Geometria em Lisboa

10 - Jan - 2023

"Lagrangians of Hecke cycles in the moduli space of Higgs bundles"

joint with André Oliveira, Johannes Horn and Robert Hanson

What? Construction of some complex Lagrangian

Why? Interest comes from the study of mirror symmetry for branes in the moduli of Higgs bundles.

Geometric Langlands Conjecture

Overview of the Moduli space of Higgs bundles

$\mathbb{C}$ field of cx numbers

X smooth projective curve over $\mathbb{C}$ (Riemann surface)

g = genus of X $\quad g \geq 2$

$K_X \cong T^*X$ Canonical bundle of X

Def Higgs bundle is pair (E, Φ)

$E \to X$ (algebraic) vector bundle on X

$\Phi \in H^0(\operatorname{End} E \otimes K_X)$ Higgs field

$$E \xrightarrow{\Phi} E \otimes K_X$$

One can define some stability notions for Higgs bundles

(E, Φ) is (semi) stable iff

$\nexists\, F \subset E$ $\Phi(F) \subset F \otimes K_X$ and such that

$$\mu(F) = \frac{\deg(F)}{rk(F)} \ (>) \geq \frac{\deg(E)}{rk(E)} = \mu(E)$$

$\mathcal{M}_X(n, d)$ = moduli space of semistable Higgs bundles of $rk = n$ and $\deg = d$

$\mathcal{M}_X = \mathcal{M}_X(n, 0)$

One can equip $\mathcal{M}_X$ with a morphism

$$\begin{aligned} h : \mathcal{M}_X &\longrightarrow \mathcal{B} = \bigoplus_{i=1}^{n} H^0(K_X^i) = H^0(K_X) \oplus \_\_ \oplus H^0(K_X^n) \\ (E, \Phi) &\longmapsto (\operatorname{tr}(\Phi), \_\_, \det(\Phi)) \end{aligned}$$

$b \in \mathcal{B}$ $b = (b_1, \_\_, b_n)$ determines a n-th covering of X in $\mathrm{tot}(K_X)$ (surface)

$$\begin{tikzcd} \mathrm{tot}(K_X) \arrow[d, "p"] \\ X \end{tikzcd} \qquad \begin{tikzcd} p^* K_X \arrow[d] \\ \mathrm{tot}(K_X) \arrow[u, bend right, "\lambda"'] \end{tikzcd}$$

λ tautological section

$$V(\lambda) = \underline{X_0} \subset \mathrm{tot}(K_X)$$

for every $b \in \mathcal{B}$ one constructs the corresponding spectral curve C_b given by the vanishing locus of the $p^* K_X^n$

$$\begin{tikzcd} C_b = V(\lambda^n + \lambda^{n-1} \cdot b_1 + \ldots + \lambda b_{n-1} + b_n) \subset \mathrm{tot}(K_X) \arrow[d, "p"] \\ X \end{tikzcd}$$

$$C_b \xrightarrow[n:1]{p_b} X$$

We consider the projectivization $\mathbb{K} = \mathbb{P}(\mathcal{O} \oplus K_X)$

$$\mathrm{tot}(K_X) \subset \mathbb{K}$$

$$\mathbb{P}(\mathcal{O} \oplus 0) = X_0 \cong X$$
$$\mathbb{P}(0 \oplus K_X) = X_\infty \cong X$$

$$\mathcal{O}_{\mathbb{K}}(X_0) = p^* K_X$$

Beauville - Narasimhan - Ramanan (BNR) correspondence

$\rightarrow$ One can describe every Higgs bundle in $\mathcal{M}_X$ in terms of rk 1 tf sheaves supported on some C_b

$\mathcal{L} \in \mathrm{Jac}(C_b)$

$p_{b*}\mathcal{L} = E$ vector bundle

$$\mathcal{L} \xrightarrow{\otimes\lambda} \mathcal{L} \otimes p^* K_X$$

$$\underbrace{p_{b*}\mathcal{L}}_{E} \longrightarrow p_{b*}(\mathcal{L} \otimes p^* K_X) = p_{b*}\mathcal{L} \otimes K_X$$

$$E \xrightarrow[\Phi]{\quad} E \otimes K_X$$

BNR correspondence

$$\}(E,\Phi)\{ \xleftrightarrow{1:1} \mathcal{L}$$ torsion-free sheaves on $\mathbb{K}$ supported on C_b

$$C_b \cap X_\infty = \emptyset$$

$$M_{\mathbb{K}} = M_{\mathbb{K}}(0, [nX_0], \delta)\Big|_{\mathrm{supp} \cap X_\infty = 0} \cong \mathcal{M}_X$$

torsion sheaves in $\mathbb{K}$ Higgs bundles.

