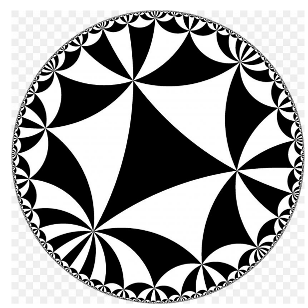
“Kähler-Einstein metrics, Archimedean Zeta functions and phase transitions”

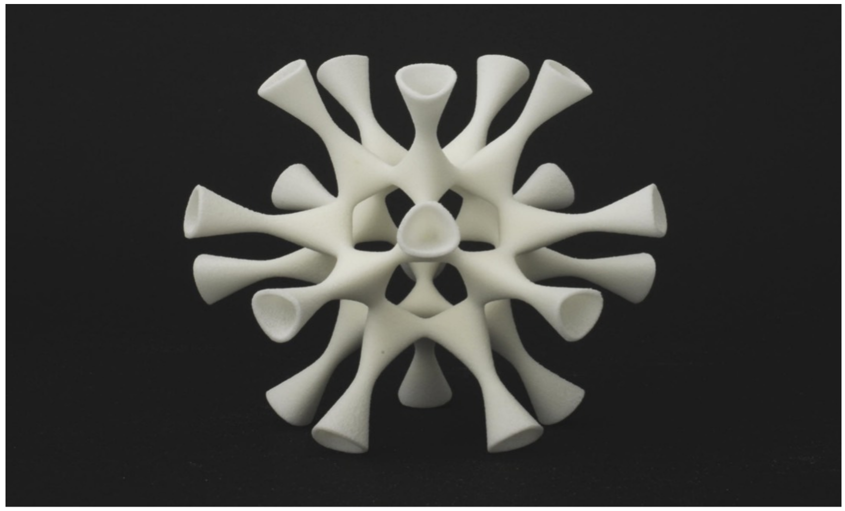
(*Robert Berman*, 2020)



Chalmers U. of Technology/U. of Gothenburg

For more details and references see

B. ”An invitation to Kähler-Einstein metrics and random point processes”. Surveys in Differential Geometry Vol. 23 (2020).



Motivation

Let X be a $n-$dim. complex projective algebraic variety (non-singular) and assume

$$K_X > 0$$

$K_X = \det(T^*X)$

(i.e. K_X ample). By the Aubin-Yau theorem X ('78) admits a unique Kähler-Einstein metric ω_{KE} with *negative* Ricci curvature:

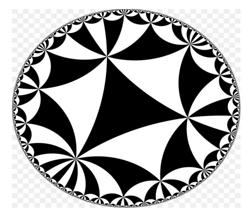
$$\mathrm{Ric}\,\omega_{KE} = -\omega_{KE}$$

But this is an *abstract* existence result.

Problem: find *explicit* formulas!

Bad news: even the case of a complex *curve* X is intractible...

Indeed, this problem is equivalent to finding an explicit *"uniformization map"* for X :



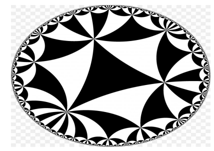
$$f: X \to \mathbb{H}/\Gamma, \quad \omega_{KE} = f^*\omega_{\mathbb{H}}$$

Special cases essentially appear in the classical works of Weierstrass, Riemann, Fuchs, Schwartz, Klein,Poincaré,...

(e.g. X is the classical modular curve, the Klein quartic, ...)

In these special cases the uniformization map

$$f : X \to \mathbb{H}/\Gamma$$



is expressed in terms of *periods*, i.e. integrals of the form

$$\int_\gamma \alpha,$$

where α is an *algebraic form* and γ is a real cycle:

$$f(x) = \frac{\int_{s\in\gamma_1} \alpha(x,s)}{\int_{s\in\gamma_2} \alpha(x,s)}.$$

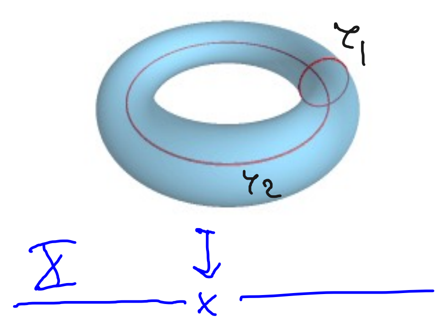
α is a relative top form on an "auxiliary" family

$$\mathcal{Y} \to X, \quad \text{fiber } \mathcal{Y}_x$$

Ex: for X the modular curve ($X \cong \mathbb{H}/SL(2,\mathbb{Z})$)

$$\mathcal{Y}_x = \text{elliptic curve},$$

$$\alpha(x,s) = \frac{ds}{\left(4s^3 - g_2(x)s - g_1(x)\right)^{1/2}} \text{ on } \mathcal{Y}_x\text{-}\{\infty\} \subsetneq \mathbb{C}^2_{s,t}$$

