Elements of Bayesian Geometry

Miguel de Carvalho[†]

†Joint with
B. J. Barney; University of Utah & G. L. Page; Brigham Young University, US

THE UNIVERSITY of EDINBURGH
School of Mathematics

Athena SWAN Bronze Award

LONDON MATHEMATICAL SOCIETY GOOD PRACTICE SCHEME

Introduction

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- Others have investigated two competing priors to specify so-called weakly informative priors (Evans and Jang, 2011; Gelman et al., 2011).

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- Interpretations are similar to those that accompany the correlation coefficient for continuous random variables.

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$$\boldsymbol{y} \mid \theta \overset{\text{iid}}{\sim} \text{Bern}(\theta), \quad \theta \sim \text{Beta}(a,b), \quad \theta \mid \boldsymbol{y} \sim \text{Beta}(a^\star, b^\star),$$

with $a^\star = \sum Y_i + a$ and $b^\star = n - \sum Y_i + b$.

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- The authors conduct the analysis picking $(a,b) = (3.44, 22.99)$.

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We provide a unified treatment to answer the questions above.

Storyboard

Plan of this Talk

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2. Bayes Geometry (Next)

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with inner-product

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and norm $\| \cdot \| = (\langle \cdot, \cdot \rangle)^{1/2}$.
- The fact that $\mathscr{H}$ is an Hilbert space is often known as the Riesz–Fischer theorem (Cheney, 2001, p. 411).

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$$p(\boldsymbol{\theta} \mid \boldsymbol{y}) = \frac{\pi(\boldsymbol{\theta}) f(\boldsymbol{y} \mid \boldsymbol{\theta})}{\int_{\Theta} \pi(\boldsymbol{\theta}) f(\boldsymbol{y} \mid \boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}}$$
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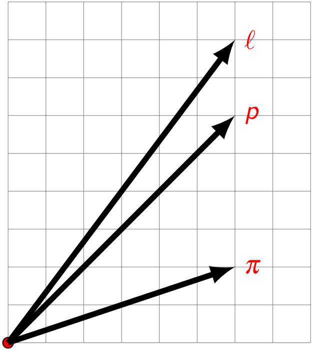

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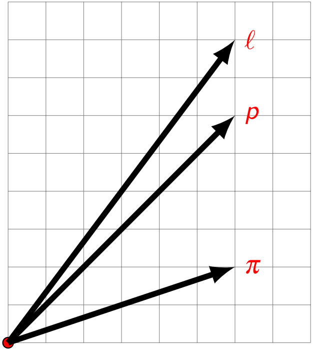

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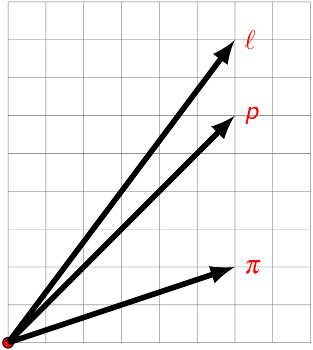

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- Define the angle measure between the prior and the likelihood as

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- Our first target object of interest is given by a standardized inner product

$$\kappa_{\pi,\ell} = \frac{\langle \pi, \ell \rangle}{\|\pi\| \|\ell\|},$$

which quantifies how much an expert's opinion agrees with the data, thus providing a natural measure of prior–data agreement.

Definition (Millman and Parker, 1991, p. 17)

An abstract geometry $\mathscr{A}$ consists of a pair $\{\mathscr{P}, \mathscr{L}\}$, where the elements of set $\mathscr{P}$ are designed as points, and the elements of the collection $\mathscr{L}$ are designed as lines, such that:

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- Our abstract geometry of interest is $\mathscr{A} = \{\mathscr{P}, \mathscr{L}\}$, where $\mathscr{P} = L_2(\Theta)$ and

$$\mathscr{L} = \{g + kh, : g, h \in L_2(\Theta)\}.$$

- In our setting points are, for example, prior densities, posterior densities, or likelihoods, as long as they are in $L_2(\Theta)$.

- Lines are elements of $\mathscr{L}$, so that for example if g and h are densities, line segments in our geometry consist of all possible mixture distributions which can be obtained from g and h, i.e.:

$$\{\lambda g + (1 - \lambda) h : \lambda \in [0, 1]\}.$$

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- If $g, h \in L_2(\Theta)$ are vectors then we say that g and h are collinear if there exists $k \in \mathbb{R}$, such that $g(\boldsymbol{\theta}) = kh(\boldsymbol{\theta})$.
- Put differently, we say g and h are collinear if $g(\boldsymbol{\theta}) \propto h(\boldsymbol{\theta})$, for all $\boldsymbol{\theta} \in \Theta$.

