On the Constant Scalar curvature Kähler metrics

X. X. Chen
Based on joint work with J.R. Cheng

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For more than six decades, we have witnessed the phenomenal success of Calabi's program. Noticeably, the celebrated theorem of Yau in 1976 and the now well known Chen-Donaldson-Sun theorem in 2012.

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What we will lecture today is one step beyond the Kähler Einstien metrics problem. Inspired by the celebrated C^0, C^2 and C^3 a priori estimate of Calabi, Yau and others on Kaehler Einstein metrics, we present a report on a priori estimates on constant scalar curvature Kaehler metrics.

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With this estimate, we prove the Donaldson conjecture on geodesic stability and the well known properness conjecture on the Mabuchi energy functional.

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With this estimate, we prove the Donaldson conjecture on geodesic stability and the well known properness conjecture on the Mabuchi energy functional.

The general setting for this talk is the interaction among algebraic geometry, partial differential equation and complex geometry.

Basic Kähler Geometry I

$(M, [\omega])$ is a polarized Kähler manifold where

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\, d\, w_\alpha \wedge d\, \bar{w}_\beta > 0 \qquad \text{on } M.$$

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In some local coordinate $U \subset M$, there is a local potential function ρ such that

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \rho}{\partial w_\alpha \partial \bar{w}_\beta}, \qquad \forall\ \alpha, \beta = 1, 2, \cdots n.$$

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A Kähler class

$$[\omega] = \{\omega_\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}$$

where φ is a real valued function.

Basic Kähler Geometry II

Ricci form:

$$
\begin{aligned}
Ric(\omega) &= -\sqrt{-1}\partial\bar{\partial}\log\omega^n \\
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Scalar curvature:

$$\begin{aligned} R &= -g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial w_\alpha \bar{\partial} w_\beta}\log\det\left(g_{\alpha\bar{\beta}}\right) \\ &= -\triangle_g \log\det\left(g_{\alpha\bar{\beta}}\right). \end{aligned}$$

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The first Chern class is positive definite (resp: negative definite) if

$$[Ric(\omega)] > (\text{resp.} <)\ 0 \ \text{ on } M.$$

Calabi Conjecture

Conjecture (Calabi 1954 ICM)

In Kähler manifold where the first Chern class is either positive, zero or negative, does there always exist a Kähler Einstein metric with positive, zero or negative scalar curvature?

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Conjecture (Calabi 1950s)

For any Kähler class in Kähler manifold, does there always exist an extremal Kähler metric?

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- In 1976, for $C_1 < 0$, Calabi conjecture is solved by S. T. Yau and T. Aubin independently.
- In 2012, for $C_1 > 0$, Chen-Donaldson-Sun proved the stability conjecture of Fano manifold which goes back to S. T. Yau.

Constant scalar curvature Kähler metrics

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Conjecture (Chen)

In Kähler manifold $(M, [\omega])$, if ω_φ is a constance scalar curvature Kähler metric and φ is bounded, then all derivatives of φ with respect to the background metric is uniformly bounded.

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Conjecture (Properness conjecture)

The existence of constant scalar curvature Kähler metrics in $(M, [\omega])$ *is equivalent to the properness of K energy functional in terms of geodesic distance in the space of Kähler potentials.*

The space of Kähler potentials

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$$\mathcal{H} = \{\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}$$

where the tangent space

$$T_\varphi \mathcal{H} = C^\infty(M).$$

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This metric is also defined by S. Semmes in 1991 and S. K. Donaldson in 1996 for different motivation.

The space of Kähler Potentials

The geodesic equation is:

$$\varphi''(t) - g_{\varphi}^{\alpha\bar{\beta}}\varphi'_{\alpha}(t)\varphi'_{\bar{\beta}}(t) = 0 \tag{1}$$

where $g_{\varphi\,\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial w_\alpha \partial \bar{w}_\beta}$.