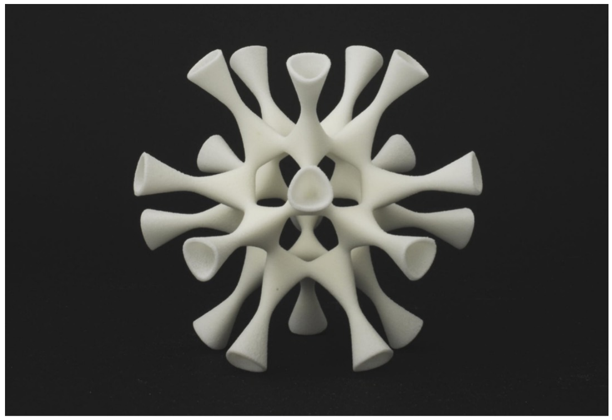


$\rightsquigarrow$ WP-metric on X

= KE metric

In higher dimensions, there are a few explicit uniformization results also using periods (Deligne-Mostow,...)

Relaxed problem: find canonical Kähler metrics ω_k approximating ω_{KE} such that ω_k is *explicitely* encoded by the *algebraic* structure of X.



More precisely: we would like the canonical approximation ω_k of ω_{KE} to be encoded by the canonical ring of X :

$$R(X) := \bigoplus_{k=0}^{\infty} H^0(X, kK_X)$$

$K_X^{\otimes k}$

i.e. ω_k should be encoded by $H^0(X, kK_X)$

(=the pluricanonical forms of “degree” k)

$\rightsquigarrow$Yau-Tian-Donaldson conjecture

Here will explain a probabilistic approach to KE-metrics, that leads to a canonical sequence of metrics ω_k approximating ω_{KE}.

- ω_k is expressed as a *period integral* over $X \times X \cdots \times X$ explicitely encoded by $H^0(X, kK_X)$

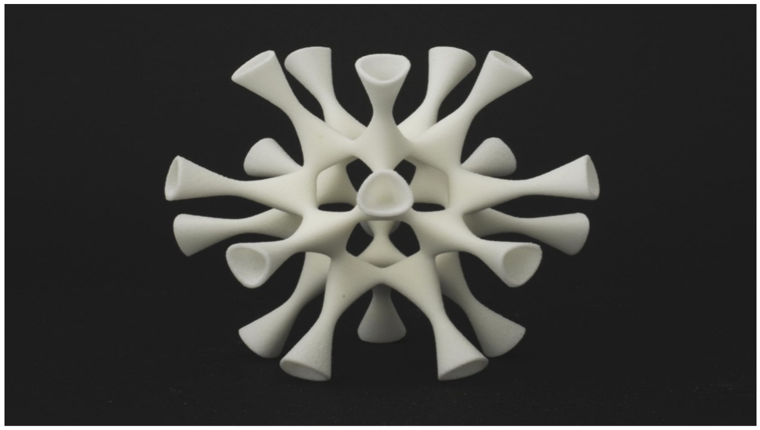
- In the case of Fano varieties $-K_X > 0$ (i.e. *positive* Ricci curvature) there are only partial results

The Fano case $\rightsquigarrow$ intruiging relations to

- *Yau-Tian-Donaldson conjecture*
- *Zeta functions*
- *The theory phase transitions (in statistical mechanics)*

$\rightsquigarrow$ A few new results ...


STAY
TUNED

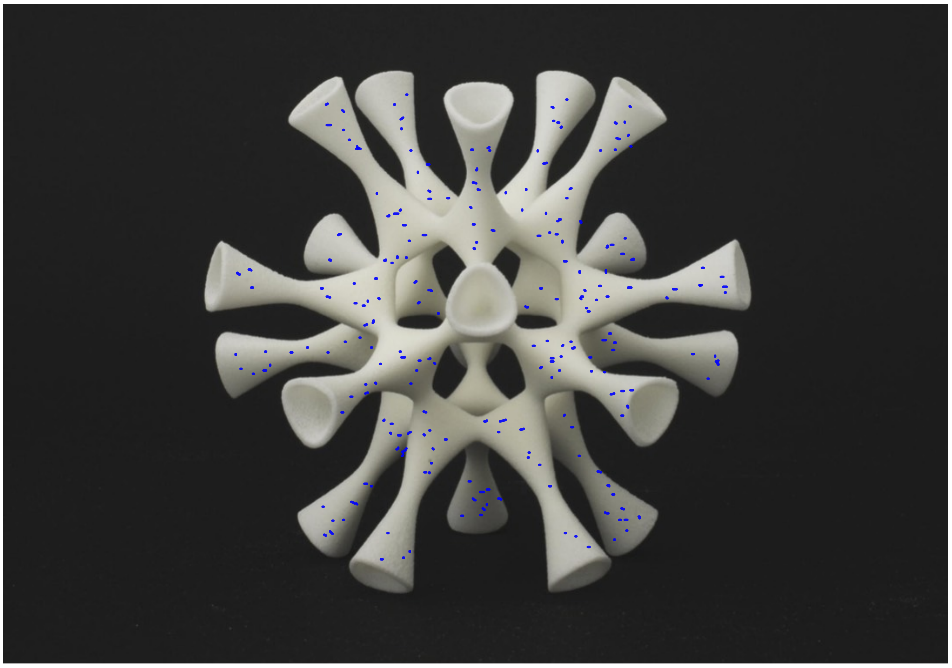


The probabilistic approach to KE metrics ($K_X > 0$)

