Investment problem with switching modes

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Planned structure

- The setup
  - Motivation
  - Mathematical framework
- Switching problem
  - Solutions
  - Experiments
- Investment problem
  - Same investment costs for both projects
  - Different investment costs for the projects
- Final comments and remarks

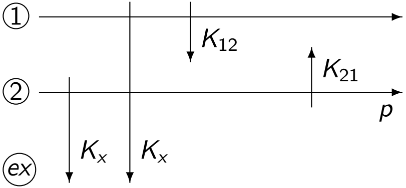
Main references

- Décamps, J.P., Mariotti, T. and Villeneuve, S., 2006. *Irreversible investment in alternative projects.*, Economic Theory, 28(2), pp.425-448.

- Zervos, M., Oliveira, C. and Duckworth, K., 2018. *An investment model with switching costs and the option to abandon*. Mathematical Methods of Operations Research, 88(3), pp.417-443.

Economic motivation

- The firm has two possible investments I_1 and I_2
- There is only one source of uncertainty the price (of the product) p



- Terminology
  - I_1 Investment ①
  - I_2 Investment ②
  - $K_{12} > 0$ Switching cost $I_1 \to I_2$
  - $K_{21} > 0$ Switching cost $I_2 \to I_1$
  - K_x exit cost (considered negative)
- The firm stays at one of the state at each moment of time $z \in \{I_1, I_2, ex\}$

Model 1/2

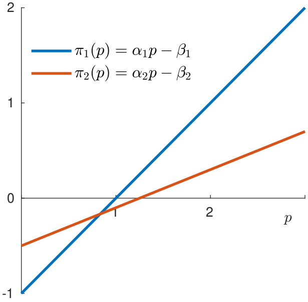
- Price process geometric Brownian motion: $dP_t = \mu P_t dt + \sigma P_t dB_t$
- Infinitesimal generator: $\mathcal{L} := \mu p \partial_p + \frac{1}{2}\sigma^2 p^2 \partial_{pp}$
- Payoff of the investment during time $[t_1, t_2]$

$$\int_{t_1}^{t_2} e^{-rs} \pi_i(P_s) ds,$$

$$\pi_i(p) = \alpha_i p - \beta_i, \quad \alpha_1 > \alpha_2, \quad \beta_1 > \beta_2$$

μ - instantaneous drift
σ - instantaneous variance
r - discount rate

- β_i can be interpreted as instantaneous fixed costs of production



Model 2/2

For strategy $\mathfrak{s}$ in the set of admissible strategies $\mathcal{S}$ the expected payoff is:

$$J_{\mathfrak{s}}(z,p) = \mathbb{E}_p \Bigg[\overbrace{\int_0^\infty e^{-rt}\left(\pi_1(P_t)\mathcal{I}_{\{Z_t=l_1\}} + \pi_2(P_t)\mathcal{I}_{\{Z_t=l_2\}}\right) dt}^{\text{production}} - \underbrace{-K_{12}\sum_{j=1}^{\infty} e^{-rT_j^{12}}\mathcal{I}_{\{T_j^{12}<\infty\}} - K_{21}\sum_{j=1}^{\infty} e^{-rT_j^{21}}\mathcal{I}_{\{T_j^{21}<\infty\}}}_{\textit{costs associated with switching}} \underbrace{-K_x e^{-r\tau}\mathcal{I}_{\{\tau<\infty\}}}_{\textit{cost of exit}} \Bigg]$$

Switching problem

Problem (Switching)

Find function V (or equivalently optimal strategy $\mathfrak{s}^*$)

$$V(z,p) = \sup_{\mathfrak{s}\in\mathcal{S}} J_s(z,p) = J_{\mathfrak{s}^*}(z,p)$$

We introduce: $v_1(p) := V(I_1, p)$ *and* $v_2(p) := V(I_2, p)$

[1] $V'(z,\cdot)$ is in particular absolutely continuous, [Zervos, 2003]

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Theorem (Verification theorem)

If $V \in Car(\{I_1, I_2, ex\} \times [0,\infty))$[1] and satisfies Hamilton-Jacobi-Bellman (HJB) equation(s) than V is solution to the optimization problem.

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Method of solution

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If $V \in Car(\{I_1, I_2, ex\} \times [0,\infty))^1$ and satisfies Hamilton-Jacobi-Bellman (HJB) equation(s) than V is solution to the optimization problem.