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According to S. Semmes, this can be written as

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)_{(n+1)\times(n+1)} = 0$$

in $[0,1] \times S^1 \times M$.

Basic work in space of potentials $\mathcal{H}$

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The K energy

T. Mabuchi in 1988 defined a 1-form in the space of Kähler potentials

$$d\,E : T\mathcal{H} \to \mathbf{R}$$

by

$$d\,E(\varphi, \psi) = -(R(\omega_\varphi) - \underline{R}, \psi)_\varphi,$$

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An important observation:
The K energy functional is convex along any smooth geodesic.

Convexity of K energy functional

Conjecture (Chen 1998)

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2017, W.Y. He-Y. Zeng confirmed this conjecture with minor assumption.

Fundamental work in $\mathcal{H}$ I

Here are the Fundamental theorem in $\mathcal{H}$ needed.

Conjecture (V. Guedj)

The completion of the space $\mathcal{H}$ of smooth potentials equipped with the L^2 metric is precisely the space $\mathcal{E}^2(M, \omega_0)$ of potentials of finite energy.

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Guedj's conjecture holds for all $p \geq 1$.

Note that the extension to $p = 1$ is crucial. Moreover, the convexity of K energy can also be extended to $\mathcal{E}^1(M, \omega_0)$ space as well.

Fundamental work in $\mathcal{H}$ II

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Theorem

[Berman-Boucksom-Eyssidieux-Guedj-Zeriahi, Berman-Darvas-Lu]
Let $\{u_i\}_i \subset \mathcal{E}^1$ be a sequence for which the following condition holds:

$$\sup_i d_1(0, u_i) < \infty, \ \sup_i E(u_i) < \infty.$$

Then $\{u_i\}_i$ contains a d_1-convergent subsequence.

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Note that if K energy is bounded from above and if the K energy is proper, this automatically implies the d_1 distance is bounded.

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where

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and

$$C_t = (1-t)\underline{\chi} - t\underline{R}.$$

On the New Continuity Path II

Theorem (X. X. Chen 2015)

For any $\chi > 0$ *and* $t \in (0, 1)$*, if there exists one solution to Equation (2) for time* $t \in (0, 1)$*, then there exists a small* $\delta > 0$ *such that for any* $t' \in (t - \delta, t + \delta)$*, there exists a solution to Equation (2) for time* t'*.*

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Theorem (Y. Zeng, Y. Hashimoto)

For any $\chi > 0$, if there exists one solution to Equation (2) for time $t = 0$, then there exists a small $\delta > 0$ such that for any $t' \in [0, \delta)$, there exists a solution to Equation (2) for time t'.

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Theorem (Chen-Paun-Zeng)

For any $\chi > 0$, if there exists one solution to Equation (2) for time $t = 1$, then there exists a small $\delta > 0$ such that for any $t' \in (1 - \delta, 1]$, there exists a solution to Equation (2) for time t'.

a Priori Estimates of Calabi, Yau

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- S. T. Yau, 1976, C^0 implies C^2 and C^0 estimate holds for $C_1 < 0$ and $C_1 = 0$.

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- S. T. Yau, 1976, C^0 implies C^2 and C^0 estimate holds for $C_1 < 0$ and $C_1 = 0$.
- S. T. Yau solve the famous Calabi Conjecture and won Fields Medal.

What about constant scalar curvature Kähler metrics

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- the first problem prevent us from adopting the celebrated work of S. T. Yau on Calabi conjecture where Maximum principle is crucial.
- The second problem prevent us from applying the Cheeger-Colding theory as in Chen-Donaldson-Sun theorem on the stability conjecture which goes back to Yau.

A Coupled Equations

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Proposition

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Conjecture (Chen 2010)

Suppose (M, ω_φ) is a constant scalar curvature Kähler metric and M compact. If $|\varphi| < C$, then any higher derivative estimate of φ is also uniformly bounded.

