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Berezin-Toeplitz quantization and star products for compact Kähler manifolds

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Abstract. For compact quantizable Kähler manifolds certain naturally defined star products and their constructions are reviewed. The presentation centers around the Berezin-Toeplitz quantization scheme which is explained. As star products the Berezin-Toeplitz, Berezin, and star product of geometric quantization are treated in detail. It is shown that all three are equivalent. A prominent role is played by the Berezin transform and its asymptotic expansion. A few ideas on two general constructions of star products of separation of variables type by Karabegov and by Bordemann–Waldmann respectively are given. Some of the results presented is work of the author partly joint with Martin Bordemann, Eckhard Meinrenken and Alexander Karabegov. At the end some works which make use of graphs in the construction and calculation of these star products are sketched.

1. Introduction

Without any doubts the concepts of quantization is of fundamental importance in modern physics. These concepts are equally influential in mathematics. The problems appearing in the physical treatments give a whole variety of questions to be solved by mathematicians. Even more, quantization challenges mathematicians to develop corresponding mathematical concepts with necessary rigor. Not only that they are inspiring in the sense that we mathematician provide solutions, but these developments will help to advance our mathematical disciplines. It is not the place here to try to give some precise definition what is quantization. I only mention that one mathematical aspect of quantization is to pass from the classical "commutative" world to the quantum "non-commutative" world. There are many possible aspects of this passage. One way is to replace the algebra of classical physical observables (functions depending locally on "position" and "momenta"), i.e. the commutative algebra of functions on the phase-space manifold, by a non-commutative algebra of operators acting on a certain Hilbert space. Another way

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is to "deform" the pointwise product in the algebra of functions into some non-commutative product $\star$. The first method is called operator quantization, the second deformation quantization and the product $\star$ is called a star product. In both cases by some limiting process the classical situation should be recovered. I did not touch the question whether it is possible at all to obtain such objects if one poses certain desirable conditions. For example, the desired properties for the star product (to be explained in the article further down) does not allow to deform the product inside of the function algebra for all functions. One is forced to pass to the algebra of formal power series over the functions and deform there. The resulting object will be a formal deformation quantization.

A special case of the operator method is geometric quantization. One chooses a complex hermitian (pre)quantum line bundle on the phase space manifold. The operators act on the space of global sections of the bundle or on suitable subspaces. In the that we can endow our phase-space manifold with the structure of a Kähler manifold (and only this case we are considering here) we have a more rigid situation. Our quantum line bundle should carry a holomorphic structure, if the bundle exists at all. The passage to the classical limit will be obtained by considering higher and higher tensor powers of the quantum line bundle. The sections of the bundle are the candidates of the quantum states. But they depend on too many independent variables. In the Kähler setting there is the naturally defined subspace of holomorphic sections. These sections are constant in anti-holomorphic directions. They will be the quantum states. This selection is sometimes called Kähler polarization.

In this review we will mainly deal with another type of operators on the space of holomorphic sections of the bundle. These will be the Toeplitz operators. They are naturally defined for Kähler manifolds. The assignment defines the Berezin-Toeplitz (BT) quantization scheme. Berezin himself considered it for certain special manifold [**11**], [**15**].

Being a quantum line bundle means that the curvature of the holomorphic hermitian line bundle is essentially equal to the Kähler form. See Section 2 for the precise formulation. A Kähler manifold is called quantizable if it admits a quantum line bundle. We will explain below that this is really a condition which not always can be fulfilled.

The author in joint work with Martin Bordemann and Eckhard Meinrenken [**18**] showed that at least in the compact quantizable Kähler case the BT-quantization has the correct semi-classical limit behavior, hence it is a quantization, see Theorem 3.3. In the compact Kähler case the operator of geometric quantization is asymptotically related to the Toeplitz operator, see (3.11). The details are presented in Section 3.

The special feature of the Berezin-Toeplitz quantization approach is that it does not only provide an operator quantization but also an intimately related star product, the Berezin-Toeplitz star product $\star_{BT}$. It is obtained by "asymptotic expansion" of the product of the two Toeplitz operators associated to the two functions to be $\star$-multiplied, see (4.4). After recalling the definition of a star product in Section 4.1, the results about existence and the properties of $\star_{BT}$ are given in Section 4.2. These are results of the author partly in joint work with Bordemann, Meinrenken, and Karabegov. The star product is a star product of separation of variables type (in the sense of Karabegov) or equivalently of Wick type (in the sense of Bordemann and Waldmann). We recall Karabegov's construction of star products of this type. In particular, we discuss his formal Berezin transform.

In Section 5 we introduce the disc bundle associated to the quantum line bundle and introduce the global Toeplitz operators. The individual Toeplitz operators for each tensor power of the line bundle correspond to its modes. The symbol calculus of generalized Toeplitz operators due to Boutet de Monvel and Guillemin [**21**] is used to prove some parts of the above mentioned results. In Section 5.3 as an illustration we explain how $\star_{BT}$ is constructed.

Other important techniques which we use in this context are Berezin-Rawnsley's coherent states, co- and contra-variant symbols [**24**] [**25**] [**26**] [**27**]. Starting from a function on M, assigning to it its Toeplitz operator and then calculating the co-variant symbol of the operator will yield another function. The corresponding map on the space of function is called Berezin transform I, see Section 7. The map will depend on the chosen tensor power m of the line bundle. Theorem 7.2, obtained jointly with Karabegov, shows that it has a complete asymptotic expansion. One of the ingredients of the proof is the off-diagonal expansion of the Bergman kernel in the neighborhood of the diagonal [**57**].

With the help of the Berezin transform I the Berezin star product can be defined

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)).$$

In Karabegov's terminology both star products are dual and opposite to each other.

In Section 8.3 a summary of the naturally defined star products are given. These are $\star_{BT}$, $\star_B$, $\star_{GQ}$ (the star product of geometric quantization), $\star_{BW}$ (the star product of Bordemann and Waldmann constructed in a manner à la Fedosov, see Section 9.1). The star products $\star_{BT}, \star_{BW}$ are of separation of variables type, $\star_B$ also but with the role of holomorphic and antiholomorphic variables switched, $\star_{GQ}$ is neither nor. The first three star products are equivalent.

How the knowledge of the asymptotic expansion of the Berezin transform will allow to calculate the coefficients of the Berezin star product and recursively of the Berezin-Toeplitz star product is explained in Section 8.4.

In the Section 9 we consider the Bordemann-Waldmann star product [**19**] and make some remarks how graphs are of help in expressing the star product in a convenient form. The work of Reshetikhin and Takhtajan [**77**], Gammelgaard [**48**], and Huo Xu [**92**], [**93**] are sketched.

In an excursion we describe Kontsevich's construction [**59**] of a star product for arbitrary Poisson structures on $\mathbb{R}^n$.

The closing Section 11 gives some applications of the Berezin-Toeplitz quantization scheme.

This review is based on a talk which I gave in the frame of the *Thematic Program on Quantization, Spring 2011*, at the University of Notre Dame, USA. Some of the material was added on the basis of the questions and the discussions of the audience. I am grateful to the organizers Sam Evens, Michael Gekhtman, Brian Hall, and Xiaobo Liu, and to the audience. All of them made this activity such a pleasant and successful event. In its present version the review supplements and updates [**85**],[**86**]. Other properties, like the properties of the coherent state embedding, more about Berezin symbols, traces and examples can be found there. In particular, [**85**] contains a more complete list of related works of other authors.

2. The geometric setup

In the following let (M,ω) be a Kähler manifold. This means M is a complex manifold (of complex dimension n) and ω, the Kähler form, is a non-degenerate closed positive $(1,1)$-form. In the interpretation of physics M will be the phase-space manifold. (But besides the jargon we will use nothing from physics here.) Further down we will assume that M is compact.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with associative product given by point-wise multiplication. After forgetting the complex structure of M, our form ω will become a symplectic form and we introduce on $C^\infty(M)$ a Lie algebra structure, the *Poisson bracket* $\{.,.\}$, in the following way. First we assign to every $f\in C^\infty(M)$ its *Hamiltonian vector field* X_f, and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$\omega(X_f,\cdot)=df(\cdot),\qquad \{\,f,g\,\}:=\omega(X_f,X_g)\ . \tag{2.1}$$

In this way $C^\infty(M)$ becomes a *Poisson algebra*, i.e. we have the compatibility

$$\{h,f\cdot g\}=\{h,f\}\cdot g+f\cdot\{h,g\},\qquad f,g,h\in C^\infty(M). \tag{2.2}$$

The next step in the geometric set-up is the choice of a quantum line bundle. In the Kähler case a *quantum line bundle* for (M,ω) is a triple (L,h,∇), where L is a holomorphic line bundle, h a Hermitian metric on L, and ∇ a connection compatible with the metric h and the complex structure, such that the (pre)quantum condition

$$\begin{gathered}\operatorname{curv}_{L,\nabla}(X,Y):=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}=-\,\mathrm{i}\,\omega(X,Y),\\ \text{in other words}\quad \operatorname{curv}_{L,\nabla}=-\,\mathrm{i}\,\omega\end{gathered} \tag{2.3}$$

is fulfilled. By the compatibility requirement ∇ is uniquely fixed. With respect to a local holomorphic frame of the bundle the metric h will be represented by a function $\hat h$. Then the curvature with respect to the compatible connection is given by $\overline{\partial}\partial\log\hat h$. Hence, the quantum condition reads as

$$\mathrm{i}\,\overline{\partial}\partial\log\hat h=\omega\ . \tag{2.4}$$

If there exists such a quantum line bundle for (M,ω) then M is called quantizable. Sometimes the pair manifold and quantum line bundle is called quantized Kähler manifold.

Remark 2.1. Not all Kähler manifolds are quantizable. In the compact Kähler case from (2.3) it follows that the curvature is a positive form, hence L is a positive line bundle. By the Kodaira embedding theorem [**83**] there exists a positive tensor power $L^{\otimes m_0}$ which has enough global holomorphic sections to embed the complex manifold M via these sections into projective space $\mathbb{P}^N(\mathbb{C})$ of suitable dimension N. By Chow's theorem [**83**] it is a smooth projective variety. The line bundle $L^{\otimes m_0}$ which gives an embedding is called very ample. This implies for example, that only those higher dimensional complex tori are quantizable which admit "enough theta functions", i.e. which are abelian varieties.

A warning is in order, let $\phi:M\mapsto\mathbb{P}^N(\mathbb{C})$ be the above mentioned embedding as complex manifolds. This embedding is in general not a Kähler embedding, i.e. $\phi^*(\omega_{FS})\neq\omega$, where ω_{FS} is the standard Fubini-Study Kähler form for $\mathbb{P}^N(\mathbb{C})$. Hence, we cannot restrict our attention only on Kähler submanifolds of projective space.

For compact Kähler manifolds we will always assume that the quantum bundle L itself is already very ample. This is not a restriction as $L^{\otimes m_0}$ will be a quantum line bundle for the rescaled Kähler form $m_0\omega$ for the same complex manifold M.

Next, we consider all positive tensor powers of the quantum line bundle: $(L^m, h^{(m)}, \nabla^{(m)})$, here $L^m := L^{\otimes m}$ and $h^{(m)}$ and $\nabla^{(m)}$ are naturally extended. We introduce a product on the space of sections. First we take the Liouville form $\Omega = \frac{1}{n!}\omega^{\wedge n}$ as volume form on M and then set for the product and the norm on the space $\Gamma_\infty(M, L^m)$ of global C^∞-sections (if they are finite)

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\, \Omega\ , \qquad ||\varphi|| := \sqrt{\langle \varphi, \varphi \rangle}\ . \tag{2.5}$$

Let $\mathrm{L}^2(M, L^m)$ be the L^2-completed space of bounded sections with respect to this norm. Furthermore, let $\Gamma^b_{hol}(M, L^m)$ be the space of global holomorphic sections of L^m which are bounded. It can be identified with a closed subspace of $\mathrm{L}^2(M, L^m)$. Denote by

$$\Pi^{(m)} : \mathrm{L}^2(M, L^m) \to \Gamma^b_{hol}(M, L^m) \tag{2.6}$$

the orthogonal projection.

If the manifold M is compact "being bounded" is of course no restriction. Furthermore, $\Gamma_{hol}(M, L^m) = \Gamma^b_{hol}(M, L^m)$ and this space is finite-dimensional. Its dimension $N(m) := \dim \Gamma_{hol}(M, L^m)$ will be given by the Hirzebruch-Riemann-Roch Theorem [**83**]. Our projection will be

$$\Pi^{(m)} : \mathrm{L}^2(M, L^m) \to \Gamma_{hol}(M, L^m)\ . \tag{2.7}$$

If we fix an orthonormal basis $s_l^{(m)}, l = 1, \ldots, N(m)$ of $\Gamma_{hol}(M, L^m)$ then[1]

$$\Pi^{(m)}(\psi) = \sum_{l=1}^{N(m)} \langle s_l^{(m)}, \psi \rangle \cdot s_l^{(m)}. \tag{2.8}$$

3. Berezin-Toeplitz operator quantization

Let us start with the compact Kähler manifold case. I will make some remarks at the end of this section on the general setting. In the interpretation of physics, our manifold M is a phase-space. Classical observables are (real-valued) functions on the phase space. Their values are the physical values to be found by experiments. The classical observables commute under pointwise multiplication. One of the aspects of quantization is to replace the classical observable by something which is non-commutative. One approach is to replace the functions by operators on a certain Hilbert space (and the physical values to be measured should correspond to eigenvalues of them). In the Berezin-Toeplitz (BT) operator quantization this is done as follows.