Method of solution

Guess solution

[1] $V'(z,\cdot)$ is in particular absolutely continuous, [Zervos, 2003]

Switching problem

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Method of solution

Guess solution $\longrightarrow$ Check HJB

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If $V \in Car(\{I_1, I_2, ex\} \times [0,\infty))$[1] and satisfies Hamilton-Jacobi-Bellman (HJB) equation(s) than V is solution to the optimization problem.

Method of solution

Guess solution $\longrightarrow$ Check HJB $\longrightarrow$ Feel smart and happy

[1] $V'(z,\cdot)$ is in particular absolutely continuous, [Zervos, 2003]

Hamilton-Jacobi-Bellman equation 1/2

$$\begin{aligned}
&\qquad\qquad (a) \qquad\qquad\qquad (b) \qquad\qquad\quad (c) \\
&\max\left\{\mathcal{L}v_1 - rv_1 + \pi_1, v_2 - v_1 - K_{12}, -v_1 - K_x\right\} = 0 \\
&\max\left\{\mathcal{L}v_2 - rv_2 + \pi_2, v_1 - v_2 - K_{21}, -v_2 - K_x\right\} = 0
\end{aligned}$$

Hamilton-Jacobi-Bellman equation 1/2

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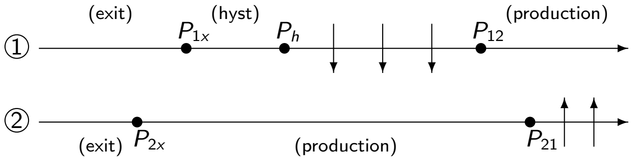


Space division

- Production region: $(a) = 0$
- Switching region: $(b) = 0$
- Exit region: $(c) = 0$
- Hysteresis region: $(a) = 0$ only for ①

Hamilton-Jacobi-Bellman equation 2/2

- In the production region: $(a) \quad \mathcal{L}v_i - rv_i + \pi_i = 0$
  Cauchy-Euler equation
  Assuming: $r > -\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2} - \mu\right)^2$
- Solution:

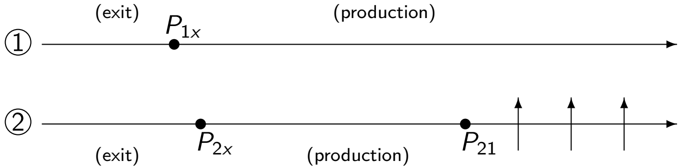
$$v = v_{hom} + v_{part}$$

$$v_{hom} = C_1 p^{d_1} + C_2 p^{d_2}, \quad C_1, C_2 \in \mathbb{R}$$

$$v_{part} = \frac{\alpha_i}{r - \mu} p - \frac{\beta_i}{r}$$

where $d_1 < 0$ and $d_2 > 1$ solve equation $\frac{\sigma^2}{2} d^2 + (\mu - \frac{\sigma^2}{2})d - r = 0$

Solution of Type I (No downgrading)



$$v_1(p) = \begin{cases} -K_x, & p < P_{1x} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r}, & p \geq P_{1x} \end{cases} \quad (1)$$

$$v_2(p) = \begin{cases} -K_x, & p < P_{2x} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu} p - \frac{\beta_2}{r}, & P_{2x} \leq p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r} - K_{21}, & p \geq P_{21} \end{cases} \quad (2)$$

Construction of solution

Find constants $A, C, D \in \mathbb{R}^+$, $P_{21} > P_{2x} > 0$, $P_{21} > P_{1x} > 0$, such that:

- v_1 and v_2 are continuous
- v_1 and v_2 have continuous derivatives (smooth pasting)

$$v_1(p) = \begin{cases} -K_x, & p < P_{1x} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}, & p \geq P_{1x} \end{cases}$$

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$$-K_{1x} = AP_{1x}^{d_1} + \frac{\alpha_1}{r-\mu}P_{1x} - \frac{\beta_1}{r}$$

$$0 = d_1 AP_{1x}^{d_1} + \frac{\alpha_1}{r-\mu}P_{1x}$$

$$0 = CP_{2x}^{d_1} + DP_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} - \frac{\beta_2}{r} + K_{2x}$$

$$0 = Cd_1 P_{2x}^{d_1} + Dd_2 P_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x}$$

$$0 = (C-A)P_{21}^{d_1} + DP_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} - \frac{\beta_2-\beta_1}{r} + K_{21}$$

$$0 = (C-A)d_1 P_{21}^{d_1} + Dd_2 P_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21}$$

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Construction of solution

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Solution exists

Construction of solution

Find constants $A, C, D \in \mathbb{R}^+$, $P_{21} > P_{2x} > 0$, $P_{21} > P_{1x} > 0$, such that:

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0 &= (C-A)d_1 P_{21}^{d_1} + Dd_2 P_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21}
\end{aligned}
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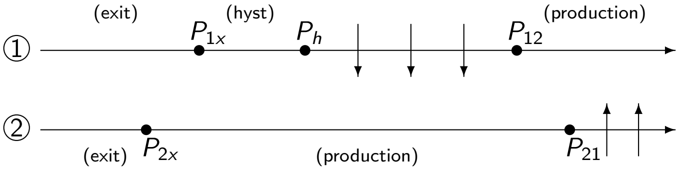
$$

Solution exists $\longrightarrow$ Verifies HJB for the set of parameters $\mathcal{O}$

Solution of Type II (Hysteresis)

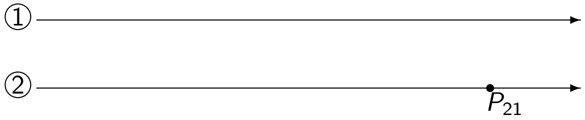


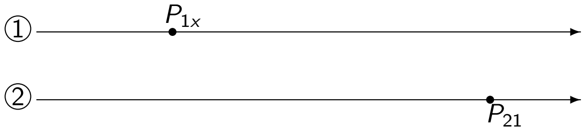
$$v_1(p) = \begin{cases} -K_x & p < P_{1x} \\ C_1 p^{d_1} + D_1 p^{d_2} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r} & P_{1x} \le p < P_h \\ C_2 p^{d_1} + D_2 p^{d_2} + \frac{\alpha_2}{r-\mu} p - \frac{\beta_2}{r} - K_{12} & P_h \le p < P_{12} \\ A p^{d_1} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r} & P_{12} \le p \end{cases} \tag{3}$$

$$v_2(p) = \begin{cases} -K_x & p < P_{2x} \\ C_2 p^{d_1} + D_2 p^{d_2} + \frac{\alpha_2}{r-\mu} p - \frac{\beta_2}{r} & P_{2x} \le p < P_{21} \\ A p^{d_1} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r} - K_{21} & P_{21} \le p \end{cases} \tag{4}$$

Some other types of solution

There are other several possible types of solutions that depends on the parameters of the problem





Division of the state space, [Zervos et al., 2018]

Conditions on $K_1 > 0$, $K_0 > 0$, $K \in \mathbb{R}$ and $h(\cdot)$		Case
$0 \le K$	$rK_1 \le h(0)$	I.1, Lemma 1
	$\max\{-rK_0, -rK\} \le h(0) < rK_1$	I.2, Lemma 2
	$K_0 \le K$ and $h(0) < -rK_0$	II.1, Lemma 4
	$K < K_0$ and $-rK_0 \le h(0) < -rK$	II.2, Lemma 5
	$K < K_0^\star \le K_0$ and $h(0) < -rK_0$	II.2, Lemma 5
	$K < K_0 < K_0^\star$ and $h(0) < -rK_0$	II.3, Lemma 6
$K < 0$	$rK_1 - rK \le h(0)$	I.1, Lemma 1
	$-rK \le h(0) < rK_1 - rK$	I.3, Lemma 3
	$-rK_0 \le h(0) < -rK$	III.1, Lemma 7
	$h(0) < -rK_0$ and $h(\delta_\dagger) \ge 0$ or $\big(h(\delta_\dagger) < 0$ and $K_1 \ge K_1^\dagger\big)$ or $\big(h(\delta_\dagger) < 0,\ K_1 < K_1^\dagger$ and $K_0 \ge K_0^\dagger\big)$	III.1, Lemma 7
	$h(0) < -rK_0$, $h(\delta_\dagger) < 0$, $K_1 < K_1^\dagger$ and $K_0 < K_0^\dagger$	III.2, Lemma 8

- Zervos, M., Oliveira, C. and Duckworth, K., 2018. *An investment model with switching costs and the option to abandon*. Mathematical Methods of Operations Research, 88(3), pp.417-443.
- Table, page 25
- Project I_2 has $\pi_2(p) = -\beta_2$, no production

Division of the parameter space

Proposition

Consider that, $rK_x + \alpha_2 P_{2x} - \beta_2 < 0$, *and let* $\delta = \frac{(\beta_1 - rK_x)(d_2 - 1)}{\alpha_1 d_2}$, *then if*

I.1 $-\beta_1 + \beta_2 + rK_{12} > 0$ *or*

I.2 $-\beta_1 + \beta_2 + rK_{12} < 0$ *and one of*

- $\pi_1(\delta) - \pi_2(\delta) > 0$
- $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} \geq K_{21}^{\dagger}$
- $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} < K_{21}^{\dagger}$ *and* $K_{12} > K_{12}^{\dagger}$

the function V is of the type I, if the opposite holds, i.e.