Core estimates I

The Key a Priori Estimates

Theorem

Suppose (M, ω_φ) is a constant scalar curvature Kähler metric. Then the following statements are mutually equivalent:

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- *There is a constant C such that $\frac{1}{C} < \frac{\omega_\varphi^n}{\omega^n} < C$;*

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Core estimates I

The Key a Priori Estimates

Theorem

Suppose (M, ω_φ) is a constant scalar curvature Kähler metric. Then the following statements are mutually equivalent:

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Theorem

(Chen-He 2010) Suppose φ is a solution of

$$\log\det(g_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}}) = F + \log\det(g_{\alpha\bar{\beta}})$$

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Theorem

Let φ *be a smooth solution to (3), (4), then for any* $1 < p < \infty$, *there exists a constant* $\alpha(p) > 0$, *depending only on* p, *and another constant* C, *depending only on* $||\varphi||_0$, *the background metric* g, *and* p, *such that*

$$\int_M e^{-\alpha(p)F}(n + \Delta\varphi)^p \le C. \tag{5}$$

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Let φ be a smooth solution to (3), (4), then for any $1 < p < \infty$, there exists a constant C, depending only on the background Kähler metric (M, g), an upper bound of $\int_M e^F F dvol_g$, and p, such that

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Theorem

Let φ be a smooth solution to (3), (4). Then there exists $p_n > 1$, depending only on n, and a constant C, depending on $||\varphi||_0$, $||F||_0$, $||n + \Delta\varphi||_{L^{p_n}(dvol_g)}$, and the background metric g, such that

$$n + \Delta\varphi \leq C. \tag{7}$$

The a priori estimate II

Our Main Compactness theorem:

Theorem

The set of Kähler potentials(suitably normalized up to a constant) with bounded scalar curvature and entropy (or geodesic distance) is bounded in $W^{4,p}$ for any $p < \infty$, hence precompact in $C^{3,\alpha}$ for any $0 < \alpha < 1$.

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The Calabi flow can be extended as long as the scalar curvature is uniformly bounded.

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Theorem

The Calabi flow can be extended as long as the scalar curvature is uniformly bounded.

Conjecture

(Calabi, Chen) Initiating from any smooth Kähler potential, the Calabi flow always exists globally.

Geodesic stability

Definition

Let $\rho(t) : [0, \infty) \to \mathcal{E}_0^1$ be a locally finite energy geodesic ray with unit speed. One can define an invariant $\yen([\rho])$ as

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Definition

Let $\varphi_0 \in \mathcal{E}_0^1$ with $K(\varphi_0) < \infty$, $(M, [\omega])$ is called geodesic stable at φ_0(resp. geodesic-semistable) if for all locally finite energy geodesic ray initiating from φ_0, their $\yen$ invariant is always strictly positive(resp. nonnegative). $(M, [\omega])$ is called geodesic stable(resp. geodesic semistable) if it is geodesic stable(resp. geodesic semistable) at any $\varphi \in \mathcal{E}_0^1$.

Main theorems I

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Alternatively,

Theorem

Suppose $Aut_0(M, J) = 0$*. Then* $(M, [\omega])$ *admits a cscK metric if and only if it is geodesic stable.*

Main theorem II

Definition

We say the K-energy is proper with respect to L^1 geodesic distance if for any sequence $\{\varphi_i\}_{i\geq 1} \subset \mathcal{H}_0$, $\lim_{i\to\infty} d_1(0,\varphi_i) = \infty$ implies $\lim_{i\to\infty} K(\varphi_i) = \infty$.

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Theorem (Properness Conjecture)

The existence of constant scalar curvature Kähler metric is equivalent to the properness of K-energy in terms of the L^1 geodesic distance.

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Theorem (One version of YTD conjecture)

In Toric Variety, the existence of constant scalar curvature Kähler metrics is equivalent to the uniform stability.

Calabi Dream Manifolds

Definition

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Is the Moduli space of Calabi Dream manifolds necessary smooth?

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What about manifold without constant scalar curvature Kähler metrics?