DEFINITION 3.1. For a function $f \in C^\infty(M)$ the associated *Toeplitz operator* $T_f^{(m)}$ *(of level m)* is defined as

$$T_f^{(m)} := \Pi^{(m)}\,(f\cdot) : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)\ . \tag{3.1}$$

[1]In my convention the scalar product is anti-linear in the first argument.

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

With respect to the explicit representation (2.8) we obtain

$$T_f^{(m)}(s) := \sum_{l=1}^{N(m)} \langle s_l^{(m)}, f\, s\rangle\; s_l^{(m)}. \tag{3.2}$$

After expressing the scalar product (2.5) we get a representation of $T_f^{(m)}$ as an integral

$$T_f^{(m)}(s)(x) = \int_M f(y) \left(\sum_{l=1}^{N(m)} h^{(m)}(s_l^{(m)}, s)(y)\, s_l^{(m)}(x)\right) \Omega(y). \tag{3.3}$$

The space $\Gamma_{hol}(M, L^m)$ is the *quantum space (of level m)*. The linear map

$$T^{(m)} : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M, L^m)\big), \quad f \to T_f^{(m)} = \Pi^{(m)}(f\cdot)\ , m \in \mathbb{N}_0 \tag{3.4}$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map (of level m)*. It will neither be a Lie algebra homomorphism nor an associative algebra homomorphism as in general

$$T_f^{(m)}\, T_g^{(m)} = \Pi^{(m)}\,(f\cdot)\,\Pi^{(m)}\,(g\cdot)\,\Pi^{(m)} \neq \Pi^{(m)}\,(fg\cdot)\,\Pi = T_{fg}^{(m)}.$$

For M a compact Kähler manifold it was already mentioned that the space $\Gamma_{hol}(M, L^m)$ is finite-dimensional. On a fixed level m the BT quantization is a map from the infinite dimensional commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. A lot of classical information will get lost. To recover this information one has to consider not just a single level m but all levels together as done in the

Definition 3.2. The Berezin-Toeplitz (BT) quantization is the map

$$C^\infty(M) \to \prod_{m\in\mathbb{N}_0} \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to (T_f^{(m)})_{m\in\mathbb{N}_0}. \tag{3.5}$$

In this way a family of finite-dimensional (matrix) algebras and a family of maps are obtained, which in the classical limit should "converges" to the algebra $C^\infty(M)$. That this is indeed the case and what "convergency" means will be made precise in the following.

Set for $f \in C^\infty(M)$ by $|f|_\infty$ the sup-norm of f on M and by

$$||T_f^{(m)}|| := \sup_{\substack{s\in\Gamma_{hol}(M,L^m)\\ s\neq 0}} \frac{||T_f^{(m)} s||}{||s||} \tag{3.6}$$

the operator norm with respect to the norm (2.5) on $\Gamma_{hol}(M, L^m)$.

That the BT quantization is indeed a quantization in the sense that it has the correct semi-classical limit, or that it is a strict quantization in the sense of Rieffel, is the content of the following theorem from 1994.

THEOREM 3.3. *[Bordemann, Meinrenken, Schlichenmaier]* [**18**]
(a) For every $f \in C^\infty(M)$ *there exists a* $C > 0$ *such that*

$$|f|_\infty - \frac{C}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ . \tag{3.7}$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$.
(b) For every $f, g \in C^\infty(M)$

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m}) \ . \tag{3.8}$$

(c) For every $f, g \in C^\infty(M)$

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m}) \ . \tag{3.9}$$

The original proof uses the machinery of generalized Toeplitz structures and operators as developed by Boutet de Monvel and Guillemin [**21**]. We will give a sketch of some parts of the proof in Section 5 and Section 7.3. In the meantime there also exists other proofs on the basis of Toeplitz kernels, Bergman kernels, Berezin transform etc. Each of them give very useful additional insights.

We will need in the following from [**18**]

PROPOSITION 3.4. *On every level* m *the Toeplitz map*

$$C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to T_f^{(m)},$$

is surjective.

Let us mention that for real-valued f the Toeplitz operator $T_f^{(m)}$ will be self-adjoint. Hence, they have real-valued eigenvalues.

REMARK 3.5. (Geometric Quantization.) Kostant and Souriau introduced the operators of geometric quantization in this geometric setting. In a first step the prequantum operator associated to the bundle L^m (and acting on its sections) for the function $f \in C^\infty(M)$ is defined as $P_f^{(m)} := \nabla^{(m)}_{X_f^{(m)}} + \mathrm{i}\, f \cdot id$. Here $X_f^{(m)}$ is the Hamiltonian vector field of f with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$ and $\nabla^{(m)}_{X_f^{(m)}}$ is the covariant derivative. In the context of geometric quantization one has to choose a polarization. This corresponds to the fact that the "quantum states", i.e. the sections of the quantum line bundle, should only depend on "half of the variables" of the phase-space manifold M. In general, such a polarization will not be unique. But in our complex situation there is a canonical one by taking the subspace of holomorphic sections. This polarization is called *Kähler polarization*. This means that we only take those sections which are constant in anti-holomorphic directions. The operator of geometric quantization with Kähler polarization is defined as

$$Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}. \tag{3.10}$$

By the surjectivity of the Toeplitz map there exists a function f_m, depending on the level m, such that $Q_f^{(m)} = T_{f_m}^{(m)}$. The Tuynman lemma [**89**] gives

$$Q_f^{(m)} = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f}, \tag{3.11}$$

where Δ is the Laplacian with respect to the Kähler metric given by ω. It should be noted that for (3.11) the compactness of M is essential.

As a consequence, which will be used later, the operators $Q_f^{(m)}$ and the $T_f^{(m)}$ have the same asymptotic behavior for $m \to \infty$.

REMARK 3.6. (The non-compact situation.) If our Kähler manifold is not necessarily compact then in a first step we consider as quantum space the space of bounded holomorphic sections $\Gamma^b_{hol}(M, L^m)$. Next we have to restrict the space of quantizable functions to a subspace of $C^\infty(M)$ such that the quantization map (3.5) (now restricted) will be well-defined. One possible choice is the subalgebra of functions with compact support. After these restrictions the Berezin-Toeplitz operators are defined as above. In the case of M compact, everything reduces to the already given objects. Unfortunately, there is no general result like Theorem 3.3 valid for arbitrary quantizable Kähler manifolds (e.g. for non-compact ones). There are corresponding results for special important examples. But they are more or less shown by case by case studies of the type of examples using tools exactly adapted to this situation. See [**85**] for references in this respect.

REMARK 3.7. (Auxiliary vector bundle.) We return to the compact manifold case. It is also possible to generalize the situation by considering an additional auxiliary hermitian holomorphic line bundle E. The sequence of quantum spaces is now the space of holomorphic sections of the bundles $E \otimes L^m$. For the case that E is a line bundle this was done, e.g. by Hawkins [**51**], for the general case by Ma and Marinescu, see [**64**] for the details. See also Charles [**32**]. By the hermitian structure of E we have a scalar product and a corresponding projection operator from the space of all sections to the space of holomorphic sections. The Toeplitz operator $T_f^{(m)}$ is defined for $f \in C^\infty(M, End(E))$. The situation considered in this review is that E equals the trivial line bundle. But similar results can be obtained in the more general set-up. This is also true with respect to the star product discussed in Section 4[2]. Of special importance, beside the trivial bundle case, is the case when the auxiliary vector bundle is a square root L_0 of the canonical line bundle K_M, i.e. $L_0^{\otimes 2} = K_M$ (if the square root exists). Recall that $K_M = \bigwedge^n \Omega_M$, where $n = \dim_{\mathbb{C}} M$ and Ω_M is the rang n vector bundle of holomorphic 1-differentials. The corresponding quantization is called quantization with metaplectic corrections. It turns out that with the metaplectic correction the quantization behaves better under natural constructions. An example is the *Quantization Commutes with Reduction* problem in the case that we have a well-defined action of a group G on the compact (quantizable) Kähler manifold with G-equivariant quantum line bundle. Under suitable conditions on the action we have a linear isomorphy of the G-invariant subspace of the quantum spaces $\mathrm{H}^0(M, L^m)^G$ with the quantum spaces $\mathrm{H}^0(M//G, (L//G)^m)$. This was shown by Guillemin and Sternberg [**49**]. But this isomorphy is not unitary. If one uses the quantum spaces with respect to the metaplectic correction then at least it is asymptotically (i.e. $m \to \infty$) unitary. This was shown independently[3] and with slightly different aspects by Ma and Zhang [**66**] (partly based on work of Zhang [**96**]) and by Hall and Kirwin [**50**]. See also [**63**].

[2]For E not a line bundle the Berezin-Toeplitz star product is a star product in $C^\infty(X, End(E))[[\nu]]$. This might be considered as a quantization with additional internal degrees of freedom, see [**64**, Remark 2.27].

[3]I am grateful to Xiaonan Ma for pointing this out to me.

For interesting details about these approaches see also the article of Kirwin [**58**] explaining some of the relations. For the general singular situation, see Li [**60**].

Another case when the quantization with metaplectic correction is more functorial is if one considers families of Kähler manifolds as they show up e.g. in the context of deforming complex structures on a given symplectic manifold. See work by Charles [**33**] and Andersen, Gammelgaard and Lauridsen [**6**].

4. Deformation quantization – star products

4.1. General definitions. There is another approach to quantization. One deforms the commutative algebra of functions "into non-commutative directions given by the Poisson bracket". It turns out that this can only be done on the formal level. One obtains a deformation quantization, also called star product. This notion was around quite a long time. See e.g. Berezin [**13**],[**15**], Moyal [**69**], Weyl [**91**], etc. Finally, the notion was formalized in [**9**]. See [**36**] for some historical remarks.

For a given Poisson algebra $(C^\infty(M), \cdot, \{\,,\,\})$ of smooth functions on a manifold M, a *star product* for M is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, the space of formal power series with coefficients from $C^\infty(M)$, such that for $f, g \in C^\infty(M)$

(1) $f \star g = f \cdot g \mod \nu$,
(2) $(f \star g - g \star f)\,/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

The star product of two functions f and g can be expressed as

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M), \tag{4.1}$$

and is extended $\mathbb{C}[[\nu]]$-bilinearly. It is called differential (or local) if the $C_k(\,,\,)$ are bidifferential operators with respect to their entries. If nothing else is said one requires $1 \star f = f \star 1 = f$, which is also called "null on constants".

Remark 4.1. (Existence) Given a Poisson bracket, is there always a star product? In the usual setting of deformation theory there always exists a trivial deformation. This is not the case here, as the trivial deformation of $C^\infty(M)$ to $\mathcal{A}$ extending the point-wise product trivially to the power series, is not allowed as it does not fulfill the second condition for the commutator of being a star product (at least not if the Poisson bracket is non-trivial). In fact the existence problem is highly non-trivial. In the symplectic case different existence proofs, from different perspectives, were given by DeWilde-Lecomte [**34**], Omori-Maeda-Yoshioka [**71**], and Fedosov [**44**]. The general Poisson case was settled by Kontsevich [**59**]. For more historical information see the review [**36**].

Two star products $\star$ and $\star'$ for the same Poisson structure are called *equivalent* if and only if there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $\quad B(f) \star' B(g) = B(f \star g)$.

To every equivalence class of a differential star product its *Deligne-Fedosov class* can be assigned. It is a formal de Rham class of the form

$$cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M,\mathbb{C})[[\nu]]). \tag{4.2}$$

This assignment gives a 1:1 correspondence between equivalence classes of star products and such formal forms.