Division of the parameter space

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- $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} < K_{21}^{\dagger}$ *and* $K_{12} > K_{12}^{\dagger}$

the function V *is of the type I, if the opposite holds, i.e.*

The thresholds $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ are constants that can be calculated from the parameters of the problem. Moreover, $K_{21}^{\dagger}$ is independent of K_{21} and K_{12}, and $K_{12}^{\dagger}$ is independent of K_{12}

Division of the parameter space

Proposition

Consider that, $rK_x + \alpha_2 P_{2x} - \beta_2 < 0$*, and let* $\delta = \frac{(\beta_1 - rK_x)(d_2 - 1)}{\alpha_1 d_2}$*, then if*

I.1 $-\beta_1 + \beta_2 + rK_{12} > 0$ *or*

I.2 $-\beta_1 + \beta_2 + rK_{12} < 0$ *and one of*

- $\pi_1(\delta) - \pi_2(\delta) > 0$
- $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} \geq K_{21}^{\dagger}$
- $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} < K_{21}^{\dagger}$ *and* $K_{12} > K_{12}^{\dagger}$

the function V is of the type I, if the opposite holds, i.e.

II $\pi_1(\delta) - \pi_2(\delta) < 0$ *and* $K_{21} < K_{21}^{\dagger}$ *and* $K_{12} < K_{12}^{\dagger}$
the function V is of the type II.

The thresholds $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ are constants that can be calculated from the parameters of the problem. Moreover, $K_{21}^{\dagger}$ is independent of K_{21} and K_{12}, and $K_{12}^{\dagger}$ is independent of K_{12}

Type II (hysteresis) solution

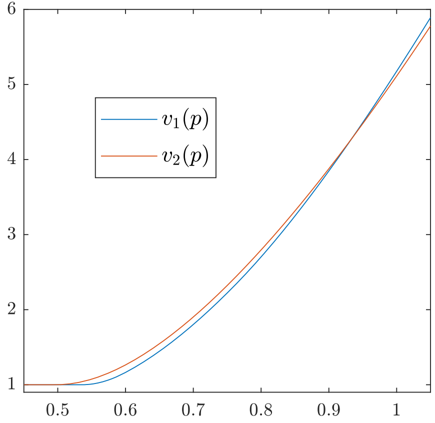


Illustration of HJB verification for Type II (hysteresis)

Example. The parameters: $\mu = 0, \sigma = 0.2, r = 0.05, \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 0.5, \beta_2 = 0.5, K_{21} = 0.3, K_{12} = 0.1$.
Auxiliary $d_1 = -1.16, d_2 = 2.16, \delta = 0.56, K_{12}^\dagger = 0.32, K_{21}^\dagger = 21.55$,
Points: $P_{2x} = 0.50, P_{1x} = 0.54, P_h = 0.60, P_{12} = 0.78, P_{21} = 1.37$,
Coefficients: $A = 5.0$, $C_1 = 4.82$, $C_2 = 2.42$, $D_1 = 1.38$, $D_2 = 2.69$.

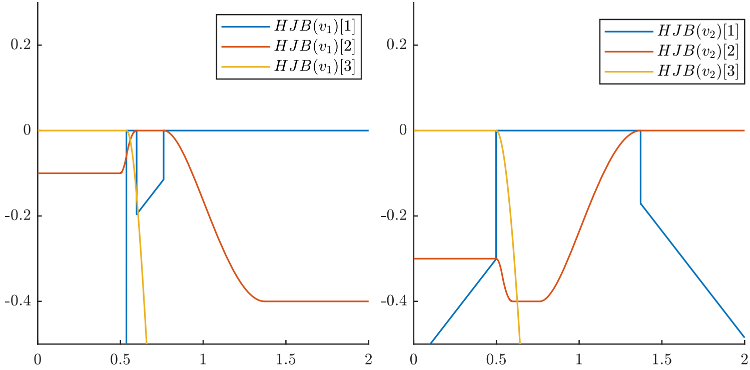
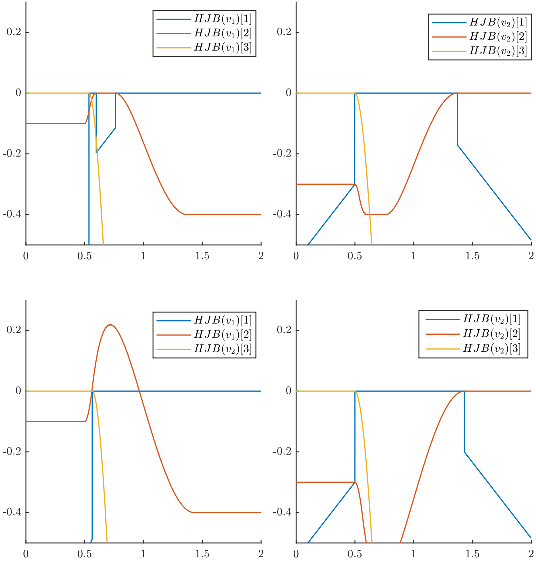