The starting point is the basic fact that the Kähler-Einstein ω_{KE} on X can be recovered from its *volume form* dV_{KE} :

$$i\partial\bar{\partial}(\log dV_{KE}) = \omega_{KE}$$

It is thus enough to construct the canonical (normalized) *volume form* dV_{KE}.



To this end we will show that there is canonical way of chosing N points on X at random, so that we get equidistribution towards dV_{KE} (almost surely)

First need to define a canonical probability measure $d\mathbb{P}_N$ on X^N.

- It has to be *symmetric*

- We want it to be encoded by

$$R(X) = \bigoplus_{k=0}^{\infty} H^0(X, kK_X)$$

To this end take N to be the sequence

$$N_k := \dim H^0(X, kK_X)(\to \infty)$$

(="plurigenera")

Pick a basis $\alpha_1(x), \ldots, \alpha_{N_k}(x)$ in $H^0(X, kK_X)$ and define

$$\det(x_1, \ldots, x_{N_k}) := \alpha_1(x_1)\alpha_1(x_2)\cdots\alpha_{N_k}(x_{N_k}) \pm \ldots$$

completely antisymmetrized in $(x_1, \ldots, x_{N_k})$.

We then get an *algebraic form* α on X^{N_k} by defining

$$\alpha(x_1, \ldots, x_{N_k}) := \det(x_1, \ldots x_{N_k})^{1/k}$$

It is complex and multivalued, but

$$i\,\alpha(x_1, \ldots, x_N) \wedge \overline{\alpha(x_1, \ldots, x_N)}$$

defines an honest positive real top form on X^{N_k}, which is symmetric. Now define

$$d\mathbb{P}_{N_k} := \frac{1}{Z_{N_k}} i\,\alpha(x_1, \ldots, x_{N_k}) \wedge \overline{\alpha(x_1, \ldots, x_{N_k})},$$

where Z_{N_k} is the *normalizing constant.*

$$= \int_{X^{N}} \alpha \wedge \bar{\alpha}$$

But is this construction really canonical? In other words, is

$\varphi_1(x_1)\varphi_2(x_2)\cdots$

$$d\mathbb{P}_{N_k} := \frac{\left(\mathrm{det}(x_1,\ldots x_{N_k})^{1/k} \wedge \overline{\mathrm{det}(x_1,\ldots x_{N_k})^{1/k}}\right)}{Z_{N_k}},$$

independent of the choice of basis in $H^0(X, kK_X)$?

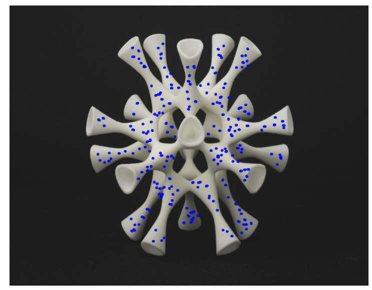
Yes! Under a change of basis $\mathrm{det}(x_1,\ldots x_{N_k}) \to C_{N_k}\mathrm{det}(x_1,\ldots x_{N_k})$.

So OK by homogeneity!

Main theorem [B. 2017]

Consider the random measure

$$\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i}$$



on X^{N_k} (endowed with $d\mathbb{P}_{N_k}$). As $N_k \to \infty$ it converges towards dV_{KE} in probability.

More precisely, $\forall \epsilon > 0$

$$\mathbb{P}_N\left(d\left(\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i}, dV_{KE}\right) > \epsilon\right) \leq e^{-C_\epsilon N}, \quad N \to \infty$$

In particular, consider the *expectations*

$$dV_k := \mathbb{E}(\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i})$$

which define a sequence of canonical *volume forms* dV_k on X.

The previous theorem implies that

$$dV_k \to dV_{KE}, \;\; k \to \infty$$

on X (weakly).