In the Kähler case we might look for star products adapted to the complex structure. Karabegov [**52**] introduced the notion of star products with *separation of variables type* for differential star products. The star product is of this type if in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in holomorphic and the second argument in anti-holomorphic directions. Bordemann and Waldmann in their construction [**19**] used the name *star product of Wick type.*[4] All such star products $\star$ are uniquely given (not only up to equivalence) by their Karabegov form $kf(\star)$ which is a formal closed $(1,1)$ form. We will return to it in Section 4.3

4.2. The Berezin-Toeplitz deformation quantization.

Theorem 4.2. [**18**],[**78**],[**80**],[**81**],[**57**] *There exists a unique differential star product*

$$f \star_{BT} g = \sum \nu^k C_k(f,g) \tag{4.3}$$

such that

$$T^{(m)}_f T^{(m)}_g \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T^{(m)}_{C_k(f,g)}. \tag{4.4}$$

This star product is of separation of variables type with classifying Deligne-Fedosov class cl and Karabegov form kf

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}), \qquad kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \tag{4.5}$$

First, the asymptotic expansion in (4.4) has to be understood in a strong operator norm sense. For $f,g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T^{(m)}_f T^{(m)}_g - \sum_{0\leq j<N} \left(\frac{1}{m}\right)^j T^{(m)}_{C_j(f,g)}|| \leq K_N(f,g) \left(\frac{1}{m}\right)^N . \tag{4.6}$$

Second, the used forms, resp. classes are defined as follows. Let K_M be the canonical line bundle of M, i.e. the n^{th} exterior power of the holomorphic bundle of 1-differentials. The canonical class δ is the first Chern class of this line bundle, i.e. $\delta := c_1(K_M)$. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1,1)$-form ω_{can}. In our sign conventions we have $\delta = [\omega_{can}]$. The Karabegov form will be introduced in Section 4.3.

[4]In Karabegov's original approach the role of holomorphic and antiholomorphic variables are switched, i.e. in the approach of Bordemann-Waldmann they are of anti-Wick type. Unfortunately we cannot simply retreat to one these conventions, as we really have to deal in the following with naturally defined star products and relations between them, which are of separation of variables type of both conventions.

REMARK 4.3. Using Theorem 3.3 and the Tuynman relation (3.11) one can show that there exists a star product $\star_{GQ}$ given by asymptotic expansion of the product of geometric quantization operators. The star product $\star_{GQ}$ is equivalent to $\star_{BT}$, via the equivalence $B(f) := (id - \nu\frac{\Delta}{2})f$. In particular, it has the same Deligne-Fedosov class. But it is not of separation of variables type, see [**81**].

4.3. Star product of separation of variables type. In [**52, 53**] Karabegov not only gave the notion of *separation of variables type*, but also a proof of existence of such formal star products for any Kähler manifold, whether compact, non-compact, quantizable, or non-quantizable. Moreover, he classified them completely as individual star product not only up to equivalence.

In this set-up it is quite useful to consider more generally pseudo-Kähler manifolds (M, ω_{-1}), i.e. complex manifolds with a non-degenerate closed $(1,1)$-form ω_{-1} not necessarily positive. (In this context it is convenient to denote by ω_{-1} the ω we use at other places of the article.)

A formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \dots \tag{4.7}$$

is called a formal deformation of the form $(1/\nu)\omega_{-1}$ if the forms ω_r, $r \geq 0$, are closed but not necessarily nondegenerate (1,1)-forms on M. Karabegov showed that to every such $\widehat{\omega}$ there exists a star product $\star$. Moreover he showed that all deformation quantizations with separation of variables on the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parameterized by the formal deformations of the form $(1/\nu)\omega_{-1}$. By definition the *Karabegov form* $kf(\star) := \widehat{\omega}$, i.e. it is taken to be the $\widehat{\omega}$ defining $\star$.

Let us indicate the principal idea of the construction. First, assume that we have such a star product $(\mathcal{A} := C^\infty(M)[[\nu]], \star)$. Then for $f, g \in \mathcal{A}$ the operators of left and right multiplication L_f, R_g are given by $L_f g = f \star g = R_g f$. The associativity of the star-product $\star$ is equivalent to the fact that L_f commutes with R_g for all $f, g \in \mathcal{A}$. If a star product is differential then L_f, R_g are formal differential operators. Now Karabegov constructs his star product associated to the deformation $\widehat{\omega}$ in the following way. First he chooses on every contractible coordinate chart $U \subset M$ (with holomorphic coordinates $\{z_k\}$) its formal potential

$$\widehat{\Phi} = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \dots, \qquad \widehat{\omega} = i\partial\bar{\partial}\widehat{\Phi}. \tag{4.8}$$

Then the construction is done in such a way that the left (right) multiplication operators $L_{\partial\widehat{\Phi}/\partial z_k}$ ($R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$) on U are realized as formal differential operators

$$L_{\partial\widehat{\Phi}/\partial z_k} = \partial\widehat{\Phi}/\partial z_k + \partial/\partial z_k, \quad \text{and} \quad R_{\partial\widehat{\Phi}/\partial \bar{z}_l} = \partial\widehat{\Phi}/\partial \bar{z}_l + \partial/\partial \bar{z}_l. \tag{4.9}$$

The set $\mathcal{L}(U)$ of all left multiplication operators on U is completely described as the set of all formal differential operators commuting with the point-wise multiplication operators by antiholomorphic coordinates $R_{\bar{z}_l} = \bar{z}_l$ and the operators $R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$. From the knowledge of $\mathcal{L}(U)$ the star product on U can be reconstructed. This follows from the simple fact that $L_g(1) = g$ and $L_f(L_g)(1) = f \star g$. The operator corresponding to the left multiplication with the (formal) function g can recursively (in the ν-degree) be calculated from the fact that it commutes with the operators $R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$. The local star-products agree on the intersections of the charts and define the global star-product $\star$ on M. See the original work of Karabegov [**52**] for these statements.

We have to mention that this original construction of Karabegov will yield a star product of separation of variables type but with the role of holomorphic and antiholomorphic variables switched. This says for any open subset $U \subset M$ and any holomorphic function a and antiholomorphic function b on U the operators L_a and R_b are the operators of point-wise multiplication by a and b respectively, i.e., $L_a = a$ and $R_b = b$.

The construction of Karabegov is on one side very universal without any restriction on the (pseudo) Kähler manifold. But it does not establish any connection to an operator representation. The existence of such an operator representation is related in a vague sense to the quantization condition. The BT deformation quantization has such a relation and singles out a unique star product. Modulo switching the role of holomorphic and anti-holomorphic variable $\star_{BT}$ corresponds to a unique Karabegov form. This form is given in (4.5). The identification is done in Section 8.1 further down. That the form starts with $(-1/\nu)\omega$ is due to the fact that the role of the variables have to be switched to end up in Karabegov's classification.

4.4. Karabegov's formal Berezin transform. Given a pseudo-Kähler manifold (M, ω_{-1}). In the frame of his construction and classification Karabegov assigned to each star products $\star$ with the separation of variables property the formal *Berezin transform* $I_\star$. It is as the unique formal differential operator on M such that for any open subset $U \subset M$, antiholomorphic functions a and holomorphic functions b on U the relation

$$a \star b = I(b \cdot a) = I(b \star a), \tag{4.10}$$

holds true. The last equality is automatic and is due to the fact, that by the separation of variables property $b \star a$ is the point-wise product $b \cdot a$. He shows

$$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \quad I_0 = id, \quad I_1 = \Delta. \tag{4.11}$$

Let us summarize. Karabegov's classification gives for a fixed pseudo-Kähler manifold a 1:1 correspondence between
(1) the set of star products with separation of variables type in Karabegov convention and
(2) the set of formal deformations (4.7) of ω_{-1}.
Moreover, the formal Berezin transform $I_\star$ determines the $\star$ uniquely.

We will introduce further down a Berezin transform in the set-up of the BT quantization. In [**57**] it is shown that its asymptotic expansion gives a formal Berezin transform in the sense of Karabegov, associated to a star product related to $\star_{BT}$ explained as follows.

4.5. Dual and opposite star products. Given for the pseudo-Kähler manifold (M, ω_{-1}) a star product $\star$ of separation of variables type (in Karabegov convention) Karabegov defined with the help of $I = I_\star$ the following associated star products. First the *dual* star-product $\tilde{\star}$ on M is defined for $f, g \in \mathcal{A}$ by the formula

$$f \,\tilde{\star}\, g = I^{-1}(I(g) \star I(f)). \tag{4.12}$$

It is a star-product with separation of variables but now on the pseudo-Kähler manifold $(M, -\omega_{-1})$. Denote by $\tilde{\omega} = -(1/\nu)\omega_{-1} + \tilde{\omega}_0 + \nu\tilde{\omega}_1 + \ldots$ the formal form parameterizing the star-product $\tilde{\star}$. By definition $\tilde{\omega} = kf(\tilde{\star})$. Its formal Berezin transform equals I^{-1}, and thus the dual to $\tilde{\star}$ is again $\star$.

Given a star product, the opposite star product is obtained by switching the arguments. Of course the sign of the Poisson bracket is changed. Now we take the opposite of the dual star-product, $\star' = \tilde{\star}^{op}$, given by

$$f \star' g = g \,\tilde{\star}\, f = I^{-1}(I(f) \star I(g)). \tag{4.13}$$

It defines a deformation quantization with separation of variables on M, but with the roles of holomorphic and antiholomorphic variables swapped - in contrast to $\star$. But now the pseudo-Kähler manifold will be (M, ω_{-1}). Indeed the formal Berezin transform I establishes an equivalence of the deformation quantizations $(\mathcal{A}, \star)$ and $(\mathcal{A}, \star')$.

How is the relation to the Berezin-Toeplitz star product $\star_{BT}$ of Theorem 4.2? There exists a certain formal deformation $\widehat{\omega}$ of the form $(1/\nu)\omega$ which yields a star product $\star$ in the Karabegov sense [**57**]. The opposite of its dual will be equal to the Berezin-Toeplitz star product, i.e.

$$\star_{BT} \;=\; \tilde{\star}^{op} \;=\; \star' \,. \tag{4.14}$$

The classifying Karabegov form $kf(\tilde{\star})$ will be the form (4.5). Here we fix the convention that we take for determining the Karabegov form of the BT star product the Karabegov form of the opposite one to adjust to Karabegov's original convention, i.e.

$$kf(\star_{BT}) := kf(\star_{BT}^{op}) = kf(\tilde{\star}). \tag{4.15}$$

As $\tilde{\star}$ is a star product for the pseudo-Kähler manifold $(M, -\omega)$ the $kf(\star_{BT})$ starts with $(-1/\nu)\omega$.

The formula (4.13) gives an equivalence between $\star$ and $\star_{BT}$ via I . Hence, we have for the Deligne-Fedosov class $cl(\star) = cl(\star_{BT})$, see the formula (4.5). We will identify $\widehat{\omega} = kf(\star)$ in Section 8.1.

5. Global Toeplitz operators

In this section we will indicate some parts of the proofs of Theorem 4.2 and Theorem 3.3. For this goal we consider the bundles L^m over the compact Kähler manifold M as associated line bundles of one unique S^1-bundle over M. The Toeplitz operator will appear as "modes" of a global Toeplitz operator. Moreover, we will need the same set-up to discuss coherent states, Berezin symbols, and the Berezin transform in the next sections.

5.1. The disc bundle. Recall that our quantum line bundle L was assumed to be already very ample. We pass to its dual line bundle $(U, k) := (L^*, h^{-1})$ with dual metric k. In the example of the projective space, the quantum line bundle is the hyperplane section bundle and its dual is the tautological line bundle. Inside the total space U, we consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

and denote by $\tau : Q \to M$ (or $\tau : U \to M$) the projections to the base manifold M.

The bundle Q is a contact manifold, i.e. there is a 1-form ν such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q. Moreover,

$$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M). \tag{5.1}$$

Denote by $\mathrm{L}^2(Q,\mu)$ the corresponding L^2-space on Q. Let $\mathcal{H}$ be the space of (differentiable) functions on Q which can be extended to holomorphic functions on the disc bundle (i.e. to the "interior" of the circle bundle), and $\mathcal{H}^{(m)}$ the subspace of $\mathcal{H}$ consisting of m-homogeneous functions on Q. Here m-homogeneous means $\psi(c\lambda) = c^m\psi(\lambda)$. For further reference let us introduce the following (orthogonal) projectors: the *Szegö projector*

$$\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}, \tag{5.2}$$

and its components the *Bergman projectors*

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}. \tag{5.3}$$

The bundle Q is a S^1-bundle, and the L^m are associated line bundles. The sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m. This identification is given on the level of the L^2 spaces by the map

$$\gamma_m : \mathrm{L}^2(M,L^m) \to \mathrm{L}^2(Q,\mu), \quad s \mapsto \psi_s \quad \text{where} \tag{5.4}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))). \tag{5.5}$$

Restricted to the holomorphic sections we obtain the unitary isomorphism

$$\gamma_m : \Gamma_{hol}(M,L^m) \cong \mathcal{H}^{(m)}. \tag{5.6}$$

5.2. Toeplitz structure. Boutet de Monvel and Guillemin introduced the notion of a Toeplitz structure (Π,Σ) and associated generalized Toeplitz operators [**21**]. If we specialize this to our situation then Π is the Szegö projector (5.2) and Σ is the submanifold

$$\Sigma := \{\, t\nu(\lambda) \mid \lambda \in Q,\, t>0 \,\} \ \subset\ T^*Q \setminus 0 \tag{5.7}$$

of the tangent bundle of Q defined with the help of the 1-form ν. It turns out that Σ is a symplectic submanifold, a symplectic cone.

