SYZ mirror symmetry for del Pezzo surfaces and rational elliptic surfaces

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Joint w/ A. Jacob and Y.-S. Lin

Outline

1. del Pezzo surfaces and rational elliptic surfaces.

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2 Mirror symmetry and divisor compliments.

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6 SYZ mirror symmetry for del Pezzo surfaces, rational elliptic surfaces and Hodge numbers.

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- For degree 8 there are two families given by the first Hirzebruch surface $\mathrm{Bl}_p\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.
- All del Pezzo surfaces admit a smooth divisor $D \in |-K_Y|$.

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del Pezzo surfaces and rational elliptic surfaces as Calabi-Yau pairs

- Let Y be a RES or a del Pezzo surface, and $D \in |-K_Y|$ a divisor. If $s \in H^0(Y, -K_Y)$ is a holomorphic section with $\{s = 0\} = D$, then $\frac{1}{s}$ is a holomorphic $(2,0)$ form on $Y \setminus D$ with a simple pole on D.

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- Therefore, $Y \setminus D$ is a natural *non-compact* Calabi-Yau manifold.
- The existence of a Ricci-flat Kähler metric does not follow from Yau's theorem, since $Y \setminus D$ is non-compact.

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- For example, an instantiation of this principle is

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- This is a particular case of mirror symmetry for the Hodge diamonds of $X, \check{X}$.

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- The basic proposal for how to *construct* mirror symmetric pairs is due to Strominger-Yau-Zaslow (SYZ).
- There are programs of Gross-Siebert and Kontsevich-Soibelman aimed at using the SYZ philosophy to construct (often formal) algebraic mirrors.

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- The notion makes sense for ω *not* the Calabi-Yau symplectic form, but in this case they are no longer volume minimizing.

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- The T-dual fibrations exchange complex and symplectic affine structures on B.

Mirror symmetry beyond CYs

- If Y is not a compact Calabi-Yau manifold, then the mirror to be a (partial compactification of) a *Landau-Ginzburg* model: ie. a non-compact Kähler manifold M together with a holomorphic function $W : M \to \mathbb{C}$.

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- If Y is compact Kähler and $D \in |-K_Y|$ is a divisor, Auroux laid out a general picture for constructing the mirror to Y by applying SYZ mirror symmetry to the non-compact CY manifold $X = Y \setminus D$.

Mirror symmetry beyond for del Pezzo surfaces and rational elliptic surfaces

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- Lunts-Przyjalkowski proved mirror symmetry for "Hodge numbers", as proposed by Katzarkov-Kontsevich-Pantev.
- Doran-Thompson studied del Pezzo $\leftrightarrow$ RES mirror symmetry at a lattice theoretic level.

The goal of this talk

In this talk I hope to:

- Explain a proof of SYZ mirror symmetry for del Pezzo surfaces of degree k and RES with an I_k fiber.
- Explain mirror symmetry for Hodge numbers in terms of moduli of complete CY metrics.

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- Explain a proof of SYZ mirror symmetry for del Pezzo surfaces of degree k and RES with an I_k fiber.
- Explain mirror symmetry for Hodge numbers in terms of moduli of complete CY metrics.
- Describe applications to existence of some new CY metrics, a question of Yau, etc.

del Pezzo surfaces: Complete Calabi-Yau metrics

The first ingredient we need is a fundamental result of Tian-Yau, which in our case gives

Theorem (Tian-Yau)

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and such that ω_{TY} is asymptotic to the Calabi model *(with estimates...)*

Remark

The Tian-Yau theorem holds in all dimensions

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$$\mathcal{C} := \{\xi \in E : 0 < |\xi|_h < 1\}, \qquad \Omega_{\mathcal{C}} := \Omega_D \wedge \frac{dw}{w}.$$

$$\omega_{\mathcal{C}} = \frac{2}{3}\sqrt{-1}\partial\overline{\partial}\left(-\log|\xi|_h^2\right)^{\frac{3}{2}}.$$

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Then $(\mathcal{C}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$ is Calabi-Yau, and furthermore complete at $0 \subset E$.

del Pezzo surfaces: Complete Calabi-Yau metrics

- The Riemannian geometry of $(\mathcal{C}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$ can be visualized by considering the level sets

$$\pi : \mathcal{C}_\varepsilon := \{\xi \in E : |\xi|_h = \varepsilon\} \to D$$

are S^1 bundles over D, fibering $\mathcal{C}$.

