Localizing the Donaldson–Futaki invariant

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Theorem (Ding–Tian (Fano case) & Donaldson (polarized case $\omega \in c_1(L)$))

Let $(\mathcal{M}, \mathcal{L})$ be a polarized test configuration over (M, L) with irreducible central fiber $(M_0, \mathcal{L}_0)$ then if there is a cscK metric in $c_1(L)$ we have

$$\mathit{Fut}_{(M_0, \mathcal{L}_0)}(V) \geq 0.$$

Here V is the vector field induced by the $\mathbb{S}^1$–action (see above).

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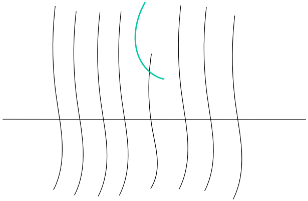

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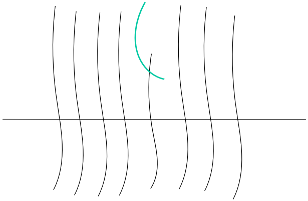

The central fiber M_0 is NOT irreductible.

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- $(\mathcal{M}, \Omega_c)$ is the clutching construction.

Few words on the proof of Ding–Tian.

- Ding–Tian proved that when M_0 has orbifold type singularities, using the embedding in $\mathbb{P}^N$ given by the polarization $L = -K_M > 0$. The test configuration $\mathcal{M}$ is the completion of an orbit of a $\mathbb{C}^*$ action on $\mathbb{P}^N$.

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- Then used Bando–Mabuchi's result : if there exists a Kähler–Einstein metric in $[\omega] = c_1(-K_M)$ then E_ω is bounded below.

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- Donaldson introduced a numerical algebraic invariant, denoted $DF(\mathcal{M}, \mathcal{L})$, on $\mathbb{C}^*$–linearized line bundle $(\mathcal{M}, \mathcal{L})$ taking into account the asymptotic of the weight on the induced $\mathbb{C}^*$–action on $H^0(\mathcal{M}, \mathcal{L})$,

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(M, L) is K–semistable if $DF(\mathcal{M}, \mathcal{L}) \geq 0$ for all test configurations* over it.

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Dervan–Ross, Sjöstrom-Dyrefelt extends the theory to Kähler (non-polarized) varieties and proved that cscK implies K-semistability.

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Recall that given a $\mathbb{S}^1$ action on a manifold M, generated by a vector field $V \in \Gamma(TM)$. A $\mathbb{S}^1$–invariant form $\alpha \in \Omega^*(TM)$ is equivariantly closed if

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By Atiyah–Bott, Berline–Vergne theory, if α is an equivariantly closed the

$$\int_M \alpha = \sum_Z \int_Z \frac{\iota_Z^* \alpha}{\chi_{\mathbb{S}^1}(E_Z^M)}$$

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$$\chi_V(E_Z^M) = (2\pi)^{\mathsf{rank}(E_Z^M)} \prod_j (c_1(L_j) - w_j).$$

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- Write the Donaldson–Futaki invariant as an intersection of equivariantly closed forms $\alpha_\Omega := (\Omega - \mu)$ and

$$\beta_\Omega = \frac{nc}{n+1}(\Omega - \mu) - \left(\rho^\Omega - \frac{1}{2}\Delta^\Omega \mu\right) + (\pi^* \omega_{FS} - \pi^* \mu_{FS})$$

  where μ_{FS} is a Hamiltonian for the standard S^1 action on $(\mathbb{P}^1, \omega_{FS})$

$$DF(\mathcal{M}, \Omega) = [\alpha_\Omega]^n \cup [\beta_\Omega]([\mathcal{M}])$$

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- Thus the Donaldson–Futaki only sees the fixed point sets lying in the central fiber and we get

$$\begin{aligned}\frac{DF(\mathcal{M},\Omega)}{n!} =& \sum_Z \int_Z \frac{nc_{[\omega]}(\Omega_Z-\mu_Z)^{n+1}}{(n+1)!\chi(E_Z^{\mathcal{M}})(V)} \\ & - \sum_Z \int_Z \frac{(\rho_Z^\Omega + \langle w, V\rangle)\wedge(\Omega_Z-\mu_Z)^n}{n!\chi(E_Z^{\mathcal{M}})(V)} \\ & + \sum_Z \int_Z \frac{(\Omega_Z-\mu_Z)^n}{n!\chi(E_Z^{\mathcal{M}})(V)}.\end{aligned}$$

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  where w is the sum, with multiplicities, of the weights of $\mathbb{S}^1$-action on E_Z^M.
- The remaining is essentially to prove that $\chi(E_Z^{\mathcal{M}}) = \chi(E_Z^{M_0})/2\pi$ and use the localization "backward" (inside M_0).