- Two different densities π_1 and π_2 cannot be collinear:

  If $\pi_1 = k\pi_2$, then $k = 1$, otherwise $\int \pi_2(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} \neq 1$.

- A density can be collinear to a likelihood:

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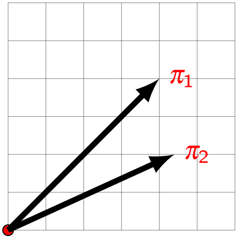

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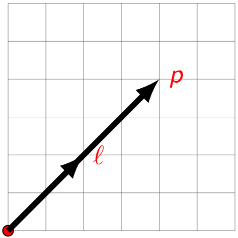

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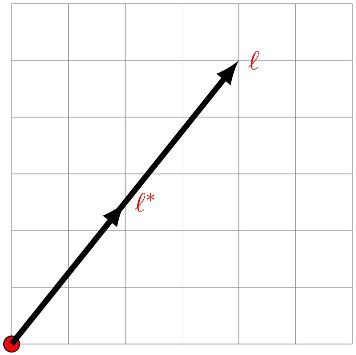

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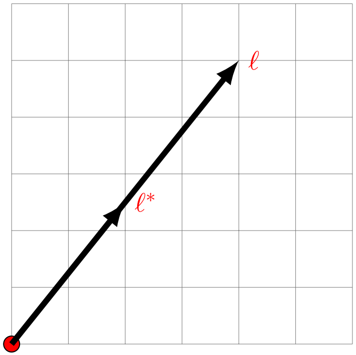

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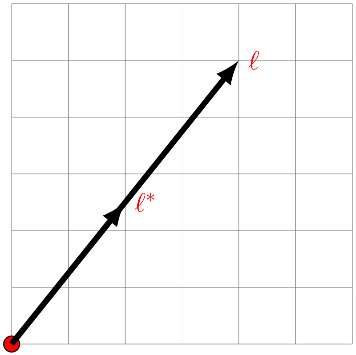

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"Likelihoods with the same direction should yield the same inference."

Definition (Compatibility)

The compatibility between points in the geometry under consideration is the mapping $\kappa : L_2(\Theta) \times L_2(\Theta) \to [0,1]$ defined as

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- $\kappa_{\pi,\ell}$: prior-data agreement.
- $\kappa_{\pi,p}$: sensitivity of the posterior to the prior specification.
- κ_{π_1,π_2}: compatibility of different priors [coherency of opinions of experts].

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Example

Let $U \sim \text{Unif}(a,b)$ and let $\pi(x) = (b-a)^{-1} I_{(a,b)}(x)$. Then,

$$\|\pi\| = 1/(12\sigma_U^2)^{1/4},$$

where the variance of U is $\sigma_U^2 = 1/12(b-a)^2$.

Example

Let $X \sim \text{N}(\mu, \sigma_X^2)$ with known variance σ_X^2. It can be shown that

$$\|\phi\| = \{\int_{\mathbb{R}} \phi^2(x;\mu,\sigma_X^2)\,\mathrm{d}\mu\}^{1/2} = 1/(4\pi\sigma_X^2)^{1/4}.$$

Proposition

Let $\Theta \subset \mathbb{R}^p$ *with* $|\Theta| < \infty$ *where* $|\cdot|$ *denotes the Lebesgue measure. Consider* $\pi : \Theta \to [0, \infty)$ *a probability density with* $\pi \in L_2(\Theta)$ *and let* $\pi_0 \sim Unif(\Theta)$ *denote a uniform density on* Θ*, then*

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- This interpretation cannot be applied to Θ's that do not have finite Lebesgue measure as there is no corresponding proper Uniform distribution.
- Yet, the notion that the norm of a density is a measure of its peakedness may be applied whether or not Θ has finite Lebesgue measure.

- To see this, evaluate $\pi(\theta)$ on a grid $\theta_1 < \cdots < \theta_D$ and consider the vector

$$p = (\pi_1, \ldots, \pi_D),$$

with $\pi_d = \pi(\theta_d)$ for $d = 1, \ldots, D$.

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- Now, think of a density as a vector with infinitely many components (its value at each point of the support) and replace summation by integration to get the L_2 norm.