Back to periods

The canonical volume form dV_k on X is *explicitely* obtained as follows:

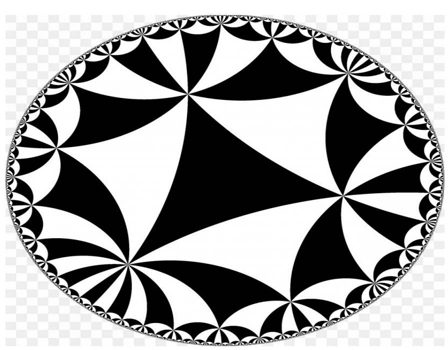
$$\begin{array}{ccc} (x_1, \ldots) & X^{N_k} & d\mathbb{P}_{N_k} \\ \downarrow & \downarrow & \downarrow \\ x_1 & X & dV_k \end{array}$$

$$dV_k = \int_{X^{N_k - 1}} d\mathbb{P}_{N_k}$$

$\alpha = \det^{\frac{1}{k}}$

$$= \frac{\int_{X^{N_k-1}} \alpha(x, \ldots, x_{N_k}) \wedge \overline{\alpha(x, \ldots, x_{N_k})}}{Z_{N_k}}$$

Thus, dV_k is indeed a quotient of two *periods*



One obtains a sequence of canonical Kähler metrics ω_k on X by setting

$$\omega_k := i\partial\bar{\partial}(\log dV_k)$$

(i.e. ω_k is the curvature form of the metric on K_X induced by dV_k).

The convergence

$$dV_k \to dV_{KE}, \quad k \to \infty$$

then implies that

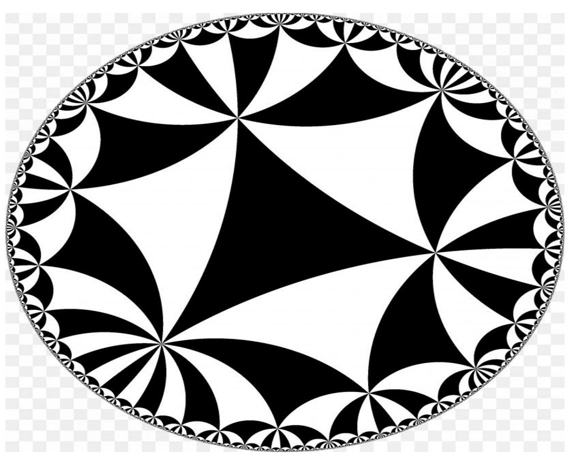
$$\omega_k \to \omega_{KE}, \quad k \to \infty \quad \text{(weakly)}$$

(using $i\partial\bar{\partial}(\log dV_{KE}) = \omega_{KE}$).

The canonical Kähler metric ω_k is *explicitely* given by

$$i\partial_x\bar{\partial}_x \log \int_{X^{N_k-1}} \alpha(x, x_2, ..., x_{N_k}) \wedge \overline{\alpha(x, x_2, ..., x_{N_k})}$$

By differentiating log this also becomes a quotient of two *periods.*



Fano varities

Now consider the "opposite case" where $-K_X > 0$, i.e. X is a Fano variety (non-singular).

Then a Kähler-Einstein metric ω_{KE} on X must have *positive* Ricci curvature:

$$\mathrm{Ric}\,\omega_{KE} = \omega_{KE}$$

However, there are *obstructions* to the existence of ω_{KE} :

YTD conjecture (/theorem) X admits a Kähler-Einstein metric ω_{KE} iff X is K-stable

(*recall*: this is a GIT-type stability condition).

The probabilistic approach when $-K_X > 0$

Recall: when $K_X > 0$ the probability measure on X^{N_k} is defined by

$$d\mathbb{P}_{N_k} := \frac{\left(\det(x_1, ...x_{N_k})^{1/k} \wedge \overline{\det(x_1, ...x_{N_k})^{1/k}}\right)}{Z_{N_k}},$$

where $\det(x_1, ...x_N) \in H(X, kK_X)^{\otimes N_k}$

- However, when $-K_X > 0$ the spaces $H^0(X, kK_X)$ are trivial!

- Instead, we need to work with the spaces $H^0(X, -kK_X)$

- But then we are forced to replace the power $1/k$ with $-1/k$

$\alpha = \det^{\frac{1}{k}}$

We thus set

$$N_k := \dim H^0(X, -kK_X) \longrightarrow \infty$$

and

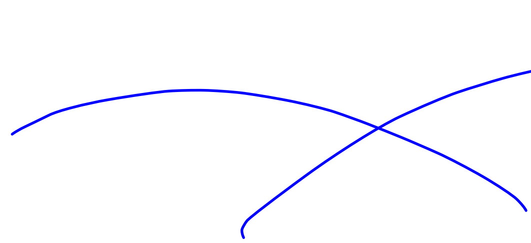
$$d\mathbb{P}_{N_k} = \frac{\left(\det(x_1, \ldots x_{N_k})^{-1/k} \wedge \overline{\det(x_1, \ldots x_{N_k})^{-1/k}}\right)}{Z_{N_k}}$$

$$= \frac{1/\alpha(x_1, \ldots, x_{N_k}) \wedge \overline{1/\alpha(x_1, \ldots, x_{N_k})}}{Z_{N_k}}$$