A (generalized) *Toeplitz operator* of order k is an operator $A : \mathcal{H} \to \mathcal{H}$ of the form $A = \Pi \cdot R \cdot \Pi$ where R is a pseudo-differential operator (ΨDO) of order k on Q. The Toeplitz operators constitute a ring. The symbol of A is the restriction of the principal symbol of R (which lives on T^*Q) to Σ. Note that R is not fixed by A, but Boutet de Monvel and Guillemin showed that the symbols are well-defined and that they obey the same rules as the symbols of ΨDOs. In particular, the following relations are valid:

$$\sigma(A_1A_2) = \sigma(A_1)\sigma(A_2), \qquad \sigma([A_1,A_2]) = \mathrm{i}\,\{\sigma(A_1),\sigma(A_2)\}_\Sigma. \tag{5.8}$$

Here $\{.,.\}_\Sigma$ is the restriction of the canonical Poisson structure of T^*Q to Σ coming from the canonical symplectic form on T^*Q. Furthermore, a Toeplitz operator of order k with vanishing symbol is a Toeplitz operator of order $k-1$.

We will need the following two generalized Toeplitz operators:

(1) The generator of the circle action gives the operator $D_\varphi = \dfrac{1}{\mathrm{i}}\dfrac{\partial}{\partial\varphi}$, where φ is the angular variable. It is an operator of order 1 with symbol t. It operates on $\mathcal{H}^{(m)}$ as multiplication by m.

(2) For $f \in C^\infty(M)$ let M_f be the operator on $\mathrm{L}^2(Q,\mu)$ corresponding to multiplication with τ^*f. We set

$$T_f = \Pi \cdot M_f \cdot \Pi : \quad \mathcal{H} \to \mathcal{H}\ . \tag{5.9}$$

As M_f is constant along the fibers of τ, the operator T_f commutes with the circle action. Hence we can decompose

$$T_f = \prod_{m=0}^{\infty} T_f^{(m)} \ , \tag{5.10}$$

where $T_f^{(m)}$ denotes the restriction of T_f to $\mathcal{H}^{(m)}$. After the identification of $\mathcal{H}^{(m)}$ with $\Gamma_{hol}(M, L^m)$ we see that these $T_f^{(m)}$ are exactly the Toeplitz operators $T_f^{(m)}$ introduced in Section 3. We call T_f the global Toeplitz operator and the $T_f^{(m)}$ the local Toeplitz operators. The operator T_f is of order 0. Let us denote by $\tau_\Sigma : \Sigma \subseteq T^*Q \to Q \to M$ the composition then we obtain for its symbol $\sigma(T_f) = \tau_\Sigma^*(f)$.

5.3. The construction of the BT star product. To give a sketch of the proof of Theorem 4.2 we will need the statements of Theorem 3.3. The part (a) of this theorem we will show with the help of the asymptotic expansion of the Berezin transform in Section 7.3. The other parts will be sketched here, too. Full proofs of Theorem 4.2 can be found in [**81**], [**80**]. Full proofs of Theorem 3.3 in [**18**].

Let the notation be as in the last subsection. In particular, let T_f be the Toeplitz operator, D_φ the operator of rotation, and $T_f^{(m)}$, resp. $(m\cdot)$ their projections on the eigenspaces $\mathcal{H}^{(m)} \cong \Gamma_{hol}(M, L^m)$.

(a) The definition of the $C_j(f, g) \in C^\infty(M)$
The construction is done inductively in such a way that

$$A_N = D_\varphi^N T_f T_g - \sum_{j=0}^{N-1} D_\varphi^{N-j} T_{C_j(f,g)} \tag{5.11}$$

is always a Toeplitz operator of order zero. The operator A_N is S^1-invariant, i.e. $D_\varphi \cdot A_N = A_N \cdot D_\varphi$. As it is of order zero his symbol is a function on Q. By the S^1-invariance the symbol is even given by (the pull-back of) a function on M. We take this function as next element $C_N(f, g)$ in the star product. By construction, the operator $A_N - T_{C_N(f,g)}$ is of order -1 and $A_{N+1} = D_\varphi(A_N - T_{C_N(f,g)})$ is of order 0 and exactly of the form given in (5.11).

The induction starts with

$$A_0 = T_f T_g, \qquad \text{and} \tag{5.12}$$

$$\sigma(A_0) = \sigma(T_f)\sigma(T_g) = \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g) = \tau_\Sigma^*(f \cdot g) \ . \tag{5.13}$$

Hence, $C_0(f, g) = f \cdot g$ as required.
It remains to show statement (4.6) about the asymptotics. As an operator of order zero on a compact manifold A_N is bounded (ΨDOs of order 0 on compact manifolds are bounded). By the S^1-invariance we can write $A = \prod_{m=0}^{\infty} A^{(m)}$ where $A^{(m)}$ is the restriction of A on the orthogonal subspace $\mathcal{H}^{(m)}$. For the norms we get $||A^{(m)}|| \le ||A||$. If we calculate the restrictions we obtain

$$||m^N T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{N-1} m^{N-j} T_{C_j(f,g)}^{(m)}|| = ||A_N^{(m)}|| \le ||A_N|| \ . \tag{5.14}$$

After dividing by m^N Equation (4.6) follows. Bilinearity is clear. For $N = 1$ we obtain (3.9) and Theorem 3.3, Part (c).

(b) The Poisson structure

First we sketch the proof for (3.8). For a fixed $t > 0$

$$\Sigma_t := \{t \cdot \nu(\lambda) \mid \lambda \in Q\} \quad \subseteq \Sigma. \tag{5.15}$$

It turns out that $\omega_{\Sigma|\Sigma_t} = -t\tau_\Sigma^*\omega$. The commutator $[T_f, T_g]$ is a Toeplitz operator of order -1. From the above we obtain with (5.8) for the symbol of the commutator

$$\sigma([T_f, T_g])(t\nu(\lambda)) = \mathrm{i}\,\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma(t\nu(\lambda)) = -\,\mathrm{i}\,t^{-1}\{f,g\}_M(\tau(\lambda)) \ . \tag{5.16}$$

We consider the Toeplitz operator

$$A := D_\varphi^2\,[T_f, T_g] + \mathrm{i}\,D_\varphi\,T_{\{f,g\}} \ . \tag{5.17}$$

Formally this is an operator of order 1. Using $\sigma(T_{\{f,g\}}) = \tau_\Sigma^*\{f,g\}$ and $\sigma(D_\varphi) = t$ we see that its principal symbol vanishes. Hence it is an operator of order 0. Arguing as above we consider its components $A^{(m)}$ and get $||A^{(m)}|| \le ||A||$. Moreover,

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m^2[T_f^{(m)}, T_g^{(m)}] + \mathrm{i}\,mT_{\{f,g\}}^{(m)}. \tag{5.18}$$

Taking the norm bound and dividing it by m we get part (b) of Theorem 3.3. Using (5.6) the norms involved indeed coincide.

For the star product we have to show that $C_1(f,g) - C_1(g,f) = -\,\mathrm{i}\,\{f,g\}$. We write explicitly (5.14) for $N = 2$ and the pair of functions (f,g):

$$||m^2 T_f^{(m)} T_g^{(m)} - m^2 T_{f\cdot g}^{(m)} - mT_{C_1(f,g)}^{(m)}|| \le K \ . \tag{5.19}$$

A corresponding expression is obtained for the pair (g,f). If we subtract both operators inside of the norm we obtain (with a suitable K')

$$||m^2(T_f^{(m)}T_g^{(m)} - T_g^{(m)}T_f^{(m)}) - m(T_{C_1(f,g)}^{(m)} - T_{C_1(g,f)}^{(m)})|| \le K' \ . \tag{5.20}$$

Dividing by m and multiplying with i we obtain

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T^{(m)}_{\mathrm{i}\left(C_1(f,g) - C_1(g,f)\right)}|| = O(\frac{1}{m}) \ . \tag{5.21}$$

Using the asymptotics given by Theorem 3.3(b) for the commutator we get

$$||T^{(m)}_{\{f,g\} - \mathrm{i}\left(C_1(f,g) - C_1(g,f)\right)}|| = O(\frac{1}{m}) \ . \tag{5.22}$$

Taking the limit for $m \to \infty$ and using Theorem 3.3(a) we get

$$||\{f,g\} - \mathrm{i}\,(C_1(f,g) - C_1(g,f))||_\infty = 0 \ . \tag{5.23}$$

Hence indeed, $\{f,g\} = \mathrm{i}\,(C_1(f,g) - C_1(g,f))$. For the associativity and further results, see [**81**].

Within this approach the calculation of the coefficient functions $C_k(f,g)$ is recursively and not really constructive. In Section 8.4 we will show another way how to calculate the coefficients. It is based on the asymptotic expansion of the Berezin transform, which itself is obtained via the off-diagonal expansion of the Bergman kernel.

In fact the Toeplitz operators again can be expressed via kernel functions also related to the Bergman kernel. In this way certain extensions of the presented results are possible. See in particular work by Ma and Marinescu for compact symplectic manifolds and orbifolds. One might consult the review [**64**] for results and further references.

For another approach (still symbol oriented) to Berezin Toeplitz operator and star product quantization see Charles [**31**], [**30**].

6. Coherent states and symbols

Berezin constructed for an important but limited classes of Kähler manifolds a star product. The construction was based on his covariant symbols given for domains in $\mathbb{C}^n$. In the following we will present their definition for arbitrary compact quantizable Kähler manifolds.

6.1. Coherent states. We look again at the relation (5.5)

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

but now from the point of view of the linear evaluation functional. This means, we fix $\alpha \in U \setminus 0$ and vary the sections s.

The *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with

$$\langle e_\alpha^{(m)}, s\rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))) \tag{6.1}$$

for all $s \in \Gamma_{hol}(M, L^m)$. A direct verification shows $e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}$ for $c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Moreover, as the bundle is very ample we get $e_\alpha^{(m)} \neq 0$.

This allows the following definition.

Definition 6.1. The *coherent state (of level m)* associated to $x \in M$ is the projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0. \tag{6.2}$$

The *coherent state embedding* is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L^m)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e_{\tau^{-1}(x)}^{(m)}]. \tag{6.3}$$

See [**10**] for some geometric properties of the coherent state embedding.

Remark 6.2. A coordinate independent version of Berezin's original definition and extensions to line bundles were given by Rawnsley [**76**]. It plays an important role in the work of Cahen, Gutt, and Rawnsley on the quantization of Kähler manifolds [**24, 25, 26, 27**], via Berezin's covariant symbols. In these works the coherent vectors are parameterized by the elements of $L \setminus 0$. The definition here uses the points of the total space of the dual bundle U. It has the advantage that one can consider all tensor powers of L together on an equal footing.

6.2. Covariant Berezin symbol.

Definition 6.3. For an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ its *covariant Berezin symbol* $\sigma^{(m)}(A)$ *(of level m)* is defined as the function

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, Ae_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \quad \alpha \in \tau^{-1}(x) \setminus \{0\}. \tag{6.4}$$

Using the *coherent projectors* (with the convenient bra-ket notation)

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \qquad \alpha \in \tau^{-1}(x) \tag{6.5}$$

it can be rewritten as $\sigma^{(m)}(A) = \mathrm{Tr}(AP_x^{(m)})$. In abuse of notation $\alpha \in \tau^{-1}(x)$ should always mean $\alpha \neq 0$.

6.3. Contravariant Symbols. We need Rawnsley's epsilon function $\epsilon^{(m)}$ [**76**] to introduce contravariant symbols in the general Kähler manifold setting. It is defined as

$$\epsilon^{(m)}: M \to C^\infty(M), \quad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})(x)}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \ \alpha \in \tau^{-1}(x). \tag{6.6}$$

As $\epsilon^{(m)} > 0$ we can introduce the modified measure $\Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x)$ on the space of functions on M. If M is a homogeneous manifold under a transitive group action and everything is invariant, $\epsilon^{(m)}$ will be constant. This was the case considered by Berezin.

DEFINITION 6.4. Given an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ then **a** contravariant Berezin symbol $\check{\sigma}^{(m)}(A) \in C^\infty(M)$ of A is defined by the representation of the operator A as an integral

$$A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)} \, \Omega_\epsilon^{(m)}(x), \tag{6.7}$$

if such a representation exists.

We quote from [**85**, Prop. 6.8] that the Toeplitz operator $T_f^{(m)}$ admits such a representation with $\check{\sigma}^{(m)}(T_f^{(m)}) = f$. This says, the function f itself is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$. Note that the contravariant symbol is not uniquely fixed by the operator. As an immediate consequence from the surjectivity of the Toeplitz map it follows that every operator A has a contravariant symbol, i.e. every operator A has a representation (6.7). For this we have to keep in mind, that our Kähler manifolds are compact.