Example (On-the-job drug usage toy example, cont. 1)

From the example in the Introduction we have $\theta \mid \boldsymbol{y} \sim \text{Beta}(a^\star, b^\star)$ with $a^\star = a + \sum Y_i = a + 2$ and $b^\star = b + n - \sum Y_i = b + 8$. The norm of the prior, posterior, and likelihood are respectively given by

$$\|\pi(a,b)\| = \frac{\{B(2a-1, 2b-1)\}^{1/2}}{B(a,b)}, \quad a, b > 1/2,$$

and

$$\|p(a,b)\| = \|\pi(a^\star, b^\star)\|.$$

Bayes Geometry

Prior and Posterior Norms: On-the-Job Drug Usage Toy Example

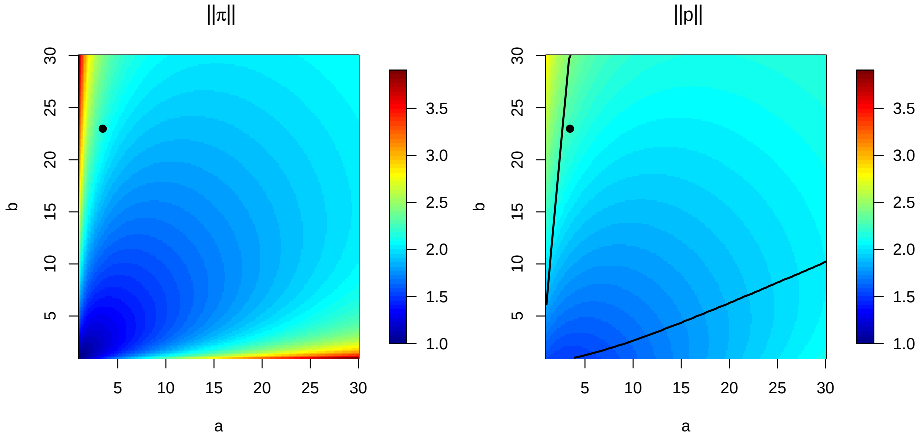

Figure: Prior and posterior norms for on-the-job drug usage toy example. The black dot corresponds to $(a, b) = (3.44, 22.99)$ (values employed by Christensen et al. 2011, pp. 26–27).

Considering κ, it follows that

$$\kappa_{\pi,\ell}(a,b) = B(a^\star, b^\star)\{B(2a-1, 2b-1)B(2\sum Y_i + 1, 2(n - \sum Y_i) + 1)\}^{-1/2}.$$

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Example (On-the-job drug usage toy example, cont. 2)

Extending a previous example, we calculate

$$\begin{aligned}\kappa_{\pi,p} = {} & B(\sum Y_i + 2a - 1, n - \sum Y_i + 2b - 1) \\ & \times \{B(2a-1, 2b-1) \\ & \times B(2\sum Y_i + 2a - 1, 2n - 2\sum Y_i + 2b - 1)\}^{-1/2},\end{aligned}$$

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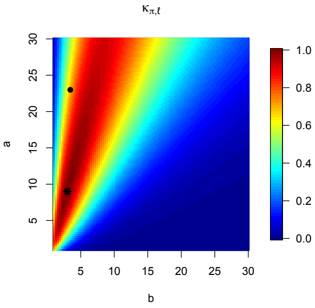
and for $\pi_1 \sim \text{Beta}(a_1, b_1)$ and $\pi_2 \sim \text{Beta}(a_2, b_2)$,

$$\kappa_{\pi_1,\pi_2} = \frac{B(a_1+a_2-1, b_1+b_2-1)}{\{B(2a_1-1,2b_1-1)B(2a_2-1,2b_2-1)\}^{1/2}}.$$

Bayes Geometry

Compatibility: On-the-Job Drug Usage Toy Example



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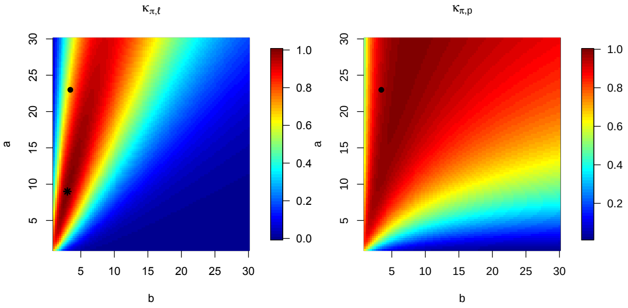

