SYZ conjecture for Fermat hypersurfaces

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- SYZ is physically motivated, and admits many interpretations. The strong version assets that the SLag fibration exists globally; this would be much harder (*cf.* Joyce). Some people adopt much softer viewpoints (algebraic, symplectic, topological, mirror symmetry).

Weak SYZ conjecture

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- Nonarchimedean viewpoint (*cf.* Boucksom).

Weak SYZ conjecture

Fermat family:

$$X_s = \{Z_0 Z_1 \dots Z_{n+1} + e^{-s} \sum_0^{n+1} Z_i^{n+2} = 0\}, s \gg 1.$$

The 'correct' normalisation on the Kähler class is $[\omega_s] = s^{-1}\mathcal{O}(1)|_{X_s}$.

Weak SYZ conjecture

Theorem

Weak SYZ holds for the Fermat family (at least subsequentially) as $s \to \infty$*.*

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Remark

We need the permutation symmetry on the Z_i variables to simplify the combinatorics. We expect the result can be generalised to many other families. The limit should be unique and passing to subsequence ought not be necessary, but that's for the future.

Nature of the Problem: collapsing metrics

Key feature: in the generic region of the CY manifold, the local complex structure is modelled on a large annulus region in $(\mathbb{C}^*)^n$, ie.

$$\{1 < |z_i| < \Lambda, \forall i\} \subset (\mathbb{C}^*)^n$$

This feature is easy to check and occurs frequently.

Nature of the Problem: collapsing metrics

Example

Consider smooth hypersurfaces $\{\sum a_m e^{\lambda(m)s} z^m = 0\}$ inside toric varieties of dimension $n+1$ with $s \gg 1$, where λ satisfies suitable convexity conditions. In the generic region, only two monomial terms dominate, so the local structure of the hypersurface is modelled on $(\mathbb{C}^*)^n$, with natural coordinates $s^{-1} \log z_i$. The reason this is not the whole $(\mathbb{C}^*)^n$ is that the monomials only dominate in some local regions.

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- If you analyze the local charts carefully, you will see tropical geometry appearing (*i.e.* the combinatorics of the logarithm map is encoded by piecewise linear objects). The Fermat family involves the least combinatorics.

Nature of the Problem: collapsing metrics

- CY metrics have an important dimensional reduction. Take a function ϕ on (a torus invariant subset of) $(\mathbb{C}^*)^n$, so $\phi = u \circ \mathrm{Log}$. Then ϕ is psh iff u is convex, and ϕ satisfies complex MA iff u satisfies real MA.

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  Metrics from this dim reduction are called semiflat, because the restriction to torus fibres are flat. The metrics on fibres can vary.

Nature of the problem: collapsing metrics

So really we want to prove that on local charts in the generic region the metric is C^∞ approximately

$$\omega_s \approx \sqrt{-1}/2 \sum \frac{\partial^2 u}{\partial x_i \partial x_j} s^{-1} d \log z^i \wedge s^{-1} d \overline{\log} z^j.$$

Here $x_i = s^{-1} \log |z_i|$. Notice this explains our scaling convention of the Kähler class: it is compatible with a finite diameter Gromov Hausdorff limit as $s \to \infty$.

The SLag fibration in such regions is more or less for free; it is a small deformation of the log map. The construction uses no more than McLean deformation theory; then you check the independence of the chart.

Proof ingredients

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- (Technical core) Near the $s \to \infty$ limit, there is a very strong tendency for any Kähler potential to be approximated by a convex potential, in the sense of a Skoda type estimate. With a little extra control on the volume density (eg some L^∞ bound suffices) this approximation holds in the C^0-sense uniformly in s, in the generic region.
  This uses pluripotential theory (cf. Kolodziej).

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  Given a psh function ϕ on an annulus in $(\mathbb{C}^*)^n$, we can average over T^n-fibres to obtain a function $\bar{\phi}$ on an open set in $\mathbb{R}^n$, which must be convex. Equivalently $\bar{\phi}$ is the zeroth Fourier coefficient function of ϕ. Intuitively, it is very unlikely for a psh function to be highly oscillatory, so ϕ should be close to $\bar{\phi}$.