Now we introduce on $\mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ the Hilbert-Schmidt norm $\langle A, C\rangle_{HS} = Tr(A^* \cdot C)$. In [**79**] (see also [**86**]), we showed that

$$\langle A, T_f^{(m)} \rangle_{HS} = \langle \sigma^{(m)}(A), f \rangle_\epsilon^{(m)} \ . \tag{6.8}$$

This says that the Toeplitz map $f \to T_f^{(m)}$ and the covariant symbol map $A \to \sigma^{(m)}(A)$ are adjoint. By the adjointness property from the surjectivity of the Toeplitz map the following follows.

PROPOSITION 6.5. *The covariant symbol map is injective.*

Other results following from the adjointness are

$$\mathrm{tr}(T_f^{(m)}) = \int_M f \, \Omega_\epsilon^{(m)} = \int_M \sigma^{(m)}(T_f^{(m)}) \, \Omega_\epsilon^{(m)}. \tag{6.9}$$

$$\dim \Gamma_{hol}(M, L^m) = \int_M \Omega_\epsilon^{(m)} = \int_M \epsilon^{(m)}(x) \, \Omega. \tag{6.10}$$

In particular, in the special case that $\epsilon^{(m)}(x) = const$ then

$$\epsilon^{(m)} = \frac{\dim \Gamma_{hol}(M, L^m)}{vol_\Omega(M)}. \tag{6.11}$$

6.4. The original Berezin star product. Under very restrictive conditions on the manifold it is possible to construct the *Berezin star product* with the help of the covariant symbol map. This was done by Berezin himself [**13**],[**14**] and later by Cahen, Gutt, and Rawnsley [**24**][**25**][**26**][**27**] for more examples. We will indicate this in the following.

Denote by $\mathcal{A}^{(m)} \le C^\infty(M)$, the subspace of functions which appear as level m covariant symbols of operators. By Proposition 6.5 for the two symbols $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed. Hence, it is possible to define the deformed product by

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B). \tag{6.12}$$

Now $\star_{(m)}$ defines on $\mathcal{A}^{(m)}$ an associative and noncommutative product.

It is even possible to give an expression for the resulting symbol. For this we introduce the *two-point function*

$$\psi^{(m)}(x,y) = \frac{\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle \langle e_\beta^{(m)}, e_\beta^{(m)}\rangle} \tag{6.13}$$

with $\alpha = \tau^{-1}(x)$ and $\beta = \tau^{-1}(y)$. This function is well-defined on $M \times M$. Furthermore, we have the *two-point symbol*

$$\sigma^{(m)}(A)(x,y) = \frac{\langle e_\alpha^{(m)}, A e_\beta^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle}. \tag{6.14}$$

It is the analytic extension of the real-analytic covariant symbol. It is well-defined on an open dense subset of $M \times M$ containing the diagonal. Then

$$\begin{aligned}\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B)(x) &= \sigma^{(m)}(A\cdot B)(x) = \frac{\langle e_\alpha^{(m)}, A\cdot B\, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \\ &= \frac{1}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \int_M \langle e_\alpha^{(m)}, A e_\beta^{(m)}\rangle \langle e_\beta^{(m)}, B e_\alpha^{(m)}\rangle \frac{\Omega_\epsilon^{(m)}(y)}{\langle e_\beta^{(m)}, e_\beta^{(m)}\rangle} \\ &= \int_M \sigma^{(m)}(A)(x,y)\cdot \sigma^{(m)}(B)(y,x)\cdot \psi^{(m)}(x,y)\cdot \Omega_\epsilon^{(m)}(y)\ .\end{aligned} \tag{6.15}$$

The crucial problem is how to relate different levels m to define for all possible symbols a unique product not depending on m. In certain special situations like those studied by Berezin, and Cahen, Gutt and Rawnsley the subspaces are nested into each other and the union $\mathcal{A} = \bigcup_{m\in\mathbb{N}} \mathcal{A}^{(m)}$ is a dense subalgebra of $C^\infty(M)$. This is the case if the manifold is a homogeneous manifold and the epsilon function $\epsilon^{(m)}$ is a constant. A detailed analysis shows that in this case a star product is given.

For related results see also work of Moreno and Ortega-Navarro [**68**], [**67**]. In particular, also the work of Engliš [**42, 41, 40, 39**]. Reshetikhin and Takhtajan [**77**] gave a construction of a (formal) star product using formal integrals (and associated Feynman graphs) in the spirit of the Berezin's covariant symbol construction, see Section 9.2

In Section 8.2 using the Berezin transform and its properties discussed in the next section (at least in the case of quantizable compact Kähler manifolds) we will introduce a star product dual to the by Theorem 4.2 existing $\star_{BT}$. It will generalizes the above star product.

7. The Berezin transform and Bergman kernels

7.1. Definition and asymptotic expansion of the Berezin transform.

DEFINITION 7.1. The map

$$I^{(m)}: C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)}), \tag{7.1}$$

obtained by starting with a function $f \in C^\infty(M)$, taking its Toeplitz operator $T_f^{(m)}$, and then calculating the covariant symbol is called the *Berezin transform (of level m)*.

To distinguish it from the formal Berezin transforms introduced by Karabegov for any of his star products sometimes we will call the above the geometric Berezin transform. Note that it is uniquely fixed by the geometric setup of the quantized Kähler manifold.

From the point of view of Berezin's approach the operator $T_f^{(m)}$ has as a contravariant symbol f. Hence $I^{(m)}$ gives a correspondence between contravariant symbols and covariant symbols of operators. The Berezin transform was introduced and studied by Berezin [**14**] for certain classical symmetric domains in $\mathbb{C}^n$. These results where extended by Unterberger and Upmeier [**90**], see also Engliš [**40, 41, 42**] and Engliš and Peetre [**43**]. Obviously, the Berezin transform makes perfect sense in the compact Kähler case which we consider here.

THEOREM 7.2. [**57**] *Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$*

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x)\frac{1}{m^i}\ , \tag{7.2}$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are linear maps given by differential operators, uniformly defined for all $x \in M$. Furthermore, $I_0(f) = f, \quad I_1(f) = \Delta f$.

Here Δ is the Laplacian with respect to the metric given by the Kähler form ω. By *complete asymptotic expansion* the following is understood. Given $f \in C^\infty(M)$, $x \in M$ and an $N \in \mathbb{N}$ then there exists a positive constant A such that

$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{N-1} I_i(f)(x)\frac{1}{m^i}\right|_\infty \quad \leq \quad \frac{A}{m^N}\ .$$

The proof of this theorem is quite involved. An important intermediate step of independent interest is the off-diagonal asymptotic expansion of the Bergman kernel function in the neighborhood of the diagonal, see [**57**]. We will discuss this in the next subsection.

7.2. Bergman kernel. Recall from Section 5 the definition of the Szegö projectors $\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}$ and its components $\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, the Bergman projectors. The Bergman projectors have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha,\beta)$ defined on $Q \times Q$, i.e.

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha,\beta)\psi(\beta)\mu(\beta). \tag{7.3}$$

The Bergman kernels can be expressed with the help of the coherent vectors.

PROPOSITION 7.3.

$$\mathcal{B}_m(\alpha,\beta) = \psi_{e_\beta^{(m)}}(\alpha) = \overline{\psi_{e_\alpha^{(m)}}(\beta)} = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle. \tag{7.4}$$

For the proofs of this and the following propositions see [**57**], or [**82**].

Let $x, y \in M$ and choose $\alpha, \beta \in Q$ with $\tau(\alpha) = x$ and $\tau(\beta) = y$ then the functions

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle, \tag{7.5}$$

$$v_m(x,y) := \mathcal{B}_m(\alpha,\beta) \cdot \mathcal{B}_m(\beta,\alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle \tag{7.6}$$

are well-defined on M and on $M \times M$ respectively. The following proposition gives an integral representation of the Berezin transform.

PROPOSITION 7.4.

$$\begin{aligned} \left(I^{(m)}(f)\right)(x) &= \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta) \\ &= \frac{1}{u_m(x)} \int_M v_m(x,y) f(y)\Omega(y)\ . \end{aligned} \tag{7.7}$$

Typically, asymptotic expansions can be obtained using stationary phase integrals. But for such an asymptotic expansion of the integral representation of the Berezin transform we will not only need an asymptotic expansion of the Bergman kernel along the diagonal (which is well-known) but in a neighborhood of it. This is one of the key results obtained in [**57**]. It is based on works of Boutet de Monvel and Sjöstrand [**23**] on the Szegö kernel and in generalization of a result of Zelditch [**95**] on the Bergman kernel on the diagonal. The integral representation is used then to prove the existence of the asymptotic expansion of the Berezin transform. See [**82**] for a sketch of the proof.

Having such an asymptotic expansion it still remains to identify its terms. As it was explained in Section 4.3, Karabegov assigns to every formal deformation quantizations with the "separation of variables" property a *formal Berezin transform I*. In [**57**] it is shown that there is an explicitely specified star product $\star$ (see Theorem 5.9 in [**57**]) with associated formal Berezin transform such that if we replace $\frac{1}{m}$ by the formal variable ν in the asymptotic expansion of the Berezin transform $I^{(m)}f(x)$ we obtain $I(f)(x)$. This will finally prove Theorem 7.2. We will exhibit the star product $\star$ in Section 8.1.

Of course, for certain restricted but important non-compact cases the Berezin transform was already introduced and calculated by Berezin. It was a basic tool in his approach to quantization [**12**]. For other types of non-compact manifolds similar results on the asymptotic expansion of the Berezin transform are also known. See the extensive work of Engliš, e.g. [**40**].

REMARK 7.5. More recently, direct approaches to the asymptotic expansion of the Bergman kernel (outside the diagonal) were given, some of them yielding low order coefficients of the expansion. As examples, let me mention Berman, Berndtsson, and Sjöstrand, [**16**], Ma and Marinescu [**63**], Dai. Lui, and Ma [**35**]. See also Engliš [**39**].

7.3. Proof of norm property of Toeplitz operators. In [**79**] I conjectured (7.2) (which we later proved in joint work with Karabegov) and showed how such an asymptotic expansion supplies a different proof of Theorem 3.3, Part (a). For completeness I reproduce the proof here.

PROPOSITION 7.6.

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ . \tag{7.8}$$

PROOF. Using Cauchy-Schwarz inequality we calculate ($x = \tau(\alpha)$)

$$|\sigma^{(m)}(T_f^{(m)})(x)|^2 = \frac{|\langle e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)}\rangle|^2}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle^2} \le \frac{\langle T_f^{(m)} e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \le ||T_f^{(m)}||^2 \ . \tag{7.9}$$

Here the last inequality follows from the definition of the operator norm. This shows the first inequality in (7.8). For the second inequality introduce the multiplication operator $M_f^{(m)}$ on $\Gamma_\infty(M, L^m)$. Then $||T_f^{(m)}|| = ||\Pi^{(m)}\, M_f^{(m)}\, \Pi^{(m)}|| \le ||M_f^{(m)}||$ and for $\varphi \in \Gamma_\infty(M, L^m)$, $\varphi \ne 0$

$$\frac{||M_f^{(m)}\varphi||^2}{||\varphi||^2} = \frac{\int_M h^{(m)}(f\varphi, f\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} = \frac{\int_M f(z)\overline{f(z)} h^{(m)}(\varphi,\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} \le |f|_\infty^2 \ . \tag{7.10}$$

Hence,

$$||T_f^{(m)}|| \le ||M_f^{(m)}|| = \sup_{\varphi \ne 0} \frac{||M_f^{(m)}\varphi||}{||\varphi||} \le |f|_\infty. \tag{7.11}$$

$\square$

PROOF. (Theorem 3.3 Part (a).) Choose as $x_e \in M$ a point with $|f(x_e)| = |f|_\infty$. From the fact that the Berezin transform has as leading term the identity it follows that $|(I^{(m)}f)(x_e) - f(x_e)| \le C/m$ with a suitable constant C. Hence, $\big||f(x_e)| - |(I^{(m)}f)(x_e)|\big| \le C/m$ and

$$|f|_\infty - \frac{C}{m} = |f(x_e)| - \frac{C}{m} \quad \le \quad |(I^{(m)}f)(x_e)| \quad \le \quad |I^{(m)}f|_\infty \ . \tag{7.12}$$

Putting (7.8) and (7.12) together we obtain

$$|f|_\infty - \frac{C}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ . \tag{7.13}$$

$\square$

8. Berezin transform and star products

8.1. Identification of the BT star product. In [**57**] there was another object introduced, the *twisted product*

$$R^{(m)}(f,g) := \sigma^{(m)}(T_f^{(m)} \cdot T_g^{(m)}) \ . \tag{8.1}$$

Also for it the existence of a complete asymptotic expansion was shown. It was identified with a twisted formal product. This allowed relating the BT star product with a special star product within the classification of Karabegov. From this the properties of Theorem 4.2 of locality, separation of variables type, and the calculation to the classifying forms and classes for the BT star product follows.