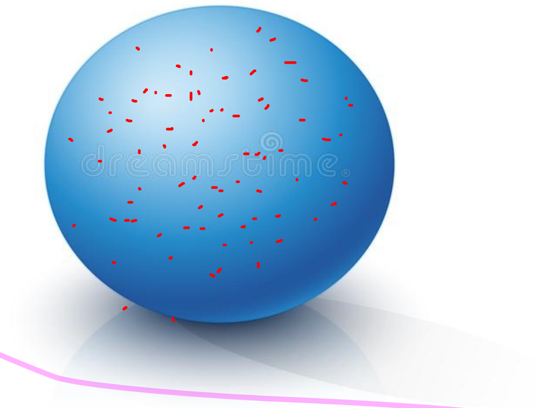
However, in this case it may be that

$$Z_{N_k} := \int_{X^{N_k}} 1/\alpha(x_1, \ldots, x_{N_k}) \wedge \overline{1/\alpha(x_1, \ldots, x_{N_k})} = \infty$$

Indeed, the integrand is singular along the divisor $\mathcal{D}_k$ in X^{N_k} cut out by $\alpha(x_1, \ldots, x_{N_k})$.



$\mathcal{D}_k \subset X^{N_k}$



Main conjecture:

- Assume that $Z_{N_k} < \infty$ for k large. Then X admits a unique KE-metric ω_{KE} and

$$\frac{1}{N_k}\sum_{i=1}^{N_k} \delta_{x_i} \to dV_{KE}, \quad N_k \to \infty$$

  in probability.

- Conversely, if X admits a unique KE-metric ω_{KE}, then $Z_{N_k} < \infty$ for k large.

The condition

$$Z_{N_k} := \int_{X^{N_k}} 1/\alpha(x_1, ..., x_{N_k}) \wedge \overline{1/\alpha(x_1, ..., x_{N_k})} < \infty$$

is of a purely algebraic nature:

let $\mathcal{D}_k$ be the anti-canonical $\mathbb{Q}-$divisor on X^{N_k} defined by

$$\mathcal{D}_k := \left\{(x_1, ..., x_{N_k}) \in X^{N_k} : \ \alpha(x_1, ..., x_{N_k}) = 0\right\}$$

$Z_{N_k} < \infty \iff \mathcal{D}_k$ has mild singularitities in the sense of birational geometry:

$$Z_{N_k} < \infty \iff (\mathcal{D}_k, X^{N_k}) \text{ is klt}$$

By definition this means that the *Log Canonical Threshold* satisfies

$$\text{lct}(\mathcal{D}_k, X^{N_k}) > 1$$

Indeed, to analytically define the lct of a divisor

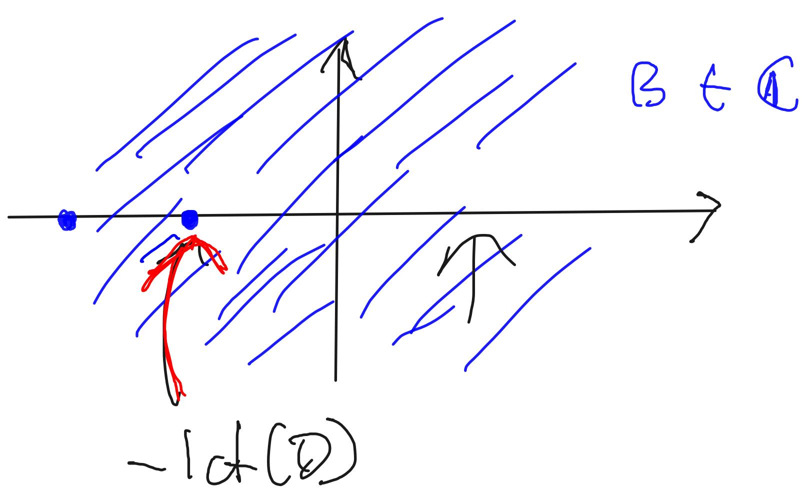
$$\mathcal{D} := \{\alpha = 0\}$$

one looks at the function

$$Z(\beta) := \int |\alpha|^{2\beta}\, dV, \quad \beta \in \mathbb{C}$$

This is a meromorphic function of β with poles in $]-\infty, 0[$:

first pole of $Z(\beta)$ at $\beta = -\mathrm{lct}\ (\mathcal{D})$

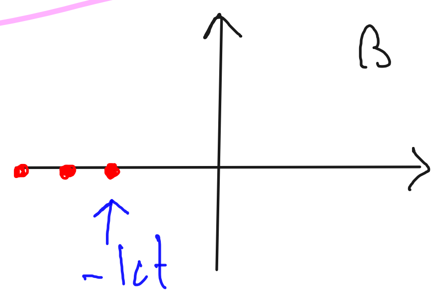


In fact, Atiyah and Gelfand-Bernstein showed in 1970 that

$$Z(\beta) := \int |\alpha|^{2\beta}\, dV, \quad \beta \in \mathbb{C}$$

has *isolated rational* poles on

$$]-\infty, 0[\subset \mathbb{C}$$



Such meromorphic functions $Z(\beta)$ are often called archimedean Zeta functions.

(non-archimedean $p-$adic version $Z_p(\beta) \rightsquigarrow$ "algebro-geometric" Zeta functions: the motivic, Hodge, topological zeta functions....)