Illustration of HJB verification



Type II

The existence of solution of the certain type is a necessary condition but not sufficient, the HJB have to be verified

Type I

Experiments μ

μ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	$K_{21}^{\dagger}$	$K_{12}^{\dagger}$
-0.1500	0.9305			0.9654	1.5650	0.0144	0.0052
-0.1000	0.8857			0.9024	1.4864	0.0908	0.0020
-0.0500	0.7944			0.7777	1.4124	0.8665	0.0054
-0.0300	0.7285			0.6938	1.3970	2.6249	0.0366
-0.0100	0.6192	0.7059	0.7710	0.5750	1.3888	9.7253	0.1613
0.0000	0.5359	0.5978	0.7608	0.4983	1.3715	21.5473	0.3183
0.0100	0.4395	0.4829	0.7504	0.4094	1.3546	59.3298	0.6130
0.0250	0.2742	0.2964	0.7345	0.2564	1.3300	942.6027	1.5882

- As μ increases (market becomes more favourable) every point moves towards zero, the firm is interested in moving faster to more risky/profitable investment I_1.
- As μ decreases and gradually moves to (downward market) the firm exits faster and delays the movement from I_2 to I_1.
- Decrease. The hysteresis region disappears, paying the cost of downgrading becomes unprofitable. $K_{12} = 0.10$ threshold is triggered.

Experiments σ

σ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	$K_{21}^{\dagger}$	$K_{12}^{\dagger}$
0.0200	0.9857			1.0011	1.0861	0.0052	
0.0250	0.9702			0.9820	1.0935	0.0330	
0.0500	0.8966			0.8906	1.1345	0.5448	0.0052
0.0800	0.8159			0.7919	1.1884	1.9002	0.0609
0.0900	0.7907	0.7907	-0.1047	0.7615	1.2071	2.5411	0.0837
0.1000	0.7654	0.8338	0.8392	0.7324	1.2251	3.2903	0.1063
0.1500	0.6398	0.7043	0.7964	0.6029	1.2999	9.2186	0.2175
0.2000	0.5359	0.5978	0.7608	0.4983	1.3715	21.5473	0.3183
0.2500	0.4509	0.5105	0.7301	0.4139	1.4416	48.7973	0.4051
0.5000	0.2088	0.2566	0.6186	0.1805	1.7905	9,964.2000	0.6456

- Increase. More uncertainty. Reluctant to make changes. The decision points spread further apart.
- Decrease. Less uncertainty. Concentration of all points at $-K_x$.
- Decrease. The hysteresis region disappears. K_{12} threshold is triggered.

Experiments K_{12}

P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	A	K_{12}
0.5015	0.5063	0.7852	0.4975	1.3377	5.0069	0.0010
0.5248	0.5647	0.7726	0.4979	1.3552	5.0069	0.0500
0.5359	0.5978	0.7608	0.4983	1.3715	5.0069	0.1000
0.5510	0.6534	0.7396	0.4990	1.4007	5.0069	0.2000
0.5618	0.7073	0.7208	0.4996	1.4263	5.0069	0.3000
0.5632	0.7156	0.7182	0.4997	1.4299	5.0069	0.3150
0.5635	0.7173	0.7177	0.4997	1.4306	5.0069	0.3180
0.5635			0.4997	1.4307	5.0069	0.3184
0.5635			0.4997	1.4307	5.0069	0.6000
0.5635			0.4997	1.4307	5.0069	10.0000

- The exit option A is not affected
- Increase. Move from I_1 to I_2 for lesser prices, until it becomes non-profitable.
- Increase, threshold $K_{12}^{\dagger} = 0.32$ is triggered, after that does not affect solution
- Decreases. Until $P_{1x} - P_h$ collapse, if negative, it is not profitable to exit from I_1, it is more profitable to move to I_2, then exit. Strategy change.

