G_2-monopoles

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Riemannian Holonomy

- (M,g) Riemannian $\to$ parallel transport of tangent vectors along paths.
- $p \in M$ and γ_p a loop, the parallel transport $P(\gamma_p) : T_pM \to T_pM$ is ortogonal
$$\mathrm{Hol}_p(M) \subset O(T_pM).$$
What are the possible $\mathrm{Hol} \subset O(n)$?
- 1926: Cartan classified symmetric spaces.
- 1953: Berger found restrictions on the remaining Hol.
- If (M,g) is simply connected, irreducible and non-symmetric. Then,

Hol	n=dim(X)	Name
SO(n)	n	Orientable manifold
U(k)	2k	Kähler manifold
SU(k)	2k	Calabi–Yau manifold
Sp(k)·Sp(1)	4k	Quaternion-Kähler manifold
Sp(k)	4k	Hyperkähler manifold
G2	7	G_2-manifold
Spin(7)	8	Spin(7) manifold

- Except for G_2 and $\mathrm{Spin}(7)$ they all appear in infinite families.

Stage

- (M^7, g_φ) complete noncompact Riemannian manifold with holonomy G_2.
- g_φ is Ricci-flat: It has only one end (Gromoll splitting theorem) and
$$r \lesssim \mathrm{Vol}(B_r(x_0)) \lesssim r^7, \quad \text{for } r \gg 1,$$
(Bishop-Gromov comparison and Yau).
- g_φ is determined by a 3-form φ satisfying
$$d\varphi = 0 = d * \varphi.$$
- $N^4 \subset M$ is coassociative if $*\varphi|_N = \mathrm{vol}_N$, equivalently calibrated w.r.t. $*\varphi$.
- (Joyce and Donaldson–Segal) Can one count coassociatives? Possibly related to a count of G_2-monopoles (perhaps easier to define)!

"This subsection is rather more speculative."

- We have now further evidence towards the program outlined in that subsection.

G_2-monopoles

- G a compact Lie group and $P \to M$ a principal G-bundle.
- A pair (∇, Φ) with ∇ a connection on P and $\Phi \in \Omega^0(X, \mathfrak{g}_P)$ such that
$$*\nabla\Phi = F_\nabla \wedge *\varphi,$$
is called a G_2*-monopole*.
- Observe that
$$\Delta_\nabla \Phi = - * d_\nabla * \nabla\Phi = - * d_\nabla(F_\nabla \wedge *\varphi) = 0,$$
as $d_\nabla F_\nabla = 0$ (Bianchi) and $d * \varphi = 0$.
- Then,
$$\Delta \frac{|\Phi|^2}{2} = \langle \Phi, \Delta_\nabla \Phi \rangle - |\nabla\Phi|^2 = -|\nabla\Phi|^2 \leq 0$$
and if M was to be compact and Φ smooth, then $|\Phi| = cst. \implies$ $\nabla\Phi = 0$, and
$$0 = F_\nabla \wedge *\varphi,$$
i.e. ∇ would be a G_2-instanton.

Intermediate energy

- The intermediate energy of a pair (∇, Φ) is the quantity

$$\mathcal{E}_M(\nabla, \Phi) = \frac{1}{2} \int_M |\nabla \Phi|^2 + |F_\nabla \wedge *\varphi|^2.$$

- Over an open set $U \subset M$ it may be rewritten as

$$\mathcal{E}_U(\nabla, \Phi) = \int_{\partial U} \langle \Phi, F_\nabla \rangle \wedge *\varphi + \frac{1}{2} \| * \nabla \Phi - F_\nabla \wedge *\varphi \|^2_{L^2(U)}.$$

- The pair (∇, Φ) has *finite mass* if

$$m := \lim_{\mathrm{dist}(x, x_0) \to \infty} |\Phi(x)| > 0,$$

is well defined and constant. In this situation, and if (M, g_φ) has maximal volume growth: (1) The integration by parts can be carried out globally; and (2) The first term in $\mathcal{E}_M$ is topological.

$$\Longrightarrow \quad \mathrm{G}_2\text{-monopoles minimize } \mathcal{E}_M.$$

Relation with coassociatives (when $\mathrm{G} = \mathrm{SU}(2)$)