As already announced in Section 4.3, the BT star product $\star_{BT}$ is the opposite of the dual star product of a certain star product $\star$. To identify $\star$ we will give its classifying Karabegov form $\widehat{\omega}$. As already mentioned above, Zelditch [**95**] proved that the function u_m (7.5) has a complete asymptotic expansion in powers of $1/m$. In detail he showed

$$u_m(x) \sim m^n \sum_{k=0}^{\infty} \frac{1}{m^k}\, b_k(x), \quad b_0 = 1. \tag{8.2}$$

If we replace in the expansion $1/m$ by the formal variable ν we obtain a formal function s defined by

$$e^s(x) = \sum_{k=0}^{\infty} \nu^k\, b_k(x). \tag{8.3}$$

Now take as formal potential (4.8)

$$\widehat{\Phi} = \frac{1}{\nu}\Phi_{-1} + s,$$

where Φ_{-1} is the local Kähler potential of the Kähler form $\omega = \omega_{-1}$. Then $\widehat{\omega} = \mathrm{i}\, \partial\bar{\partial}\widehat{\Phi}$. It might also be written in the form

$$\widehat{\omega} = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\, \partial\bar{\partial} \log \mathcal{B}_m(\alpha, \alpha)). \tag{8.4}$$

Here we denote the replacement of $1/m$ by the formal variable ν by the symbol $\mathbb{F}$.

8.2. The Berezin star products for arbitrary Kähler manifolds. We will introduce for general quantizable compact Kähler manifolds the Berezin star product. We extract from the asymptotic expansion of the Berezin transform (7.2) the formal expression

$$I = \sum_{i=0}^{\infty} I_i\, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \tag{8.5}$$

as a *formal Berezin transform*, and set

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)). \tag{8.6}$$

As $I_0 = id$ this $\star_B$ is a star product for our Kähler manifold, which we call the *Berezin star product*. Obviously, the formal map I gives the equivalence transformation to $\star_{BT}$. Hence, the Deligne-Fedosov classes will be the same. It will be of separation of variables type but with the role of the variables switched. We showed in [**57**] that $I = I_\star$ with star product given by the form (8.4). We can rewrite (8.6) as

$$f \star_{BT} g := I^{-1}(I(f) \star_B I(g)). \tag{8.7}$$

and get exactly the relation (4.13). Hence, $\star = \star_B$ and both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other.

When the definition with the covariant symbol works (explained in Section 6.4) $\star_B$ will coincide with the star product defined there.

8.3. Summary of naturally defined star products for compact Kähler manifolds. By the presented techniques we obtained for quantizable compact Kähler manifolds three different naturally defined star products $\star_{BT}$, $\star_{GQ}$, and $\star_B$. All three are equivalent and have classifying Deligne-Fedosov class

$$cl(\star_{BT}) = cl(\star_B) = cl(\star_{GQ}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}). \tag{8.8}$$

But all three are distinct. In fact $\star_{BT}$ is of separation of variables type (Wick-type), $\star_B$ is of separation of variables type with the role of the variables switched (anti-Wick-type), and $\star_{GQ}$ neither. For their Karabegov forms we obtain (see [**57**],[**85**])

$$kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \qquad kf(\star_B) = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m). \tag{8.9}$$

The function u_m was introduced above as the Bergman kernel evaluated along the diagonal in $Q \times Q$.

Remark 8.1. Based on Fedosov's method Bordemann and Waldmann [**19**] constructed also a unique star product $\star_{BW}$ which is of Wick type, see Section 9.1. The opposite star product has Karabegov form $kf(\star_{BW}^{opp}) = -(1/\nu)\,\omega$ and it has Deligne-Fedosov class $cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$ [**54**]. It will be equivalent to $\star_{BT}$ if the canonical class is trivial.

More precisely, in [**54**] Karabegov considered the "anti-Wick" variant of the Bordemann-Waldmann construction. This yields a star product with separation of variables in the convention of Karabegov. It has Karabegov form $(1/\nu)\omega$ and the same Deligne-Fedosov class as (8.8). Hence, it is equvialent to $\star_{BT}$. Recently, in [**55**], [**56**] Karabegov gave a more direct construction of the star product with Karabegov form $(1/\nu)\omega$. Karabegov calls this star product *standard star product*.

8.4. Application: Calculation of the coefficients of the star products. The proof of Theorem 4.2 gives a recursive definition of the coefficients $C_k(f,g)$. Unfortunately, it is not very constructive. For their calculation the Berezin transform will also be of help. Theorem 7.2 shows for quantizable compact Kähler manifolds the existence of the asymptotic expansion of the Berezin transform (7.2). We get the formal Berezin transform $I = \mathbb{F}(I^{(m)})$, see (8.5), which is the formal Berezin transform of the star product $\star_B$

$$I = \sum_{i=0}^{\infty} I_i\,\nu^i, \quad I_i : C^\infty(M) \to C^\infty(M).$$

We will show that if we know I explicitely we obtain explicitly $\star_B$ by giving the coefficients $C_k^B(f,g)$ of $\star_B$. For this the knowledge of the coefficients $C_k^{BT}(f,g)$ for $\star_{BT}$ will not be needed. All we need is the existence of $\star_{BT}$ to define $\star_B$. The operators I_i can be expressed (at least in principle) by the asymptotic expansion of expressions formulated in terms of the Bergman kernel.

As I is the formal Berezin transform in the sense of Karabegov assigned to $\star_B$ we get for local functions f, g , f anti-holomorphic, g holomorphic

$$f \star g = I(g \cdot f) = I(g \star f). \tag{8.10}$$

Expanding the formal series for $\star_B$ (4.1) and for I (8.5) we get for the coefficients

$$C_k^B(f,g) = I_k(g \cdot f). \tag{8.11}$$

Let us take local complex coordinates. As $\star_B$ is a differential star product, the C_k^B are bidifferential operators. As $\star_B$ is of separation of variables type, in C_k^B the first argument is is only differentiated with respect to anti-holomorphic coordinates, the second with respect to holomorphic coordinates. Moreover, it was shown by Karabegov that the C_k are bidifferential operators of order $(0,k)$ in the first argument and order $(k,0)$ in the second argument and that I_k is a differential operator of type (k,k).

As f is anti-holomorphic, in I_k it will only see the anti-holomorphic derivatives. The corresponding is true for the holomorphic g. By locality it is enough to consider the local functions z_i and $\overline{z}_i$ and we get that C_k^B can be obtained by "polarizing" I_k.

In detail, if we write I_k as summation over multi-indices (i) and (j) we get

$$I_k = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(i)+(j)}}{\partial z_{(i)} \partial \overline{z}_{(j)}}, \quad a^k_{(i),(j)} \in C^\infty(M) \tag{8.12}$$

and obtain for the coefficient in the star product $\star_B$

$$C_k^B(f,g) = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(j)} f}{\partial \overline{z}_{(j)}} \frac{\partial^{(i)} g}{\partial z_{(i)}}, \tag{8.13}$$

where the summation is limited by the order condition. Hence, knowing the components I_k of the formal Berezin transform I gives us C_k^B.

From I we can recursively calculate the coefficients of the inverse I^{-1} as I starts with id. From $f \star_{BT} g = I^{-1}(I(f) \star_B I(g))$, which is the Relation (8.6) inverted, we can calculate (at least recursively) the coefficients C_k^{BT}. In practice, the recursive calculations turned out to become quite involved.

The chain of arguments presented above was based on the existence of the Berezin transform and its asymptotic expansion for every quantizable compact Kähler manifold. The asymptotic expansion of the Berezin transform itself is again based on the asymptotic off-diagonal expansion of the Bergman kernel. Indeed, the Toeplitz operator can also be expressed via the Bergman kernel. Based on this it is clear that the same procedure will work for those non-compact manifolds for which we have at least the same (suitably adapted) objects and corresponding results.

Remark 8.2. In the purely formal star product setting studied by Karabegov [**52**] the set of star products of separation of variables type, the set of formal Berezin transforms, and the set of formal Karabegov forms are in 1:1 correspondence. Given $I_\star$ the star product $\star$ can be recovered via the correspondence (8.12) with (8.13). What generalizes $\star_{BT}$ in this more general setting is the dual and opposite of $\star$.

Example 8.3. As a simple but nevertheless instructive case let us consider $k = 1$. Recall that n is the complex dimension of M. Starting from our Kähler form ω expressed in local holomorphic coordinates z_i as $\omega = \mathrm{i} \sum_{i,j=1}^n g_{ij} dz_i \wedge d\overline{z}_j$ the Laplace-Beltrami operator is given by

$$\Delta = \sum_{i,j} g^{ij} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}, \tag{8.14}$$

here (g^{ij}) is as usual the inverse matrix to (g_{ij}). [5] The Poisson bracket is given (up to ϵ which is a factor of signs, complex units, and factors of $1/2$ due to preferred conventions) by

$$\{f, g\} = \epsilon \cdot \sum_{i,j} g^{ij} \left(\frac{\partial f}{\partial \bar{z}_i} \frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \bar{z}_i} \right) \tag{8.15}$$

From $I_1 = \Delta$ we deduce immediately with (8.14)

$$C_1^B(f, g) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial \bar{z}_i} \frac{\partial g}{\partial z_j}. \tag{8.16}$$

The inverse of I starts with $id - \Delta\nu + \ldots$.. If we isolate using (8.7) from

$$(id - \Delta\nu)(((id + \Delta\nu)f) \star_B ((id + \Delta\nu)g)) \tag{8.17}$$

the terms of order one in ν we get

$$C_1^{BT}(f, g) = C_1^B(f, g) + (\Delta f)g + f(\Delta g) - \Delta(fg) = -\sum_{i,j} g^{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_j}. \tag{8.18}$$

This is of course not a surprise. We could have it deduced also directly. Our star products are of separation of variables type and the C_1 have to have a form like (8.16) (or (8.18)) with coefficients a^{ij} which a priori could be different from g^{ij} and $-g^{ij}$ respectively. From $C_1(f, g) - C_1(g, f) = -\mathrm{i}\,\{f, g\}$ it follows that they are equal.

Calculating the higher orders can become quite tedious. First of course the Berezin transform is only known in closed form for certain homogeneous spaces. For general (compact) manifolds by Proposition 7.4 its asymptotic expansion can be expressed in terms of asymptotic expansions of the Bergman kernel. The Bergman kernel can be expressed locally with respect to adapted coordinates via data associated to the Kähler metric. Hence the coefficients C_k^B and C_k^{BT} can be also expressed in these data. In case that the Berezin transform exists it was an important achievement of Mirek Engliš to exploit this in detail also in the noncompact case, under the condition that the Berezin transform exists [**39**], [**42**]. He calculated small order terms in the star products.

Later, Marinescu and Ma used also Bergman kernel techniques in a different way even in the case of compact symplectic manifolds and orbifolds and allowing an auxiliary vector bundles. In their approach they introduced Toeplitz kernels and calculated small order terms for the Berezin-Toeplitz star product [**65**]. A Berezin transform does not show up. See [**64**] for a review of their techniques, results and further reference to related literature. See also results of Charles [**30**], [**31**], [**32**], [**33**].

9. Other constructions of star products – Graphs

9.1. Bordemann and Waldmann. [**19**] Fedosov's proof of the existence of a star product for every symplectic manifold was geometric in its very nature [**44**]. He considers a certain infinite-dimensional bundle $\hat{W} \to M$ of formal series of symmetric and antisymmetric forms on the tangent bundle of M. For this bundle

[5] From the context it should be clear that g and g_{ij} are unrelated objects.

he defines the fiber-wise Weyl product. Denote by $\hat{\mathcal{W}}$ the sheaf of smooth sections of this bundle, with $\circ$ as induced product.

Starting from a symplectic torsion free connection he constructs recursively what is called the Fedosov derivation D for the sheaf of sections. It is flat, in the sense that $D^2 = 0$. The kernel of D is a $\circ$-subalgebra. Let $\mathcal{W}$ be the elements of $\hat{\mathcal{W}}$ for which the values have antisymmetric degree zero. The natural projection to the symmetric degree zero part gives a linear isomorphism from the $\circ$-subalgebra $\sigma : \mathcal{W}_D = \ker D \cap \mathcal{W} \to C^\infty(M)[[\nu]]$. The algebra structure of $\mathcal{W}_D$ gives the star product we were looking for, i.e. $f \star g := \sigma(\tau(f) \circ \tau(g))$ with τ the inverse of σ which recursively can by calculated.

In case that M is an arbitrary Kähler manifold, Bordemann and Waldmann [**19**] were able to modify the set-up by taking the fiber-wise Wick product. By a modified Fedosov connection a star product $\star_{BW}$ is obtained which is of Wick type, i.e. $C_k(.,.)$ for $k \geq 1$ has only holomorphic derivatives in the first argument and anti-holomorphic arguments in the second argument. Equivalently, it is of separation of variables type. As already remarked earlier, its Karabegov form is $-(1/\nu)\omega$ and it has Deligne-Fedosov class $cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$. It will be equivalent to the BT star product if the canonical class is trivial.