Switching problem: Feel smart and happy

Investment problem: Set up

- The firm is not on the market, can enter investing in one of the projects (and then has a possibility to switch)
- K_1 - Cost of entering in the project I_1
- K_2 - Cost of entering in the project I_2

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Problem (Investment problem - different costs)

Find the value function $W_d \in Car[0, +\infty)$

$$W_d(p) = \max\left\{\sup_{\tau\in\mathcal{T}} E_p\left[e^{-r\tau}\max\{v_1(P_\tau) - K_1, v_2(P_\tau) - K_2\}\right], 0\right\},$$

or introducing $\quad v^*(p) = \max\{v_1(p) - K_1, v_2(p) - K_2\}$

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- Hamilton-Jacobi-Bellman: $\max\left\{\mathcal{L}W_d - rW_d, v^* - W_d\right\} = 0$

Same investment costs

Problem (Investment problem - same costs)

Find the value function $W_s \in Car[0, +\infty)$

$$W_s(p) = \max\left\{\sup_{\tau\in\mathcal{T}} E_p\left[e^{-r\tau}(v^*(P_\tau))\right], 0\right\},$$

where $K_e = K_1 = K_2$ *and* $v^* = max(v_1, v_2) - K_e$

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Method of solution

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Guess solution

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Method of solution

Guess solution $\longrightarrow$ *Check HJB*

Same investment costs

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$$W_s(p) = \max\left\{\sup_{\tau\in\mathcal{T}} E_p\left[e^{-r\tau}(v^*(P_\tau))\right], 0\right\},$$

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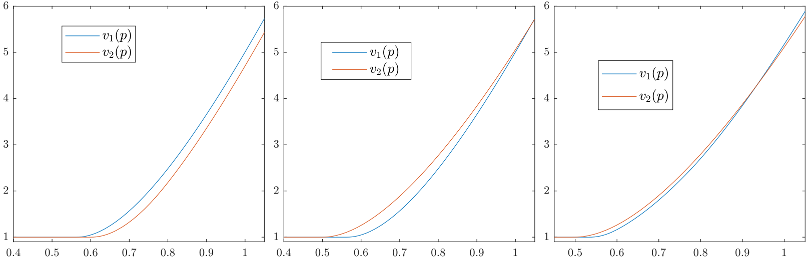
Method of solution

Guess solution $\longrightarrow$ *Check HJB* $\longrightarrow$ *Feel smart and happy*

Same investment costs: smart guess 1/2

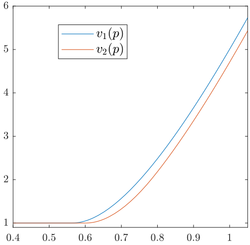


Case 0
No hysteresis,
$P_{1x} < P_{2x}$

Case 1
No hysteresis,
$P_{1x} > P_{2x}$

Case 2
Hysteresis,
$P_{1x} > P_{2x}$

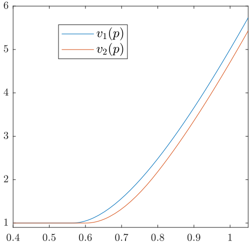
Same investment costs: smart guess 2/2

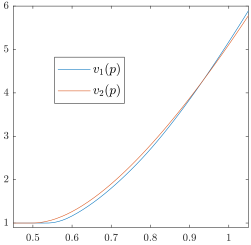


$$v^*(p) \begin{cases} -K_x & p < P_{1x} \\ f_1(p) & p \geq P_{1x} \end{cases}$$

$$f_1(p) = Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r},$$

Same investment costs: smart guess 2/2





$$v^*(p) \begin{cases} -K_x & p < P_{1x} \\ f_1(p) & p \geq P_{1x} \end{cases} \qquad v^*(p) = \begin{cases} -K_x & p < P_{2x} \\ f_2(p), & P_{2x} \leq p < \hat{p} \\ f_1(p) & p \geq \hat{p} \end{cases}$$

$$f_1(p) = Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}, \qquad f_2(p) = Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r}$$

$$\hat{p} \in (\max(P_{1x}, P_{2x}), +\infty) : f_1(\hat{p}) = f_2(\hat{p})$$

Same investment costs: solution

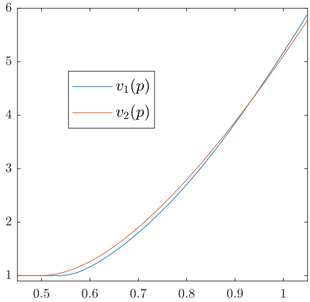
Proposition

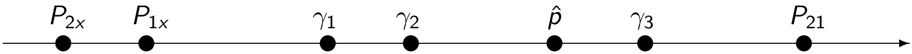
For the case $P_{2x} < P_{1x}$ *and for:*