$\alpha_k = \det^{\frac{1}{k}}$ on X^{N_k}

$D_k = \{ \alpha_k = 0 \}$

Here, fixing a volume form dV on X we can globally express

$$Z_{N_k}(\beta) := \int_{X^{N_k}} \|\alpha(x_1, ..., x_N)\|^{2\beta} dV^{\otimes N_k},$$

where $\|\cdot\|$ is the metric on $-K_{X^{N_k}}$ induced by dV.

- Hence, $Z_{N_k} := Z_{N_k}(-1)$.

By basic properties of log canonical thresholds:

$$\text{lct}(\mathcal{D}_k, X^{N_k}) > |\beta_0|$$

for some negative β_0, namely $\beta_0 = -\text{lct}(K_X)$ (Tian's $\alpha-$invariant).

This means that

$$Z_{N_k}(\beta) < \infty$$

for any $\beta > -\text{lct}(K_X)$. In fact, in this case one gets a *quantitative* estimate:

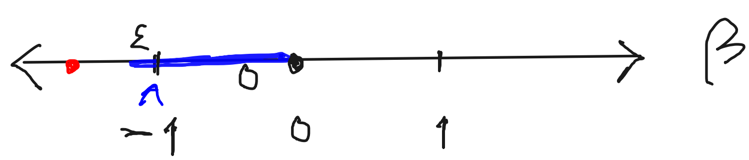
$$Z_{N_k}(\beta) \leq C^{N_k}, \quad (\text{if } \beta > -\text{lct}(K_X))$$

However, for a general Fano X such an estimate does not hold down to $\beta = -1$.

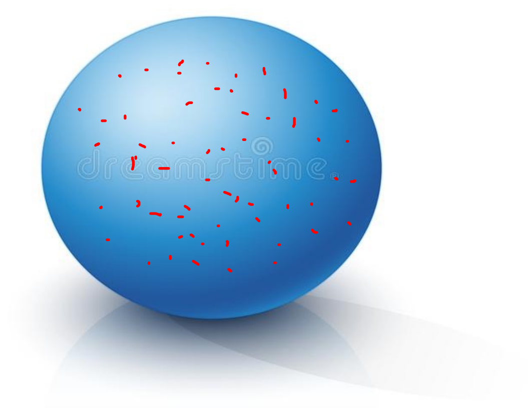
Thm 1 **(B. 2017)** Assume that there exists $\epsilon > 0$ such that

$$Z_{N_k}(\beta) \leq C^{N_k}$$

for all $\beta > -(1+\epsilon)$. Then X admits a unique KE-metric ω_{KE}.

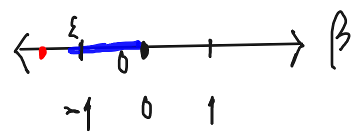


What about the (random) equidistribition towards dV_{KE} as $N_k \to \infty$?



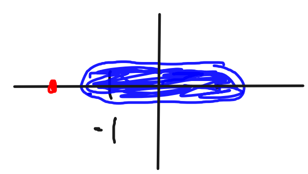
Thm 2 (B. 2020). Equidistribution towards dV_{KE} holds if there exists $\epsilon > 0$:

- $Z_{N_k}(\beta) \leq C^{N_k} \quad \forall \beta > -(1+\epsilon)$



and the following "*zero free hypothesis*" holds:

- $Z_{N_k}(\beta) \neq 0, \;$ on $[-1,0] + D_\epsilon,$



where D_ϵ is the disc of radius ϵ centered at $0 \in \mathbb{C}$

independent of N!

Stability

Recall that the probability measure $d\mathbb{P}_{N_k}$ on X^{N_k} is well-defined iff

$$\text{lct}\ (\mathcal{D}_k, X^{N_k}) > 1.$$

If this is the case for $k >> 1$, then the Fano variety X is called Gibbs stable.

There is also a stronger notion: if there exists $\epsilon > 0$:

$$\text{lct}(\mathcal{D}_k, X^{N_k}) > 1 + \epsilon, \ \ k >> 1$$

then X is called uniformly Gibbs stable.

Algebraic version of the conjecture (without convergence statement)

Let X be a Fano variety (possibly singular)

- X is Gibbs stable iff X is K-stable
- X is uniformly Gibbs stable iff X is uniformly K-stable

Theorem [Fujita-Odaka 2018, Fujita 2016]:

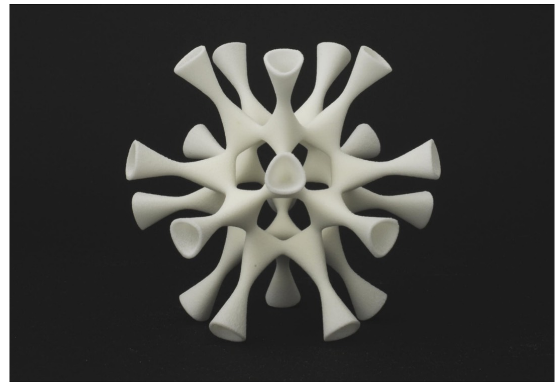
X uniformly Gibbs stable $\Longrightarrow$ X uniformly K-stable

The proof shows that

$$\liminf_{k\to\infty} \mathrm{lct}(\mathcal{D}_k, X^{N_k}) \le \delta(X) \quad (*)$$

where $\delta(X)$ is a valuative invariant of X (aka the "stability threshold" [Blum-Jonsson'20])

Recall: X is uniformly K-stable $\iff$ $\delta(X) > 1$. Hence, the main conjecture would follow from equality in (*).