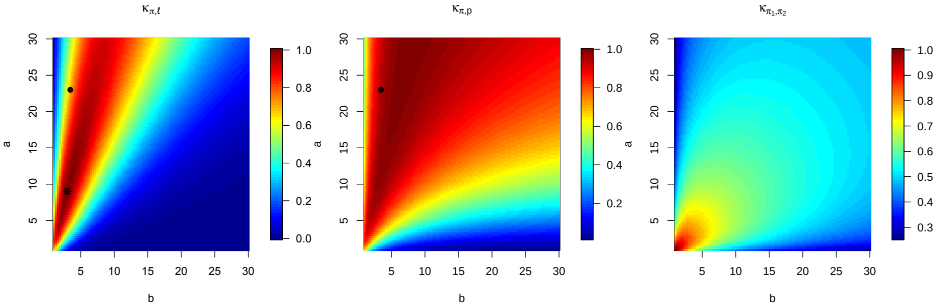
$\kappa_{\pi,\ell}$
$\kappa_{\pi,p}$
κ_{π_1,π_2}
a
b
5 10 15 20 25 30
0.0 0.2 0.4 0.6 0.8 1.0
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0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

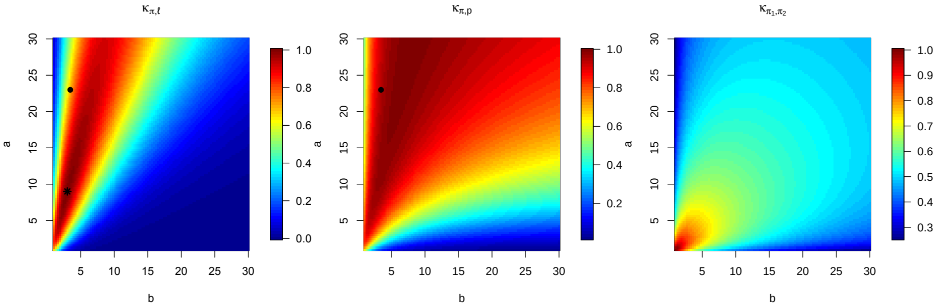

Figure: Compatibility (κ) for on-the-job drug usage toy example. In (i) and (ii) the black dot corresponds to $(a, b) = (3.44, 22.99)$ (values employed by Christensen et al. 2011, pp. 26–27).

Definition (Max-compatible prior)

Let $\boldsymbol{y} \sim f(\cdot \mid \boldsymbol{\theta})$, and let $\mathscr{P} = \{\pi(\boldsymbol{\theta} \mid \boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathscr{A}\}$ be a family of priors for $\boldsymbol{\theta}$. If there exists $\boldsymbol{\alpha}_{\boldsymbol{y}}^* \in \mathscr{A}$, such that $\kappa_{\pi,\ell}(\boldsymbol{\alpha}_{\boldsymbol{y}}^*) = 1$, the prior $\pi(\boldsymbol{\theta} \mid \boldsymbol{\alpha}_{\boldsymbol{y}}^*) \in \mathscr{P}$ is said to be max-compatible, and $\boldsymbol{\alpha}_{\boldsymbol{y}}^*$ is said to be a max-compatible hyperparameter.

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- Geometrically: A prior is max-compatible iff it is collinear to the likelihood in the sense that

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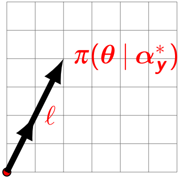

Example (Beta–Binomial)

Let $\sum_{i=1}^n Y_i \sim \text{Bin}(n, \theta)$, and suppose $\theta \sim \text{Beta}(a, b)$. It can be shown that the max-compatible prior is $\pi(\theta \mid a^*, b^*) = \beta(\theta \mid a^*, b^*)$, where $a^* = 1 + \sum_{i=1}^n Y_i$, and $b^* = 1 + n - \sum_{i=1}^n Y_i$, so that

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$$\widehat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta} \in \Theta} f(\boldsymbol{y} \mid \boldsymbol{\theta}) = m_\pi(\boldsymbol{\alpha}_{\boldsymbol{y}}^*) := \arg\max_{\boldsymbol{\theta} \in \Theta} \pi(\boldsymbol{\theta} \mid \boldsymbol{\alpha}_{\boldsymbol{y}}^*).$$

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In this case the max-compatible prior is $f_\Gamma(\theta \mid a^*, b^*)$, where $(a^*, b^*) = (1 + \sum_{i=1}^n Y_i, n)$. The max-compatible hyperparameter in this case is different from the one in the previous example, but still

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Posterior and Prior Mean-Based Estimators of Compatibility
Introduction