One can consider a holomorphic symplectic form on $\mathcal{M}_X$

$$T_{(E,\Phi)}\mathcal{M}_X = T_{\mathcal{L}} M_{\mathbb{K}} = \mathrm{Ext}^1_{\mathbb{K}}(\mathcal{L},\mathcal{L})$$

$$T_{(E,\Phi)}\mathcal{M}_X = T^*_{\mathcal{L}} M_{\mathbb{K}} = \mathrm{Ext}^1_{\mathbb{K}}(\mathcal{L}, \mathcal{L}.K_{\mathbb{K}})$$

$\mathbb{K}$ ruled surface $\quad K_{\mathbb{K}} = \mathcal{O}_{\mathbb{K}}(-2X_\infty)$

$\Theta_{\mathbb{K}}$ Poisson form on $\mathbb{K}$

Mukai - Bottacin - Markman Poisson form on $\mathcal{M}_X$

$$\Theta_{\mathcal{M}} : \mathrm{Ext}^1(\mathcal{L}, \mathcal{L} K_{\mathbb{K}}) \wedge \mathrm{Ext}^1(\mathcal{L}, \mathcal{L} K_{\mathbb{K}}) \xrightarrow{\text{Yoneda}, \mathrm{tr}, \Theta_{\mathbb{K}}} \mathbb{C}$$

$\Omega_{\mathcal{M}}$ $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \wedge \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$ Symplectic form on $\mathcal{M}_X$

on $\mathcal{M}_X = \mathcal{M}_{\mathbb{K}} \big|_{\mathrm{supp} \cap X_\infty = \emptyset}$

One can see that $h^{-1}(b)$ are Lagrangian wrt $\Omega_{\mathcal{M}}$.

Critical loci of the Hitchin fibration

$$\mathcal{B} \cong |nX_0|_{\mathrm{supp} \cap X_\infty = \emptyset}$$

$D_b = \mathrm{sing}(C_b)$

$\mathcal{B}^1 = \} C_b \in \mathcal{B}$ such that $\deg(D_b) = 1 \{$

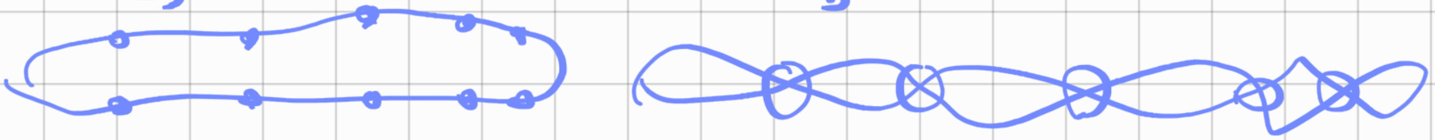
$\vdots$

$\mathcal{B}^m = \} C_b \in \mathcal{B}$ s.t. $\deg(D_b) = m \{$

Observe that every $C_b \in \mathcal{B}^m$ is normalized by $\Sigma_b \longrightarrow C_b$ with $g(\Sigma_b) = g(C_b) - m$

$\mathcal{B}^m_{nod} \subset \mathcal{B}^m$ open subset of nodal curves

$$\Sigma_b \xrightarrow{\quad \nu_b \quad} C_b$$



In 2019 Hitchin considered a family of sublinear systems of $h: \mathcal{M} \to \mathcal{B}$

for this, we need to understand $\boxed{T_b \mathcal{B}^m}$
we use deformation theory

Every curve $C_b \in \mathcal{B}^m$ $\quad \nu : \Sigma_b \longrightarrow \mathbb{K} \supset \mathrm{tot}\, K$

Deformations of $\nu : \Sigma \longrightarrow \mathbb{K}$ = 1-st order deformations of C_b remain in $\mathcal{B}^m$

[Horikawa] in 70's

1-st order deformations of $\nu : \Sigma \longrightarrow \mathbb{K}$ is parametrized by $H^0(\Sigma, N_\nu)$

$$0 \to T_\Sigma \longrightarrow \nu^* T_{\mathbb{K}} \longrightarrow N_\nu = \nu^* T_{\mathbb{K}} / T_\Sigma \longrightarrow 0$$