Proof strategy $(K_X > 0)$

Fix a volume form dV on X. Then we can express

$$d\mathbb{P}_N = \frac{1}{Z_N}\|\alpha(x_1,...,x_N)\|^2 \, dV^{\otimes N} =$$

$$\frac{1}{Z_N(\beta)} e^{-\beta N E_N(x_1,...,x_N)} dV^{\otimes N} \quad \text{on } X^N$$

where

$$E_N(x_1,...,x_N) := -N^{-1}\log\|\alpha(x_1,...,x_N)\|^2, \quad \beta = 1$$

Statistical mechanics: this is the *equilibrium distribution of N interacting particles:*

$E_N(x_1,...,x_N) =$ energy/particle, $1/\beta =$ temperature

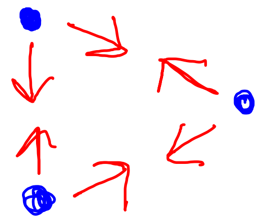
$d\mathbb{P}_N$ is called the *Gibbs measure* and $Z_N(\beta)$ the *partition function:*

Entropy!

Energy

$$d\mathbb{P}_N = \frac{1}{Z_N(\beta)} e^{-\beta N E_N(x_1,\dots,x_N)} dV^{\otimes N} \text{ on } X^N$$

n Energy - Entropy = Free energy



The general "free energy principle" "mean field"

Assume that

$$E_N(x_1,...,x_N) = E(\mu) + o(1), \quad \mu = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \in \mathcal{P}(X)$$

($E(\mu)$ is the *macroscopic energy*). Then one gets convergence in probability:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \to \mu_\beta$$

where μ_β is the minimizer of the "free energy" on $\mathcal{P}(X)$

$$F_\beta(\mu) := \beta E(\mu) - S_{dV}(\mu), \quad S_{dV}(\mu) = -\int_X \log\frac{\mu}{dV}\mu,$$

assuming that $F_\beta(\mu)$ has a unique minimizer on $\mathcal{P}(X)$:

$$\text{Free energy} = \beta\text{Energy-Entropy}$$

Here

$$E_N(x_1, ..., x_N) := -N^{-1} \log \|\alpha(x_1, ..., x_N)\|^2$$

is strongly *repulsive.*

Indeed,

$$\|\alpha(x_1, ..., x_N)\| := \|\det(x_1, ...x_N)\|^{1/k}$$

and $\det(x_1, ...x_N)$ vanishes when two points coincide.

Do we get

$$E_N(x_1, ..., x_N) = E(\mu) + o(1), \quad \mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$$

for some functional $E(\mu)$?

$$E_N(x_1, ..., x_N) := -N^{-1} \log \|\alpha(x_1, ..., x_N)\|^2$$

Step 1: The following approximation holds (wrt "$\Gamma-$convergence"):

$$E_N(x_1, ..., x_N) = E(\mu) + o(1), \quad \mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \in \mathcal{P}(X)$$

where $E(\mu)$ is the *pluricomplex energy.*

B. Boucksom - Witt ... ~ Fekete..

Step 2: The "free energy principle" applies.

Step 3: dV_{KE} is the *unique* minimizer of $F_\beta(\mu)$, $\beta = 1$.

$\parallel$

$\beta E(\mu) - S(\mu)$

In fact, F_1 is, essentially, *Mabuchi's* *K-energy functional*

Proof strategy when $-K_X > 0$

X Fano

In this case

$$d\mathbb{P}_N = \frac{1}{Z_N} e^{-\beta N E_N(x_1,\ldots,x_N)} dV^{\otimes N} \text{ on } X^N$$

where now $\beta = -1$, i.e. *negative* (absolute!) temperature.

Equivalently, can set $\beta = +1$ if

$$E_N(x_1, \ldots, x_N) \to -E_N(x_1, \ldots, x_N),$$

i.e. if the interaction energy is made *attractive.*

Formally, this makes no difference,....

...but the devil is in the details.