- $K_e^+ > K_e > -K_x$

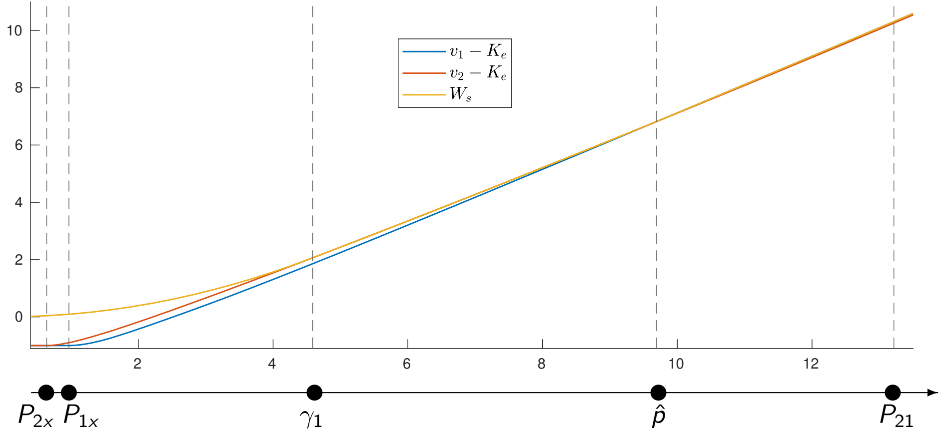
there are constants $A_1, B_1, B_2 > 0$ *and* $\gamma_3 > \hat{p} > \gamma_2 > \gamma_1 > 0$ *such that*

$$W_s(p) = \begin{cases} B_1 p^{d_2} & p \in [0, \gamma_1) \\ f_2(p) - K_e & p \in [\gamma_1, \gamma_2] \\ A_1 p^{d_1} + B_2 p^{d_2} & p \in (\gamma_2, \gamma_3) \\ f_1(p) - K_e & p \in [\gamma_3, +\infty). \end{cases}$$

