Symmetric closed Reeb orbits on the sphere

Leonardo Macarini
(Joint work with Miguel Abreu and Hui Liu)

Basic setup

- $(\mathbb{R}^{2n+2}, \omega)$, $\omega = \sum_i dq_i \wedge dp_i = d\lambda$ where $\lambda = \frac{1}{2} \sum_i (q_i dp_i - p_i dq_i)$.

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- A contact form on S^{2n+1} supporting ξ is a 1-form α given by $f\lambda|_{S^{2n+1}}$ for some positive function $f : S^{2n+1} \to \mathbb{R}$. Its Reeb vector field is the unique vector field R_α s.t. $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

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- We want to study the dynamics of Reeb flows on the standard contact sphere (S^{2n+1}, ξ).

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- This is a **very hard** question in Hamiltonian Dynamics. Notice that we are **not** supposing any generic condition here.

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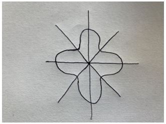
- In higher dimensions this question is widely open.

Results assuming convexity

- There is a bijection between contact forms α on (S^{2n+1}, ξ) and starshaped hypersurfaces Σ_α in $\mathbb{R}^{2n+2}$:

$$\alpha = f\lambda|_{S^{2n+1}} \longleftrightarrow \Sigma_\alpha = \{\sqrt{f(x)}x;\ x \in S^{2n+1}\}.$$



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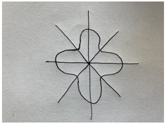

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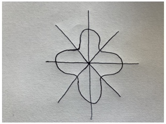

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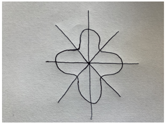

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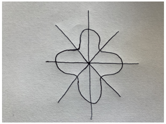

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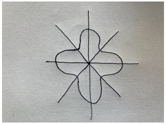

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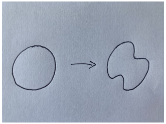
- Wang'2016: $\#\mathcal{P} \geq \lceil \frac{n+1}{2} \rceil + 1$ ($\lceil x \rceil = \inf\{k \in \mathbb{N};\ k \geq x\}$).

Dynamical convexity

- The hypothesis of convexity is not natural from the point of view of Contact Topology since it is not a condition invariant by contactomorphisms.



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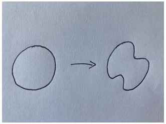

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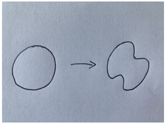

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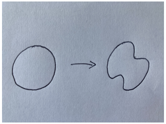

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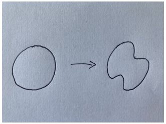

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- It is not hard to see that if α is convex then it is DC.
- Dynamical convexity is more general than convexity: there are DC contact forms that are not contactomorphic to convex ones (Chaidez-Edtmair, Abreu-M., Ginzburg-M.).

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- Ginzburg-Gurel'2020 and Duan-Liu'2017 independently: $\#\mathcal{P} \geq \lceil \frac{n+1}{2} \rceil + 1$.

A refinement of the problem

- Given an integer $p \geq 1$, consider the $\mathbb{Z}_p$-action on S^{2n+1}, regarded as a subset of $\mathbb{C}^{n+1}$, generated by the map

$$\psi(z_0, \dots, z_n) = \left(e^{\frac{2\pi i \ell_0}{p}} z_0, e^{\frac{2\pi i \ell_1}{p}} z_1, \dots, e^{\frac{2\pi i \ell_n}{p}} z_n \right),$$

  where $\ell_0, \dots, \ell_n$ are integers called the weights of the action. Such an action is free when the weights are coprime with p (that we will assume from now on) and in that case we have a lens space obtained as the quotient of S^{2n+1} by the action of $\mathbb{Z}_p$. We denote this lens space by $L_p^{2n+1}(\ell_0, \ell_1, \dots, \ell_n)$.

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- A closed orbit γ of α is **symmetric** if $\psi(\gamma(\mathbb{R})) = \gamma(\mathbb{R})$.

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- Zhang'2013: If α is convex then $\#\mathcal{P}_s \geq 2$.
- Liu-Zhang'2022: If α is DC and $p = 2$ then $\#\mathcal{P}_s \geq 2$.

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Preliminaries

- Let β be the contact form on $L_p^{2n+1}(\ell_0, \ldots, \ell_n)$ induced by α. We have a bijection between simple symmetric closed orbits of α and simple closed orbits of β whose homotopy classes are generators of $\pi_1(L_p^{2n+1}(\ell_0, \ldots, \ell_n))$.

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- To find these orbits we will use equivariant symplectic homology of orbifolds and Lusternik-Schnirelmann theory (to find two).

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- These homologies fit into the exact triangle

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- The Chen-Ruan cohomology encodes information about the singularities of the orbifold.
- In general it has a rational grading.

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- Gironella-Zhou: $\mathrm{SH}^*(\mathbb{C}^{n+1}/G) = 0$.
- Therefore, from the previous exact triangle, $\mathrm{SH}_+^*(\mathbb{C}^{n+1}/G) \simeq H_{CR}^{*+1}(\mathbb{C}^{n+1}/G; \mathbb{Q})$.

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- The equivariance here is related to the S^1-symmetry of the action functional for autonomous Hamiltonians.

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- We have that $\mathrm{SH}^*(M) \simeq \mathrm{SH}_+^*(W) \simeq H_{CR}^{*+1}(W; \mathbb{Q})$.
- Thus, since $H_{CR}^k(W; \mathbb{Q}) = 0 \ \forall k \geq 2n+2$, we have that D is an isomorphism whenever $* \geq 2n+1$.

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- Then, using an argument similar to Ekeland-Hofer, we can prove Theorem 2.

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- But, by DC of α, $\Delta(\bar{\gamma}^p) > 2 \iff p\Delta(\bar{\gamma}) > 2 \iff 1/\Delta(\bar{\gamma}) < p/2$, contradiction.