- Maximal volume growth: Let ∂M_∞ be the link of the cone to which (X, g_φ) is asymptotic to, and L the cx. line bundle over ∂M_∞ to which (∇, Φ) reduces at infinity. Then,

$$\mathcal{E}_M = 4\pi m \, \langle \alpha \cup [*\varphi|_{\partial M_\infty}], [\partial M_\infty] \rangle + \frac{1}{2} \| F_\nabla \wedge *\varphi - *\nabla\Phi \|^2_{L^2},$$

  with $\alpha = c_1(L) \in H^2(\partial M_\infty, \mathbb{Z})$ is called the *monopole class* (or charge).
- As $m \to +\infty$, we expect G_2-monopoles with monopole class α to concentrated on compact coassociatives $\{N_l\}_l$, with

$$\sum n_l Pd[N_l] = i(\alpha) \in H^3_{cs}(M, \mathbb{Z}),$$

  where

$$\ldots \to H^2(\partial M_\infty, \mathbb{Z}) \xrightarrow{i} H^3_{cs}(X, \mathbb{Z}) \xrightarrow{j} H^3(X, \mathbb{Z}) \to \ldots \ .$$

- The putative monopole invariant W_α may be recast from local data around the $\{N_l\}_l$, say $w(n_l, N_l) =$ *a count of Fueter sections*, and

$$W_\alpha \sim \sum w(n_l, N_l).$$

- (Joyce) A similar story for special Lagrangians in Calabi–Yau 3-folds.

Evidence

- $\exists$ two (M, g_φ) containing a unique compact coassociative N. Consider

$$\mathcal{M}_{inv} = \{\text{finite mass, invariant, irreducible monopoles}\}/\mathcal{G}_{inv}.$$

Theorem (–)

For all $(\nabla, \Phi) \in \mathcal{M}_{inv}$, $\Phi^{-1}(0) = N$ is the unique coassociative submanifold, and the mass gives a bijection

$$m : \mathcal{M}_{inv} \to \mathbb{R}^+.$$

Furthermore, if $\{(\nabla_m, \Phi_m)\}_{m \in [\Lambda, +\infty)} \in \mathcal{M}_{inv}$ with masses $m \nearrow +\infty$, then:

1. *After rescaling, a BPS-monopole on $\mathbb{R}^3$ bubbles off transversely to N.*
2. *A translated sequence converges to a reducible monopole away from N.*
3. $m^{-1} e(\nabla_m, \Phi_m) \rightharpoonup 4\pi\delta_N + e_\infty$.

- Are these features general phenomena? (joint work with Daniel Fadel)
- Also consider (M, g_φ) with no compact coassociative submanifold N.

When do the hypothesis hold? (for $\mathrm{G} = \mathrm{SU}(2)$)

- Can one replace the hypothesis that (∇, Φ) has finite mass by the more natural hypothesis of finite intermediate energy?
- Consider polynomial volume growth: $\mathrm{Vol}(B_r(x_0)) \sim r^l$, for $l \in [1, 7]$.

Theorem (Daniel Fadel, Ákos Nagy , –)

Suppose $l > 7/2$, (∇, Φ) has finite intermediate energy, and F_∇^{14} is bounded. Then, (∇, Φ) has finite mass.

Theorem (Daniel Fadel, Ákos Nagy , –)

Suppose $l = 7$, (∇, Φ) has finite intermediate energy, and $|F_\nabla^{14}|$ decays. Then,

1. $|\nabla\Phi| = O(r^{-6})$ *and* $|[\Phi, \nabla\Phi]|$, $|[\Phi, F_\nabla]|$ *decay exponentially.*
2. $(\nabla, \Phi) \to (\nabla_\infty, \Phi_\infty)$ *with* ∇_∞ *pseudo HYM and* $\nabla_\infty \Phi_\infty = 0$.

Corollary (Daniel Fadel, Ákos Nagy , –)

When $l = 7$, there is a Fredholm setup describing the moduli space of finite intermediate energy G_2-monopoles with fixed monopole class.

(Doable open problem): Compute the index.

Some other open problems

- Monopoles on ALC manifolds (examples).
- Monopoles and coassociative fibrations.
- The Fueter equation (with the remaining adiabatic limit equation) for (charge $k \geq 1$) transverse monopoles can be cast into a 4-dimensional problem. This probably develops concentration-compactness phenomena associated with: non-compactness of the monopole moduli space $\mathcal{M}_k$; and holomorphic spheres in $\mathcal{M}_k$ (none for $k = 1, 2$ -> good news?) – related to work of Doan, Haydys, Taubes, Walpuski and others.
- Extension to compact manifolds with a fixed coassociative N -> Theory is associated with the pair (M, N) and (∇, Φ) required to have Dirac type singularities along N.
- Can one try to do the coassociative count directly using weights from the aforementioned 4 dimensional problem?

Thank You!