Later Neumaier [**70**] was able to show that each star product of separation of variables type (i.e. the star products opposite to the Karabegov star product from Section 4.3) can be obtained by the Bordemann-Waldmann construction by adding a formal closed $(1,1)$ form as parameter in the construction.

9.2. Reshetikhin and Takhtajan. [**77**] In the following subsections we will indicate certain relations between the question of existence and/or the calculation of coefficients of star products and their description by graphs. One of the problems in the context of star products is that the questions reduce often to rather intricate combinatorics of derivatives of the involved functions and other "internal" geometrical data coming from the manifold, like Poisson form, Kähler form, etc. One has to keep track of multiple derivations of many products and sums involving tensors related to the Poisson structure, metric, etc. and the functions f and g. In this respect graphs are usually a very helpful tool to control the combinatorics and to find "closed expressions" in terms of graphs.

Berezin in his approach to define a star product for complex domains in $\mathbb{C}^n$ used analytic integrals depending on a real parameter $\hbar$. Compare this to (6.15) where due to compactness we have a discrete parameter $1/m$. In these integrals scalar products of coherent states show up. Similar to Proposition 7.3 they are identical to the Bergman kernel. Under the condition that the Kähler form is real-analytic its Kähler potential Φ admits an analytic continuation $\Phi(z, \overline{w})$ on $\mathbb{C}^n \times \mathbb{C}^n$. [6] The Bergman kernel can be rewritten with a suitable complementary factor $e_\hbar(z, \overline{w})$ as

$$\mathcal{B}_\hbar(z, \overline{w}) = \mathrm{e}^{\Phi(v, \overline{w})} e_\hbar(z, \overline{w}). \tag{9.1}$$

Moreover, one considers Calabi's diastatic function

$$\Phi(z, \overline{z}, w, \overline{w}) = \Phi(z, \overline{w}) + \Phi(w, \overline{z}) - \Phi(z, \overline{z}) - \Phi(w, \overline{w}). \tag{9.2}$$

[6] In this subsection for the formalism of analytic continuation, it is convenient to write $f(z, \overline{z})$ for a function f on M to indicate its dependence on holomorphic and anti-holomorphic directions.

The corresponding integral rewrites as

$$(9.3)\qquad (f \star_\hbar g)(z,\overline{z}) = \int_{\mathbb{C}^n} f(z,\overline{w})g(w,\overline{z})\frac{e_\hbar(z,\overline{w})e_\hbar(w,\overline{z})}{e_\hbar(z,\overline{z})}e^{(\Phi(z,\overline{z},w,\overline{w})/\hbar}\,\Omega_\hbar,$$

where $\Omega_\hbar$ is the $\hbar$ normalized Liouville form. To show that the integral gives indeed a star product Berezin needs the crucial assumption $e_\hbar(z,\overline{w})$ is constant. The desired results are obtained via the Laplace method.

Reshetikhin and Takhtajan consider now such type of integrals (still ignoring the $e_\hbar(z,\overline{w})$) as formal integrals and make a formal Laplace expansion to obtain a "star" product, which we denote for the moment by $\bullet$. The coefficients of the expansion for $f \bullet g$ can be expressed with the help of partition functions of a restricted set $\mathcal{G}$ of locally oriented graphs (Feynman diagrams) fulfilling some additional conditions and equipped with additional data. In particular, each $\Gamma \in \mathcal{G}$ contains two special vertices, a vertex R with only incoming edges and and a vertex L with only outgoing edges. Furthermore, the other vertices are divided into two sets, the solid and the hollow vertices. The "star" product for $\mathbb{C}^n$ as formal power series in ν can be written as

$$(9.4)\qquad f \bullet g = \sum_{\Gamma \in \mathcal{G}} \frac{\nu^{\chi(\Gamma)}}{|Aut(\Gamma)|} D_\Gamma(f,g).$$

Here $Aut(\Gamma)$ is the subgroup of automorphism of the graph Γ respecting the special structure, $\chi(\Gamma)$ is the number of edges of Γ minus the number of "solid" vertices. The crucial part is $D_\Gamma(f,g)$ *the partition function* of the graph Γ equipped with certain additional data. It encodes the information from the formal expansion of the integral associated to this graph. The special vertex L is responsible for differentiating f with respect to anti-holomorphic coordinates and R for differentiating g with respect to holomorphic coordinates. It is sketched that the product $\bullet$ is "functorial" with respect to holomorphic changes of coordinates and that it defines a formal deformation quantization for any arbitrary complex manifold M with Kähler form ω. But as in general $1 \bullet f \neq f \neq f \bullet 1$, i.e. it is not null on constants. Essentially this is due to the fact, that the complementary factors $e_\hbar(z,\overline{w})$ (9.1) were not taken into account. But the obtained algebra contains a unit element $e_\nu(z,\overline{z})$ which is invertible. This unit is used to twist $\bullet$

$$(9.5)\qquad (f \star g)(z,\overline{z}) = e_\nu^{-1}(z,\overline{z})((f \cdot e_\nu) \bullet (g \cdot e_\nu))$$

to obtain a star product $\star$ which is null on constants. As the notation already indicates, the unit $e_\nu(z,\overline{z})$ is related to the formal Bergman kernel evaluated along the diagonal.

9.3. Gammelgaard. [**48**] His starting point is the formal deformation $\widehat{\omega}$ of the pseudo-Kähler form $\omega = \omega_{-1}$ given by (4.7). Let $\star$ be the unique star product of separation of variables type (in the convention of Karabegov) associated to $\widehat{\omega}$ which exists globally. Gammelgaard gives a local expression of this star product by

$$(9.6)\qquad f \star g = \sum_{\Gamma \in \mathcal{A}_2} \frac{\nu^{W(\Gamma)}}{|Aut(\Gamma)|} D_\Gamma(f,g).$$

This looks similar to (9.4) but of the set of graphs to be considered are different. Also the partition functions will be different. Local means that he chooses for every point a contractible neighborhood such that $\widehat{\omega}$ has a formal potential (4.8). The set $\mathcal{A}_2$ is the set of isomorphy classes of directed acyclic graphs (parallel edges are allowed) which have exactly one vertex which is a sink (i.e. has only incoming edges) and one vertex which is a source (i.e. has only outgoing edges). These two vertices are called external vertices, the other internal. As usual we denote by E the set of edges and by V the set of vertices of the graph Γ. The graphs are weighted by assigning to every internal vertex v an integer $w(v) \geq -1$. Each internal vertex has at least one incoming and one outgoing edge. If $w(v) = -1$ then at least 3 edges are connected with v. The total weight $W(\Gamma)$ of the graph Γ is defined as the sum $W(\Gamma) := |E| + \sum_{v \text{ internal}} w(v)$. Isomorphism are required to respect the structure. Also in this sense $|Aut(\Gamma)|$ has to be understood.

To each such graph a certain bidifferential operator is assigned. It involves the geometric data and the functions f and g. The function f corresponds to the external vertex which is a source and g to the sink. The internal vertices of weight k involve $-\Phi_k$ from (4.8). Incoming edges correspond to taking derivatives with respect to holomorphic coordinates, outgoing with respect to anti-holomorphic coordinates. Hence f is only differentiated with respect to anti-holomorphic and g with respect to holomorphic. The partition function is now obtained by contracting the tensors with the help of the Kähler metric.

In the main part of the paper [**48**] Gammelgaard shows that this definition is indeed associative and defines locally a star product with the (global) Karabegov form $\widehat{\omega}$ he started with. Hence it is the local restriction of $\star$.

The formula is particularly nice if there are not so many higher order terms in $\widehat{\omega}$. For example for $\widehat{\omega} = (1/\nu)\omega_{-1}$, i.e. the "standard star product" only those graphs contribute for which all vertices have weight -1. For the Berezin star product we will have in general higher degree contributions, see (8.9). But the opposite of the Berezin-Toeplitz star product has Karabegov form $-(1/\nu)\omega + \omega_{can}$, hence only graphs which have only vertices of weight -1 or 0 will contribute. As Gammelgaard remarks this allows to give explicit formulas for the coefficients of the BT star product. Recall that for the opposite star product only the role of f and g is switched.

As an example let me derive the "trivial coefficients". The only graph of weight zero is the one consisting on the two external vertices and no edge. Hence $C_0(f, g) = f \cdot g$ as required. The only graph of weight one consists of the two external vertices and a directed edge between them. Hence, we obtain for every $\widehat{\omega} = (1/\nu)\omega_{-1} + ...$ the expression (8.16), and for the Berezin-Toeplitz star product (8.18) (note that we have to take the pseudo-Kähler form $-\omega_{-1}$ and switch the role of f and g). Internal vertices will only show up for weights ≥ 2.

9.4. Huo Xu. [**92**],[**93**] His starting point is the Berezin transform. Let us assume it exists, which at least is true in the case of compact quantizable Kähler manifolds. As explained in Section 8.4 via the formula (8.13) the coefficients of the Berezin star product are given. Based on Engliš's work [**39**] Huo Xu found a very nice way to deal with the Bergman kernel [**92**] in terms of certain graphs. In [**93**] he applies the result to the Berezin transform and Berezin star product. His formula for the product is

$$f \star_B g = \sum_{\Gamma \in \mathcal{G}} \frac{\det(A(\Gamma_-) - Id)}{|\mathrm{Aut}'(\Gamma)|} \nu^{|E|-|V|} D_\Gamma(f, g) = \sum_{k=0}^{\infty} C_k^B(f, g)\nu^k. \tag{9.7}$$

Here $\mathcal{G}$ is a certain subset of pointed directed graphs (i.e. in technical terms it is the set of strongly connected pointed stable graphs – loops and cycles are allowed) consisting of the vertices $V \cup v$ (with v the distinguished vertex) and edges E. After erasing the vertex v the graph Γ_- is obtained. Now $A(\Gamma_-)$ is its adjacency matrix. $|\mathrm{Aut}'(\Gamma)|$ is the number of automorphisms of the pointed graph fixing the distinguished vector. The only object which is a function is again the *partition function* $D_\Gamma(f,g)$ of the graph defined like follows. Each such graph Γ encodes a "Weyl invariant" given in terms of partial derivatives and contractions of the metric. This defines the partition function, whereas the distinguished vertex is replaced by "f" and "g". All incoming edges are associated to f and correspond to $\frac{\partial}{\partial \bar z_i}$ derivatives and all outgoing are associated to g and correspond to $\frac{\partial}{\partial z_i}$. For the precise formulations of his results I refer to his work.

For small orders he classifies the graphs and calculates for k up to three the $C_k^B(f,g)$ and $C_k^{BT}(f,g)$ in terms of the metric data. But again the reformulation to explicit formulas tend to become quite involved with increasing k.

The approaches via graphs presented in Sections 9.2.,9.3, and 9.4 for sure are in some sense related as they center around the same objects. But the set of graphs considered are completely different. Further investigation is necessary to understand this relation. See in this direction the very recent preprint of Xu [**94**].

10. Excursion: The Kontsevich construction

Kontsevich showed in [**59**] the existence of a star product for every Poisson manifold $(M, \{.,.\})$. In fact he proves the more general formality conjecture which implies the existence. It is not my intention even to give a sketch of this here. Furthermore, in the Kähler case we are in the symplectic case and there are other existence and classification proofs obtained much earlier. Nevertheless, as we are dealing with graphs and star product in the previous section, it is very interesting to sketch his explicit formula for the star product in terms of Feynman diagrams.

He considers star products for open sets in $\mathbb{R}^d$ with arbitrary Poisson structure given by the Poisson bivector $\alpha = (\alpha^{ij})$. In local coordinates $\{x_i\}$ the Poisson bracket is given as

$$\{f,g\}(x) = \sum_{i,j=1}^{d} \alpha^{ij}(x)\partial_i f \partial_j g, \quad \partial_i := \frac{\partial}{\partial x_i}. \tag{10.1}$$

The star product is defined by

$$f \star g = f \cdot g + \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}\,\nu}{2}\right)^n \sum_{\Gamma \in \mathcal{G}_n} w_\Gamma D_\Gamma(f,g). \tag{10.2}$$

Here $\mathcal{G}_n$ is a certain subset of graphs of order n, and the partition function D_Γ is a bidifferential operator involving the Poisson bivector α (of homogeneity n). The graph Γ encodes which derivatives have to be taken in D_Γ and w_Γ is a weight function.