- In many situations closed form estimators of κ and $\|\cdot\|$ are not available.
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- As most Bayes methods resort to using MCMC methods it would be appealing to express $\kappa_{\cdot,\cdot}$ and $\|\cdot\|$ as functions of posterior expectations and employ MCMC iterates to estimate them.
- For example, $\kappa_{\pi,p}$ can be expressed as

$$\kappa_{\pi,p} = E_p\,\pi(\boldsymbol{\theta})\left[E_p\left\{\frac{\pi(\boldsymbol{\theta})}{\ell(\boldsymbol{\theta})}\right\}E_p\{\ell(\boldsymbol{\theta})\pi(\boldsymbol{\theta})\}\right]^{-1/2},$$

where $E_p(\,\cdot\,) = \int_\Theta \cdot\, p(\boldsymbol{\theta} \mid \boldsymbol{y})d\boldsymbol{\theta}$.

- A natural Monte Carlo estimator would then be

$$\hat{\kappa}_{\pi,p} = \frac{1}{B}\sum_{b=1}^{B}\pi(\boldsymbol{\theta}^b)\left[\frac{1}{B}\sum_{b=1}^{B}\frac{\pi(\boldsymbol{\theta}^b)}{\ell(\boldsymbol{\theta}^b)}\frac{1}{B}\sum_{b=1}^{B}\ell(\boldsymbol{\theta}^b)\pi(\boldsymbol{\theta}^b)\right]^{-1/2},$$

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where $\boldsymbol{\theta}^b$ denotes the bth MCMC iterate of $p(\boldsymbol{\theta} \mid \boldsymbol{y})$.

- Consistency of such an estimator follows trivially by the ergodic theorem and the continuous mapping theorem, but there is an important issue regarding its stability.

Posterior and Prior Mean-Based Estimators of Compatibility

Problems with Previous Attempt

- Unfortunately, the previous estimator includes an expectation that contains $\ell(\boldsymbol{\theta})$ in the denominator and therefore (28) inherits the undesirable properties of the so-called harmonic mean estimator (Newton and Raftery, 1994).

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- It has been shown that even for simple models this estimator may have infinite variance (Raftery et al. 2007), and has been harshly criticized for, among other things, converging extremely slowly.
- As argued by Wolpert and Schmidler (2012, p. 655):

  "the reduction of Monte Carlo sampling error by a factor of two requires increasing the Monte Carlo sample size by a factor of $2^{1/\varepsilon}$, or in excess of $2.5 \cdot 10^{30}$ when $\varepsilon = 0.01$, rendering [the harmonic mean estimator] entirely untenable."

Posterior and Prior Mean-Based Estimators of Compatibility

Solution

- An alternate strategy is to avoid writing $\kappa_{\pi,p}$ as a function of harmonic mean estimators and instead express it as a function of posterior and prior expectations. For example, consider

$$\kappa_{\pi,p} = E_p\,\pi(\boldsymbol{\theta})\left[\frac{E_\pi\{\pi(\boldsymbol{\theta})\}}{E_\pi\{\ell(\boldsymbol{\theta})\}}\,E_p\{\ell(\boldsymbol{\theta})\pi(\boldsymbol{\theta})\}\right]^{-1/2},$$

  where $E_\pi(\,\cdot\,) = \int_\Theta \cdot\,\pi(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}$.

- Now the Monte Carlo estimator is

$$\tilde{\kappa}_{\pi,p} = \frac{1}{B}\sum_{b=1}^{B}\pi(\boldsymbol{\theta}^b)\left\{\frac{B^{-1}\sum_{b=1}^{B}\pi(\boldsymbol{\theta}_b)}{B^{-1}\sum_{b=1}^{B}\ell(\boldsymbol{\theta}_b)}\,\frac{1}{B}\sum_{b=1}^{B}\ell(\boldsymbol{\theta}^b)\pi(\boldsymbol{\theta}^b)\right\}^{-1/2},$$

  where $\boldsymbol{\theta}_b$ denotes the bth draw of $\boldsymbol{\theta}$ from $\pi(\boldsymbol{\theta})$, which can also be sampled within the MCMC algorithm.

