

- Egbert Brieskorn and Horst Knörrer (1986). *Plane Algebraic Curves*. Springer.
- G.M. Greuel, C.Lossen and E.Shustin (2007). *Introduction to Singular Deformations*. Springer.
- J. Lipman (1965). *Quasi-ordinary singularities of embedded surfaces*. Thesis, Hardvard University.
- Pedro D. González Pérez (2003). *The semigroup of a quasi-ordinary hypersurface*. Cambridge University Press.
- J. Milnor (1963). *Morse Theory*. Princeton University Press.