In our case $\Lambda^2 \nu^* T_{\mathbb{K}} = \nu^*(\Lambda^2 T_{\mathbb{K}}) = \nu^* \mathcal{O}_{\mathbb{K}}(-2X_\infty)$

Since $\mathrm{im}(\nu) = C \subset \mathbb{K}$ $\quad C \cap X_\infty \Longrightarrow$

$$N_\nu = T^*_\Sigma = K_\Sigma$$

$$T_b B^m = H^0(\Sigma_b, K_{\Sigma_b})$$ where $\Sigma_b \xrightarrow{\nu} C_b$ normalization of C_b.

Hitchin considered

$$\begin{array}{l} \mathcal{M}^m = \{ (E, \Phi) \text{ associated } \mathcal{L} = \nu_* \mathcal{F} \} \\ \cap \\ \mathcal{M}_X \end{array}$$

<u>lemma</u>

$$\begin{array}{ccc} \mathcal{M}^m & \subset & \mathcal{M}_X \\ \downarrow & & \downarrow \\ B^m & \subset & B \end{array}$$

these are subintegrable systems.

Hitchin proved for the case of $\mathrm{rk} = 2$

We can prove for $\mathrm{rk} = n$.

Subintegrable system $\quad \Omega_{\mathcal{M}}\big|_{\mathcal{M}^m} = \Omega_{\mathcal{M}^m}$ non-degenerate symplectic form.

<u>Key</u> We consider sheaves coming from

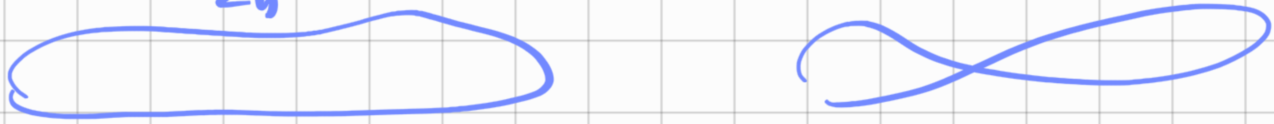
$\underline{\mathrm{Jac}(\Sigma_b)}$ $\qquad T\, h_m^{-1}(b) = H^1(\Sigma_b, \mathcal{O}_{\Sigma_b})$

fibres of $\mathcal{M}^m$

$$\begin{array}{c} H^1(\Sigma_b, \mathcal{O}_{\Sigma_b}) \\ \Big\updownarrow \text{ pairs} \\ T_b\mathcal{B}^m = \underline{H^0(\Sigma_b, K_{\Sigma_b})} \end{array}$$

In the case $\mathcal{B}^1$

$$\Sigma_b \xrightarrow{\ \nu\ } C_b$$



$$\downarrow$$

$$T\mathrm{Jac}(C_b) = H^1(C_b) \qquad T\mathrm{Jac}(\Sigma_b) = \underline{H^1(\mathcal{O}_{\Sigma_b})}$$

$$\mathbb{C}^* \hookrightarrow \mathrm{Jac}(C_b) \xrightarrow{\ \nu^*\ } \mathrm{Jac}(\Sigma_b)$$

The closure of the fibres of ν^* are Hecke cycles of Higgs bundles.

complex Lagrangians $\mathcal{HC} = \bigcup_{b \in \mathcal{B}^m} (\nu^*)^{-1}(\mathcal{F})$, $\quad \mathcal{F} \in \mathrm{Jac}(\Sigma_b)$

- Isotropic because pairing $H^1(\Sigma_b)\ \underbrace{H^0(K_{\Sigma_b})}_{T^*\mathcal{B}^m}$ is 0

- mid.

$\mathcal{M}^1$ is not a HK not BBB.

$\mathcal{HC}^1$ is complex Lagrangian BAA

Examples of BAA

MS

Examples of BBB

(V, ∇) flat

$\downarrow$

$N \subset \mathcal{M}$

complex Lagrangian for $\Omega_{\mathcal{M}}$

(W, ∇) hyperkahe $\nabla^{(0,1)}$ for

$\downarrow$

$R \subset \mathcal{M}$

Hyperkahler subvars of $\mathcal{M}_X$

MS($\mathcal{HC}$)