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- The above is just the local picture. We need to globalize this by gluing approximately the convex functions on the local charts, to produce a global Kähler potential with 'global convexity property'. This new Kähler potential is thought as the regularisation of the original one. This gluing step requires tropical combinatorics, which is managable in the Fermat case.

Proof ingredients

- The Kolodziej pluripotential theory package in its usual form allows one to estimate the Kähler potential in L^∞ under extremely weak assumption on the volume density. In our proof a slight variant is used to estimate the difference of two Kähler potentials ('stability estimate'). The key advantage of Kolodziej's method is that it is very robust under the degeneration of the complex structure, and in some sense only improves under our degeneration.

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- The effect is that we can compare the CY Kähler potential with its regularisation and show the deviation is C^0-small at least in the good region.

Proof ingredients

- Techical detail: the Skoda inequality on a compact Kähler manifold (X, ω) means that for any $\omega - psh$ function u normalised to $\sup u = 0$, then

$$\int e^{-\alpha u} d\mu \leq C,$$

  with uniform positive constants α, C independent of u. This α is related to the α-invariant important in KE metrics.
- In our collapsing setting, we should replace integrals by average integrals. The robustness of the Kolodziej estimate roughly means the above two constants α, C are the only information you need about the Kähler manifold, and the amazing thing is that they can be chosen uniformly even when the complex structure is highly degenerate.

Proof ingredients

- Techical detail: when you compare the C^0-deviation of ϕ and $\bar{\phi}$, the easy direction is to bound $\phi - \bar{\phi}$ from above by $O(s^{-1/2})$ (because of mean value inequality). My improved Skoda inequality says

$$\int e^{-\alpha s^{1/2}(\phi - \bar{\phi})} d\mu \leq C,$$

meaning that $\phi - \bar{\phi}$ can only fail to be bounded below by $-O(s^{-1/2})$ with exponentially small probability (where $d\mu$ is the Calabi-Yau measure normalised to volume 1). It is quite striking that for our Fermat case, this holds for any (suitably normalised) Kähler potential ϕ without any assumption on its volume measure. In this sense 'Kähler potentials have a strong tendency to be approximated by convex functions', and 'Kolodziej's method only improves under our degeneration'.

Proof ingredients

- Now we know $\|\phi - \bar{\phi}\|_{C^0}$ is small for large s. Using convexity we easily get Lipschitz bounds on $\bar{\phi}$, so by Arzela-Ascoli we can extract a subsequential limit ϕ_∞ as $s \to \infty$; this is also the limit of the local psh function ϕ.
- Caveat: for different s the local potentials are a priori defined on different manifolds. Here we are talking about convergence in the preferred coordinate system from $s^{-1} \log z_i$.

Proof ingredients

- Argue that ϕ_∞ solves the real MA equation. Morally this is because the MA operator has weak continuity under C^0-convergence of the potential; here we need a little extra work because the manifolds are changing.

Proof ingredients

- Solutions of real MA automatically have a lot of regularity. For example it is C^∞ near any strictly convex point, and the complement of C^∞-locus has $(n-1)$-Hausdorff measure zero. Our generic region deletes the nonsmooth locus of ϕ_∞.

Proof ingredients

- How to bootstrap to C^2 and higher regularity?
  Savin's small perturbation theorem roughly says that for a large class of fully nonlinear 2nd order elliptic equation, including complex MA, if u is a smooth solution in B_2, and v is another (viscosity) solution with $\|u - v\|_{C^0} \ll 1$, then v has C^∞ bounds and $u - v$ is C^∞-small.

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- Savin's theorem is actually a difficult fully nonlinear result.
- We apply Savin to the local universal cover of the annuli in $(\mathbb{C}^*)^n$. The effect is that the high regularity of ϕ_∞ is transferred to the CY local potential ϕ for large s, and that's what we need for weak SYZ.