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SYZ fibrations on del Pezzo surfaces

Using this model geometry we prove:

Theorem (C.-Jacob-Lin)

Let Y be a del Pezzo surface, D a smooth elliptic curve, and equip $X = Y \setminus D$ with the Tian-Yau metric ω_{TY}.

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Remark

In fact, X admits countably many distinct special Lagrangian fibrations, one for each choice of simple closed loop $\gamma \in H_1(D, \mathbb{Z})$.

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- technical issue: The geometry of the Lagrangian fibers degenerates near ∞: they are collapsing circle bundles with fixed volume and unbounded diameter.
- key point: The degeneration of the geometry is *polynomial* in the distance to a fixed point, while the mean curvature decays *exponentially*.

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11. In every compact set $K \subset X$ there can be only countably many singular holomorphic curves obtained as limits of smooth fibers. From this it follows easily that the fibration extends to a global fibration. Picture time!

Applications of SYZ fibrations on del Pezzo surfaces

Theorem (C.-Jacob-Lin)

Let Y be del Pezzo of degree k, $D \in |-K_Y|$ smooth, and $X = Y \setminus D$. Let $\pi_{SYZ} : X \to \mathbb{R}^2$ be a SYZ fibration of (X, ω_{TY}).

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Question (Yau $\sim$ 80s): What is the symplectic form for the HK rotated Tian-Yau metric?

Some Remarks

- The Lagrangian mean curvature flow argument works in all dimensions and applies to Fano manifolds Y, with $D \in |-K_Y|$ assuming D has a special Lagrangian submanifold with respect to the CY metric in $c_1(Y)|_D$.

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- If $Y = \mathbb{P}^2$ or a generic rational elliptic surface we can identify the singular fibers of the SYZ fibration as the appropriate number of nodal special Lagrangian spheres verifying some conjectures of Auroux.

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The model geometry: Let $\Delta^* = \{z \in \mathbb{Z} : 0 < |z| < 1\}$, consider

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$$\Omega = \frac{\kappa(z) dx \wedge dz}{z} \qquad \kappa(z) : \Delta \to \mathbb{C} \text{ hol'c }, \quad \kappa(0) \neq 0$$

Let's assume: $\kappa = 1$ for simplicity.

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$$\omega_{sf,\varepsilon} = \sqrt{-1}\frac{k|\log|z||}{2\pi\varepsilon}\frac{dz \wedge d\bar{z}}{|z|^2} + \frac{\sqrt{-1}}{2}\frac{2\pi\varepsilon}{k|\log|z||}(dx + B(x,z)dz) \wedge \overline{(dx + B(x,z)dz)}$$

where $B(x, z) = -\frac{\mathrm{Im}(x)}{\sqrt{-1}z|\log|z||}$, ε = volume of the fibers.

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For all $\alpha > \alpha_0$ there exists a CY metric in the Bott-Chern cohomology class of ω_0 converging exponentially fast to $\alpha\omega_{sf,\sigma,\frac{\varepsilon}{\alpha}}$ at infinity (with very precise estimates to all orders)

This applies, for example, to Kähler metrics restricted from Y.

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Main problem: If we want to understand the Kähler moduli (to do mirror symmetry, define Hodge numbers etc.) on a RES pair (Y, D) in terms of moduli of Calabi-Yau metrics, then we need a parameter space.

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Outstanding questions from Hein's theorem:

- $H^2_{dR}(X, \mathbb{R})$ has dimension $11 - k$.

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- Recall that Bott-Chern cohomology is a refinement of de Rham cohomology given by

$$H^{p,q}_{BC} := \frac{\{\mathrm{Ker}\, d : \Lambda^{p,q} \to \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}\}}{\{\mathrm{Im}(\sqrt{-1}\partial\overline{\partial} : \Lambda^{p-1,q-1} \to \Lambda^{p,q})\}}$$

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  Hein's theorem depends on a construction which leaves open the possibility of (infinitely many) distinct CY metrics *even in a fixed Bott-Chern class*.