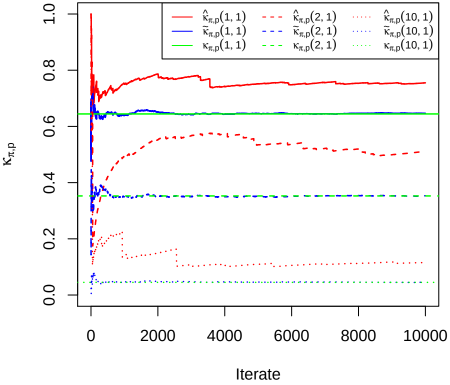

Figure: Running point estimates of prior-posterior compatibility, $\kappa_{\pi,p}$, for the on-the-job drug usage toy example. Green lines correspond to the true $\kappa_{\pi,p}$ values, blue represents $\tilde{\kappa}_{\pi,p}$ and red denotes $\hat{\kappa}_{\pi,p}$.

Proposition

The following equalities hold:

$$\|p\|^2 = \frac{E_p\{\ell(\theta)\pi(\theta)\}}{E_\pi\,\ell(\theta)}, \quad \|\pi\|^2 = E_\pi\,\pi(\theta), \quad \|\ell\|^2 = E_\pi\,\ell(\theta)\,E_p\left\{\frac{\ell(\theta)}{\pi(\theta)}\right\},$$

$$\kappa_{\pi_1,\pi_2} = E_{\pi_1}\,\pi_2(\theta)\Big[E_{\pi_1}\,\pi_1(\theta)\,E_{\pi_2}\,\pi_2(\theta)\Big]^{-1/2}, \quad \kappa_{\pi,\ell} = E_\pi\,\ell(\theta)\left[E_\pi\,\pi(\theta)\,E_\pi\,\ell(\theta)\,E_p\left\{\frac{\ell(\theta)}{\pi(\theta)}\right\}\right]^{-1/2},$$

$$\kappa_{\pi,p} = E_p\,\pi(\theta)\left[\frac{E_\pi\,\pi(\theta)}{E_\pi\,\ell(\theta)}E_p\{\ell(\theta)\pi(\theta)\}\right]^{-1/2}, \quad \kappa_{\ell,p} = E_p\,\ell(\theta)\left[E_p\left\{\frac{\ell(\theta)}{\pi(\theta)}\right\}E_p\{\ell(\theta)\pi(\theta)\}\right]^{-1/2},$$

$$\kappa_{\ell_1,\ell_2} = E_\pi\,\ell_2(\theta)\,E_{p_2}\left\{\frac{\ell_1(\theta)}{\pi(\theta)}\right\}\left[E_\pi\{\ell_1(\theta)\}E_{p_1}\left\{\frac{\ell_1(\theta)}{\pi(\theta)}\right\}E_\pi\,\ell_2(\theta)\,E_{p_2}\left\{\frac{\ell_2(\theta)}{\pi(\theta)}\right\}\right]^{-1/2}.$$

Paper

Bayesian Analysis (2019)

Bayesian Analysis (2019) **14**, Number 4, pp. 1013–1036

On the Geometry of Bayesian Inference

Miguel de Carvalho*, Garritt L. Page†, and Bradley J. Barney‡

Abstract. We provide a geometric interpretation to Bayesian inference that allows us to introduce a natural measure of the level of agreement between priors, likelihoods, and posteriors. The starting point for the construction of our geometry is the observation that the marginal likelihood can be regarded as an inner product between the prior and the likelihood. A key concept in our geometry is that of compatibility, a measure which is based on the same construction principles as Pearson correlation, but which can be used to assess how much the prior agrees with the likelihood, to gauge the sensitivity of the posterior to the prior, and to quantify the coherency of the opinions of two experts. Estimators for all the quantities involved in our geometric setup are discussed, which can be directly computed from the posterior simulation output. Some examples are used to illustrate our methods, including data related to on-the-job drug usage, midge wing length, and prostate cancer.

Keywords: Bayesian inference, geometry, Hellinger affinity, Hilbert space, marginal likelihood.

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Discussion

Final Remarks

- We discussed a natural geometric framework to Bayesian inference which motivated a simple, intuitively appealing measure of the agreement between priors, likelihoods, and posteriors: compatibility (κ).

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- We developed MCMC-based estimators of these metrics that are easily computable and, by avoiding the estimation of harmonic means, are reasonably stable.

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- In this geometric framework, we also discuss a related measure of the "informativeness" of a distribution, $\|\cdot\|$.
- We developed MCMC-based estimators of these metrics that are easily computable and, by avoiding the estimation of harmonic means, are reasonably stable.
- Our concept of compatibility can be used to evaluate how much the prior agrees with the likelihood, to measure the sensitivity of the posterior to the prior, and to quantify the level of agreement of elicited priors.

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