The case $-K_X > 0$ but with $\beta > 0$

$$d\mathbb{P}_{N,\beta} = \frac{1}{Z_N(\beta)} \|\alpha(x_1, ..., x_N)\|^{2\beta} dV^{\otimes N}$$

Then the previous proof gives

$$\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{x_i} \to dV_\beta, \quad N_k \to \infty,$$

where dV_β is the unique minimizer in $\mathcal{P}(X)$ of

$$F_\beta(\mu) := \beta E(\mu) - S(\mu), \quad S(\mu) = -\int_X \log \frac{\mu}{dV} \mu,$$

Concretely, writing

$$dV_\beta = \omega_\beta^n / V$$

the minimizing property gives

$$dV_\beta = \frac{\omega_\beta^n}{n!} \Big/ V$$

$$\text{Ric } \omega_\beta = -\beta\omega_\beta + (1+\beta)\text{Ric } \omega_0 \quad \text{on } X$$

This is *Aubin's continuity equation* with "time-parameter"

$$t := -\beta$$

Note that $\beta = -1$ gives the KE-equation on X!

If ω_{KE} exists, then

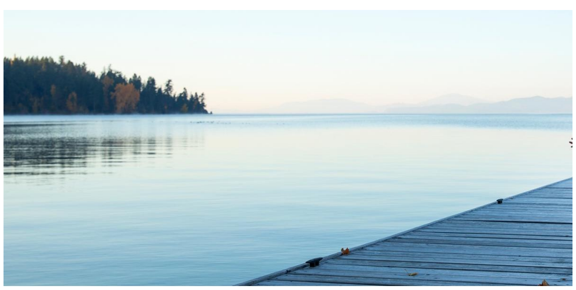
$$\beta \mapsto \omega_\beta, \quad \beta \in [-1, \infty[$$

is a *real-analytic curve* and $\omega_{-1} = \omega_{KE}$

BUT, when $N \to \infty$ the theorem only gives convergence towards ω_β when $\beta > 0$.

What about "analytic continuation"?

However, in physical terms, there could be a *"phase transition"* as the sign of β is switched and β decreases towards -1.



But a *phase transition is ruled out* by the zero-free hypothesis in Thm 2.

Indeed, one can then do “analytic continuation” from $\beta > 0$ to $\beta = -1$.

$$f_N(\beta) := \left(Z_N(\beta)\right)^{\frac{1}{N}} \leq C$$

is holomorphic and bounded

$$\beta > 0 \Rightarrow f_N(\beta) \longrightarrow f(\beta)$$ real-analytic in β

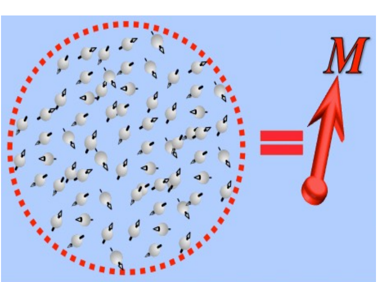
$$\overset{?}{\rightsquigarrow} \beta < 0$$

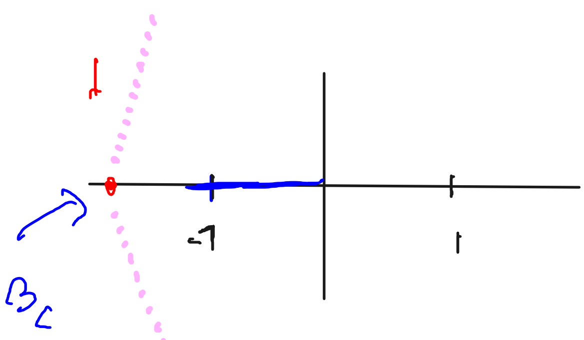
The "zero-free hypothesis" is the analog of the *"Lee-Yang property"* in physics, which rules out phase transitions by controlling the zeros of the *partition function* Z_N.

- The zeros of $\beta \mapsto Z_N(\beta)$ are usually called "Fisher zeroes"

- The "zero-free property" of $Z_N(\beta)$ is known to hold for *spin systems* iff $|\beta| < |\beta_c|$ (the "critical" inverse temperature)

- $T_c := 1/\beta_c$ is the temperature where spontanous magnetization arises).



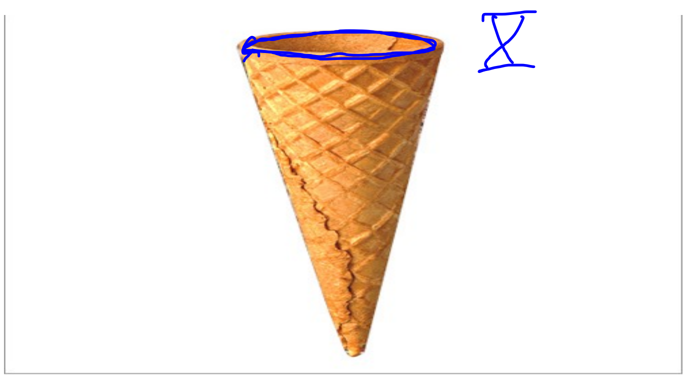


Connections to quantum gravity in:

“Emergent Sasaki-Einstein geometry and AdS/CFT”

joint with Tristan Collins and Daniel Persson (ArXiv)

X Fano appears as the base of a *Calabi-Yau cone*



Thank you!