More precisely, $\mathcal{G}_n$ consists of oriented graphs with $n+2$ vertices, labeled by $1, 2, \ldots, n, L, R$, such that at each numbered vertex $[i]$, $i = 1, \ldots, n$ exactly two edges $e_i^1 = (i, v_1(i))$ and $e_i^2 = (i, v_2(i))$ start and end at two different other vertices (including L and R) but not at $[i]$ itself. Each such graphs has $2n$ edges. Denote by E_Γ the set of edges. The number of graphs in G_n is $(n(n+1))^2$ for $n \geq 1$ and

1 for $n = 0$. The bidifferential operator is defined by

$$
\begin{aligned}
D_\Gamma(f,g) := \sum_{I:E_\Gamma \to \{1,2,\ldots,d\}} \left(\prod_{k=1}^{n} \left(\prod_{\substack{e\in E_\Gamma \\ e=(*,k)}} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right) \times \\
\times \left(\prod_{\substack{e\in E_\Gamma \\ e=(*,L)}} \partial_{I(e)} \right) f \cdot \left(\prod_{\substack{e\in E_\Gamma \\ e=(*,R)}} \partial_{I(e)} \right) g.
\end{aligned} \tag{10.3}
$$

The summation can be considered as assigning to the $2n$ edges independent indices $1 \le i_1, i_2, \ldots, i_{2n} \le d$ as labels.

EXAMPLE 10.1. Let Γ be the graph with vertices $(1, 2, L, R)$ and edges

$$
e_1^1 = (1,2), \quad e_1^2 = (1,L), \quad e_2^1 = (2,L), \quad e_2^2 = (2,R).
$$

Then

$$
D_\Gamma(f,g) = \sum_{i_1,i_2,i_3,i_4=1}^{d} (\alpha^{i_1 i_2})(\partial_{i_1}\alpha^{i_3 i_4})(\partial_{i_2}\partial_{i_3} f)(\partial_{i_4} g).
$$

The weights $w(\Gamma)$ are calculated by considering the upper half-plane $H := \{z \in \mathbb{C} \mid Im(z) > 0\}$ with the Poincare metric. Let $C_n(H) := \{u \in H^n \mid u_i \neq u_j, \text{ for } i \neq j\}$ be the configuration space of n ordered distinct points on H. For any two points z and w on H we denote by $\phi(z, w)$ the (counterclock-wise) angle between the geodesic connecting z and $\mathrm{i}\infty$ (which is a straight line) and the geodesic between z and w. Let $d\phi(z,w) = \frac{\partial}{\partial z}\phi(z,w)dz + \frac{\partial}{\partial w}\phi(z,w)dw$ be the differential. The weight is then defined as

$$
w_\Gamma = \frac{1}{(2\pi)^{2n} n!} \int_{C_n(H)} \wedge_{i=1}^{n} d\phi(u_i, u_{v_1(i)}) \wedge d\phi(u_i, u_{v_2(i)}), \tag{10.4}
$$

with the convention that for L and R the values at the boundary (of H) $u_L = 0$ and $u_R = 1$ are taken.

REMARK 10.2. In [**29**] Cattaneo and Felder gave a field-theoretical interpretation of the formula (10.2). They introduce a sigma model defined on the unit disc D (conformally equivalent to the upper half-plane) with values in the Poisson manifold M as target space. The model contains two bosonic fields: (1) X, which is function on the disc, and (2) η, which is a differential 1-form on D taking values in the pullback under X of the cotangent bundle of M, i.e. a section of $X^*(T^*M) \otimes T^*D$.

In local coordinates X is given by d functions $X_i(u)$ and η by d differential 1-forms $\eta_i(u) = \sum_\mu \eta_{i,\mu}(u)du^\mu$. The boundary condition for η is that for $u \in \partial D$, $\eta_i(u)$ vanishes on vectors tangent to ∂D. The action is defined as

$$
S[X,\eta] = \int_D \sum_i \eta_i(u) \wedge dX^i(u) + \frac{1}{2}\sum_{i,j} \alpha^{ij}(X(u))\eta_i(u) \wedge \eta_j(u). \tag{10.5}
$$

If $0, 1, \infty$ are any three cyclically ordered points on the boundary of the disc, the star product can be given (at least formally) as the semi-classical expansion of the path-integral

$$f \star g\ (x) = \int_{X(\infty)=x} f(X(1))g(X(0)) \exp(\frac{i}{\hbar}S[X,\eta])dXd\eta\ . \tag{10.6}$$

To make sense of the expansion and to perform the quantization a gauge action has to be divided out. After this the same formula as by Kontsevich is obtained, except that in the sum over the graphs also graphs with loops (also called tadpoles) appear. The corresponding integrals which supply the weights associated to the graphs with loops are not absolutely convergent. These graphs are removed by a certain technique called finite renormalization. In this way Cattaneo and Felder give a very elucidating (partly heuristic) approach to Kontsevich formula for the star product.

How the Kontsevich construction is related to the other graph construction presented in Section 9 is unclear at the moment.

11. Some applications of the Berezin-Toeplitz operators

In this closing section we will give some references indicating some applications of the Berezin-Toeplitz quantization scheme. The interested reader is invited to check the quoted literature for full details, and more references. This list of applications and references is rather incomplete.

11.1. Pull-back of the Fubini-Study metric, extremal metrics, balanced embeddings. Let (M,ω) be a Kähler manifold with very ample quantum line bundle L. After choosing an orthonormal basis of the space $\Gamma_{hol}(M,L^m)$ we can use them to construct an embedding $\phi^{(m)}: M \to \mathbb{P}^{N(m)}$ of M into projective space of dimension $N(m)$, see Remark 2.1. On $\mathbb{P}^{N(m)}$ we have as standard Kähler form the Fubini-Study form ω_{FS} (and its associated metric). The pull-back $(\phi^{(m)})^*\omega_{FS}$ will define a Kähler form on M. It will not depend on the orthogonal basis chosen for the embedding. In general it will not coincide with a scalar multiple of the Kähler form ω we started with (see [**10**] for a thorough discussion of the situation).

It was shown by Zelditch [**95**], by generalizing a result of Tian [**88**] and Catlin [**28**], that $(\Phi^{(m)})^*\omega_{FS}$ admits a complete asymptotic expansion in powers of $\frac{1}{m}$ as $m \to \infty$.

In fact it is related to the asymptotic expansion of the Bergman kernel (7.5) along the diagonal. The pullback calculates as [**95**, Prop.9]

$$\left(\phi^{(m)}\right)^* \omega_{FS} = m\omega + \mathrm{i}\,\partial\bar\partial \log u_m(x)\ . \tag{11.1}$$

In our context of star products it is interesting to note that if in (11.1) we replace $1/m$ by ν we obtain the Karabegov form of the star product $\star_B$ (8.9)

$$\widehat{\omega} = \mathbb{F}(\left(\phi^{(m)}\right)^* \omega_{FS}). \tag{11.2}$$

The asymptotic expansion of $(\phi^{(m)})^*\omega_{FS}$ is called Tian-Yau-Zelditch expansion. Donaldson [**37**], [**38**] took it as the starting point to study the existence and uniqueness of constant scalar curvature Kähler metrics ω on compact manifolds. If they exists at all he approximates them by using so-called balanced metrics on sequences of powers of the line bundle L obtained by balanced embeddings. Balanced embeddings are embeddings fulfilling certain additional properties introduced by Luo [**62**]. They are related to stability of the embedded manifolds in the sense of classifications in algebraic geometry.

It should be remarked that the "balanced condition" is equivalent to the fact that Rawnsley's [**76**] epsilon function (6.6) is constant. See also [**85**, Prop.6.6]. This function was introduced in 1975 by Rawnsley in the context of quantization of Kähler manifolds and further developed by Cahen, Gutt, and Rawnsley [**24**]. In particular it will be constant if the quantization is "projectively induced", i.e. coming from the projective space of the coherent state embedding (6.3). See Section 6.4 for consequences about the possibility of Berezin's original construction of a star product.

Let me give beside the already mentioned a few more names related to the existence and uniqueness of constant scalar curvature Kähler metrics: Lu [**61**], Arezzo and Loi [**8**], Fine [**45**]. For sure much more should be mentioned, but space limitation do not allow.

11.2. Topological quantum field theory and mapping class groups. In the context of Topological Quantum Field Theory (TQFT) the moduli space M of gauge equivalence classes of flat $SU(n)$ connections (possibly with monodromy around a fixed point) over a compact Riemann surface Σ plays an important role. This moduli space carries a symplectic structure ω and a complex line bundle L. After choosing a complex structure on Σ this moduli space will be endowed with a complex structure, ω will become a Kähler form and L get a holomorphic structure. Moreover L will be a quantum line bundle in the sense discussed in this review. Hence, we can employ the Berezin-Toeplitz quantization procedure to it. The quantum space of level m will be as above the (finite-dimensional) space of holomorphic sections of the bundle L^m over M. If we vary the complex structure on Σ the differentiable (symplectic) data will stay the same, but the complex geometric data will vary. In particular, our family of quantum spaces will define a vector bundle over the Teichmüller space (which is the space of complex structures on Σ). This bundle is called the *Verlinde bundle* of level m. There is a canonical projectively flat connection for this bundle, the Axelrod-de la Pietra-Witten/Hitchin connection.

Via the projection to the subspace of holomorphic section, the Toeplitz operators will depend on the complex structure. For a fixed differentiable function f on the moduli space of connections the Toeplitz operators will define a section of the endomorphism bundle of the Verlinde bundle.

The mapping class group acts on the geometric situation. In particular, it acts on the space of projectively covariant constant sections of the Verlinde bundle. This yields a representation of the mapping class group. By general results about the order of the elements in the mapping class group it cannot act faithfully. But it was a conjecture of Tuarev, that at least it acts asymptotically faithful. This says that given a non-trivial element of the mapping class group there is a level m such that the element has a non-trivial action on the space of projectively covariant constant sections of the Verlinde bundle of level m.

This conjecture was shown by J. Andersen in a beautiful proof using Berezin-Toeplitz techniques. For an exact formulation of the statement see [**2**], resp. the overview by Andersen and Blaavand [**4**], and [**84**].

With similar techniques Andersen could show that the mapping class groups Γ_g for genus $g \geq 2$ do not have Property (T) [**3**]. Roughly speaking Property (T) means that the trivial representation is isolated (with respect to a certain topology) in the space of all unitary representations.

There are quite a number of other interesting results shown and techniques developed by Andersen using Berezin-Toeplitz quantization operators and star products, e.g. in the context of Abelian Chern-Simons Theory [**1**], modular functors (joint with K. Ueno) [**7**], and formal Hitchin connections [**5**].

11.3. Spectral theory – quantum chaos. The large tensor power behaviour of the sections of the quantum bundle and of the Toeplitz operators are of interest.

Shiffman and Zelditch considered in [**87**] the limit distribution of zeros of such sections. The results are related to models in *quantum chaos*. See also other publications of the same authors.

As mentioned in Section 3, the Toeplitz operators associated to real-valued functions are self-adjoint. Hence, they have a real spectrum. With respect to this the following result on the trace is of importance

$$\mathrm{Tr}^{(m)}\,(T_f^{(m)}) = m^n \left(\frac{1}{\mathrm{vol}(\mathbb{P}^n(\mathbb{C}))}\int_M f\,\Omega + O(m^{-1})\right)\ . \tag{11.3}$$

Here $n = \dim_{\mathbb{C}} M$ and $\mathrm{Tr}^{(m)}$ denotes the trace on $\mathrm{End}(\Gamma_{hol}(M, L^m))$. See [**18**], resp. [**81**] for a detailed proof.

On the spectral analysis of Toeplitz operators see e.g. articles by Paoletti [**72**], [**73**], [**74**]. For relation to index theory see e.g. work of Boutet de Monvel, Leichtnam, Tang, and Weinstein [**22**], and Bismut, Ma, and Zhang [**17**].

11.4. Automorphic forms. Another field where the set-up developed in this review shows up in a natural way is the theory of automorphic forms. For example, let $B^n = SU(n,1)/S(U(n)\times U(1))$ be the open unit ball and Γ a discrete, cocompact subgroup of $SU(n,1)$ then the quotient $X = \Gamma/B^n$ is a compact complex manifold. Moreover, the invariant Kähler form on B^n will descends to a Kähler form ω on the quotient. The canonical line bundle (i.e. the bundle of holomorphic n-forms) is a quantum line bundle for (X, ω).

By definition the sections of the tensor powers of this line bundle correspond to functions on B^n which are equivariant under the action of Γ with a certain factor of automorphy. In other words they are automorphic forms. The power of the factor of automorphy is related to the tensor power of the bundle. An important problem is to construct sections, resp. automorphic forms. For example, Poincaré series are obtained by an averaging procedure and give naturally such sections. But it is not clear that they are not identically zero. T. Foth [**46**] worked in the frame-work of Berezin-Toeplitz operators to show that at least for higher tensor powers there are non-vanishing Poincaré series. In this process she used techniques proposed by Borthwick, Paul, and Uribe [**20**] and assigns to Legendrian tori sections of the bundles. By asymptotic expansion the non-vanishing follows. See also [**47**].

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