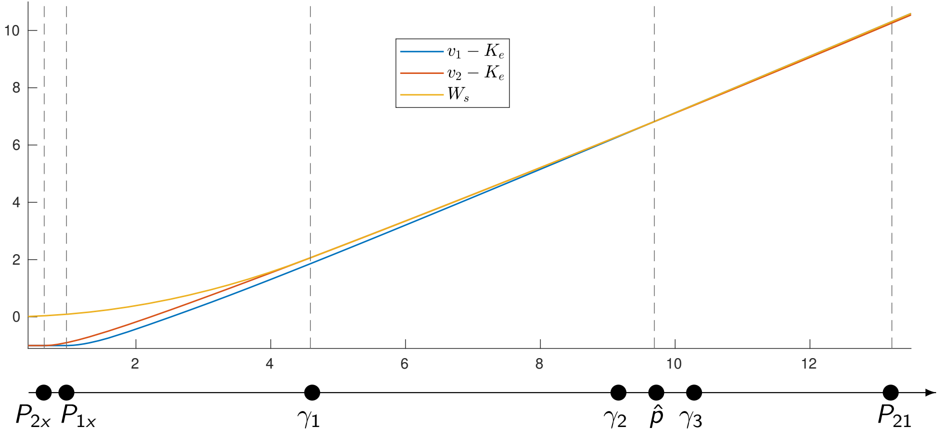




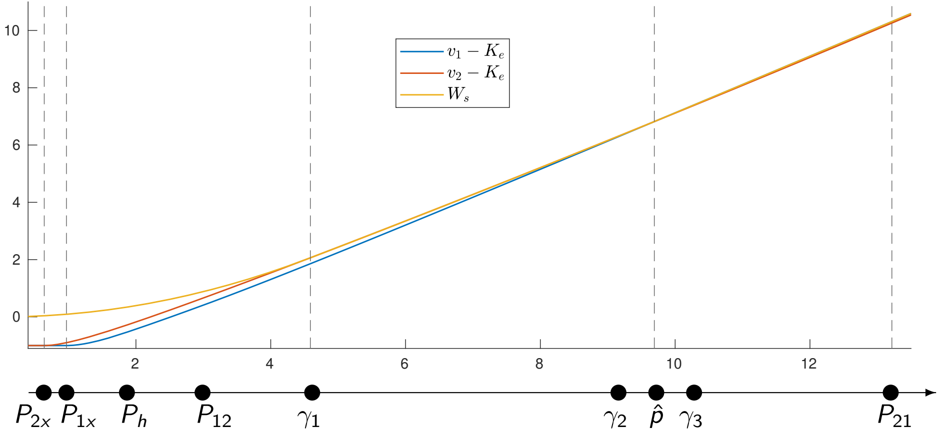
Cases 1 and 2, Illustration 1/3



Cases 1 and 2, Illustration 2/3



Cases 1 and 2, Illustration 3/3



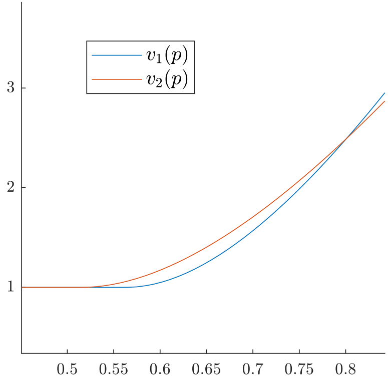
Same investment costs: Feel smart and happy

Different investment costs, No switching [Décamps et al., 2006]

Décamps, J.P., Mariotti, T. and Villeneuve, S., 2006. *Irreversible investment in alternative projects.*, Economic Theory, 28(2), pp.425-448.

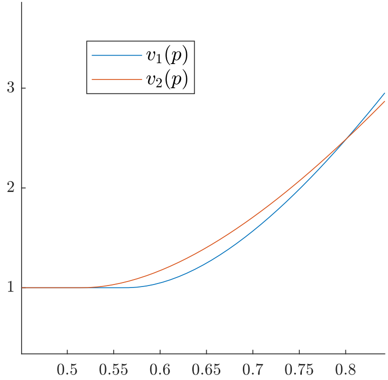
- Basic model (same as in [Dixit et al., 1994])
  - Once invested you stay in the same project forever
  - No exit option, no cost of production
- Project switching model
  - The only profitable/existing possibility is to move from I_2 to I_1, since there is no production costs.

Different investment costs: Type I 1/3



- $-K_1 - K_x < 0$ and $-K_2 - K_x < 0$
  no 'free lunch'.
- $K_1 < K_2 + K_{21}$ and $K_2 < K_1 + K_{12}$
  no 'cheat switching'

Different investment costs: Type I 1/3



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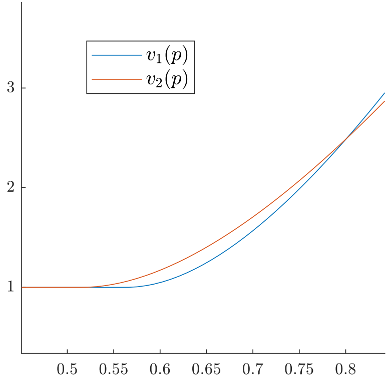
Solution of the form:

$$W_d^*(p) = \begin{cases} B_1 p^{d_2} & p \in [0, \gamma_1) \\ f_2(p) - K_2 & p \in [\gamma_1, \gamma_2] \\ A_1 p^{d_1} + B_2 p^{d_2} & p \in (\gamma_2, \gamma_3) \\ f_1(p) - K_1 & p \in [\gamma_3, +\infty) \end{cases}$$

Different investment costs: Type I 1/3



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Can we define the conditions?

Different investment costs: Type I 2/3

Proposition

For the functions of the Type I, i.e. Cases 0 and 1, there are bounds K_2^+ *and* K_1^-

- $K_2^+ > K_2 > -K_x$, *where* K_2^+ *is independent of* K_1 *or* K_2*, and*
- $K_2 + K_{21} > K_1 > K_1^-$, *where* K_1^- *is independent of* K_1*,*

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Recall:

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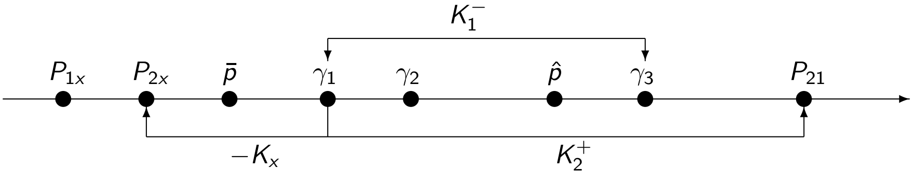
Different investment costs: Type I 3/3
Bounds

Note

Bounds K_2^+ and K_1^-



Investment problem with switching (2+1) modes

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- Applications to test! Real markets, real data lets predict!


thank you!

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