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- By Leray spectral sequence calculations one can show that

$$H^{1,1}_{BC}(X) \sim H^2_{dR}(X) \times H^0(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

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- Lunts-Przyjalkowski computed these Hodge numbers for del Pezzos of degree k and RES with an I_k fiber, and obtained $10 - k$ on both sides (proving mirror symmetry at the level of KKP Hodge numbers).

A brief digression on sections

- a section $\sigma : \Delta^* \to X_{mod}$ can be written as

$$\sigma(z) = h(z) + \frac{a}{2\pi\sqrt{-1}} \log z + \frac{b}{(2\pi\sqrt{-1})^2} (\log(z))^2$$

where $h : \Delta^* \to \mathbb{C}$ is holomorphic, $\frac{2b}{k} \in \mathbb{Z}$, $a + b \in \mathbb{Z}$.

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- Key point: Pulling back the semi-flat metric by a *multivalued* section still yields a well-defined, semi-flat, Calabi-Yau metric. We call these non-standard semi-flat metrics and say they are quasi-regular in the $\mathbb{Q}$ case, and irregular in the $\mathbb{R}$ case.

Calabi-Yau metrics on a RES with an I_k fiber

- If $\omega_{\sigma,sf,\varepsilon}$ is a quasi-regular semi-flat metric, then there is still a family of special Lagrangian "bad cycles" $C_r \subset X_{mod}$, $r \in (0,1)$, but C_r covers the circle $|z| = r$ in the base more than once.

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- As we will see, the quasi-regular metrics are important for mirror symmetry.

an Application to a question of Yau

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- *ω_γ is the symplectic form of the Ricci-flat metric produced by the previous theorem.*

an Application to a question of Yau

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Let (Y, D) be a del Pezzo pair, $\gamma \in H_1(D, \mathbb{Z})$, and $(X, g_{TY}, \omega_{TY}, J)$ be the Tian-Yau Ricci-flat Kähler structure on X. Let $\pi_\gamma : X \to \mathbb{R}^2$ be the SYZ fibration induced by this data, and let $(X, g_{TY}, \omega_\gamma, I)$ be the hyper-Kähler rotated space so that $\pi_\gamma : X \to \mathbb{C}$ is holomorphic. Then

- *ω_γ is the symplectic form of the Ricci-flat metric produced by the previous theorem.*
- *ω_γ is asymptotic to a non-standard semi-flat metric* unless *D is the torus with fundamental domain determined by the lattice $\mathbb{Z} + \sqrt{-1}\lambda\mathbb{Z}$ for $\lambda \in \mathbb{R}_{>0}$, and γ is one of the cycles generating the lattice. Generically, ω_γ is irregular.*

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Corollary

If we define the Kähler moduli to be

$$\mathcal{M}_{K\ddot{a}h} := \{ \text{ CY metrics asymptotic to } \omega_{sf,\varepsilon} \} / \mathrm{Aut}_0(X)$$

where $\mathrm{Aut}_0(X)$ are automorphisms homotopic to the identity. Then $\mathcal{M}_{K\ddot{a}h}$ is a cone with non-empty interior in $H^2_{dR}(X, \mathbb{R}) \sim \mathbb{R}^{11-k}$.

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$\Rightarrow$ we have a hope to define the Hodge numbers in terms of moduli of Calabi-Yau metrics.

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- argument is the same as that for del Pezzo surfaces using the "bad cycle" of the quasi-regular semi-flat metric as the model.

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- For $k \neq 8$ there is a unique deformation family of RES with an I_k fiber. For $k = 8$ there are two families, but these can be distinguished topologically.
- Using the Torelli theorem of Gross-Hacking-Keel there is a distinguished pair (Y_e, D_e) in the deformation family (Y', D') with trivial periods. This pair was also used by Hacking-Keating to study HMS for log CY surfaces.

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- Moreover, $(X_e, \check{\omega}_{CY})$ admits an SYZ fibration with fibers in the class α.
- By direct calculation, the SYZ fibrations on $Y \setminus D$ and $Y_e \setminus D_e$ are dual, in the sense that they exchange the complex and symplectic affine structures on the base $\mathbb{R}^2$, and that their volumes are inverse to one another.

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- Comparing the *complex* moduli of rational elliptic surfaces with the *symplectic* moduli of del Pezzos is an interesting question for future work.

The End