Lag Chow (w I. Smith)

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$$\begin{tikzcd} K(D^b\,Coh(Y))_{\mathbb{Q}} \arrow[r, phantom, "\simeq"] & K(D^{\pi}Fuk(X))_{\mathbb{Q}} \\ K(Y)_{\mathbb{Q}} \arrow[u, phantom, "\cong"] \arrow[d, "s|ch"] & Cob^{un}(X) \arrow[u, "\eta"] \arrow[dl, dashed, "??"] \\ CH(Y)_{\mathbb{Q}} & \end{tikzcd}$$

Defn: $CH_k(Y) := \mathbb{Z}\langle V \underset{alg.}{\subset} Y\rangle/\sim$, where
k-dim subvar.

$\sim$ is gen'd by $V(0)\sim V(\infty)$ for any
k+1-dim
$V \subset Y\times\mathbb{P}^1$ s.t. $V \to \mathbb{P}^1$ dominant, where
$V(z) := V \cap Y\times\{z\}$.

There's a homomorphism $CH_k(Y) \xrightarrow{cyc.} H_{2k}(Y)$
'cycle class map' $V \longmapsto [V]$
whose image lies in $PD(H^{n-k,n-k}(Y))$, and whose kernel is denoted $CH_k(Y)_{hom}$.

E.g. Y = curve, $CH_1(Y) \underset{\sim}{\xrightarrow{cyc.}} H_2(Y) = \mathbb{Z}$

$CH_0(Y) \xrightarrow{cyc.} H_0(Y) = \mathbb{Z}$

$CH_0(Y)_{hom} \underset{\sim}{\xrightarrow{AJ}} Jac(Y)$.

$\Rightarrow CH(Y) \simeq \mathbb{Z}^2 \oplus Jac(Y)$

E.g. $Y = K3$, $CH_2 \underset{\sim}{\longrightarrow} H_4(Y)$

$CH_1 \xrightarrow{\sim} Pic(Y) \subset H_2(Y)$

CH_0 = "∞-dim'l" (Mumford)

(i.e. $Y^{(d)} \times Y^{(d)} \longrightarrow CH_0(Y)_{hom}$

$(y_i) \times (y_i') \longmapsto \Sigma y_i - \Sigma y_i'$

not surj. for any d)

'unobstructed'

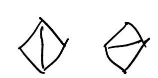
Defn: $Cob^{\substack{un.\\ or}}(X) := \mathbb{Z}\langle L \subset X \text{ un. or. Lag.}\rangle/\sim$

where $\sim$ gen'd by $\Sigma_i L_i \sim 0$ for any $V \subset X\times\mathbb{C}$ un. or. Lag., $V = \bigcup_i L_i \times \{\arg = \theta_i\}$ outside compact set:

There's a homomorphism $Cob^{or}(X) \xrightarrow{cyc.} H_n(X)$

$L \longmapsto [L]$

mirror to cycle class map.

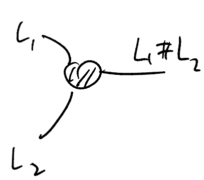


Lem: $L \sim \varphi(L)$

φ Ham. isot.

($L \simeq \varphi(L)$ in $Fuk(X)$)

Lem: $L_1 \# L_2 \sim L_1 + L_2$



Lag. surgery (Polterovich)

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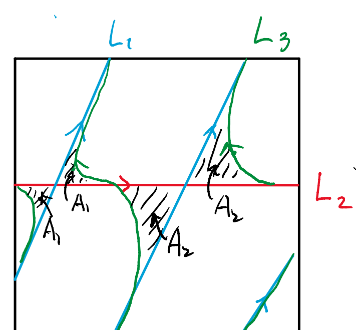
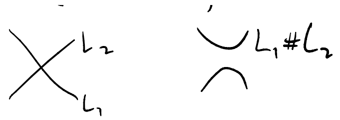
E.g. $X = T^2$

$Cob^{or}(X) \xrightarrow{cyc.} H_1(T^2) \cong \mathbb{Z}^2$

(cf. $CH(E) \xrightarrow{cyc.} H_0(E)\oplus H_2(E) \cong \mathbb{Z}^2$)



$L_3 = L_1 \# L_2$



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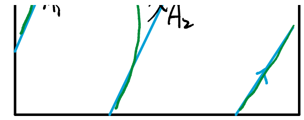
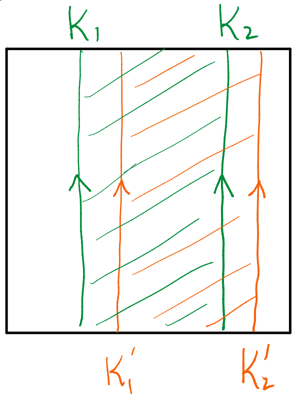


$(L_1 \# L_2 \simeq \mathrm{Cone}(L_1 \longrightarrow L_2)$ in $\mathrm{Fuk}(X)$



$K_1 + K_2 = L_3$

$= L_1 - L_2$ (indep't of A_i)

$= L_3'$ (change A_i)

$= K_1' + K_2'$

$\mathrm{Area}(///) = \mathrm{Area}(///)$

Lem: If $u \in H_2(V, \partial V)$, then
(V: cobordism)

$$\int_{\pi_X \circ u} \omega = 0$$

Cor: $\exists$ homomorphism (in fact, an iso)

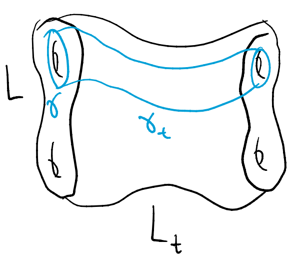
$$\mathrm{Cob}^{or}(T^2)_{hom} \longrightarrow \mathbb{R}/\mathbb{Z}\cdot \mathrm{area}(T^2)$$

$$\partial V \longmapsto \omega(V),$$

(cf. $CH(E)_{hom} \xrightarrow{AJ} E \xrightarrow{\mathrm{proj}_{syz}} S^1$).

Lem: If $L \subset X$ Lag., then

$\{\text{def. of } L\}/_{\text{Ham. isot.}} \xrightarrow[\text{`flux'}]{\sim} \mathcal{U}_L =$ nbhd of $0 \in H^1(L)$



$\{L_t\} \longmapsto \{\gamma \mapsto \int_{\gamma_t} \omega\}$.

Def: $f_L : \mathcal{U}_L \longrightarrow \mathrm{Cob}(X)$

$\alpha \longmapsto L(\alpha)$

If $\mathbb{L} = \{L_i\}_{i=1,\dots,N}$

$f_{\mathbb{L}} : \mathcal{U}_{\mathbb{L}} \longrightarrow \mathrm{Cob}(X)$

$(\alpha_i) \longmapsto \sum_i L_i(\alpha_i)$.

$\prod_i \mathcal{U}_{L_i} \subset H^1(\mathbb{L}) := H^1(\coprod_i L_i)$.

Lem: Suppose $\dim_{\mathbb{R}}(X) = 4$, $\mathbb{L}^{\pm} := \{L_1^{\pm}, \dots, L_{N^{\pm}}^{\pm}\}$.

Then

$\{(\alpha^-, \alpha^+) \in \mathcal{U}_{\mathbb{L}^-} \times \mathcal{U}_{\mathbb{L}^+} : f_{\mathbb{L}^-}(\alpha^-) = f_{\mathbb{L}^+}(\alpha^+)\}$

$= \bigcup_V Z_V$

where the union is <u>countable</u>, and each $Z_V \subset H^1(\mathbb{L}^- \sqcup \mathbb{L}^+)$ is an open subset of an affine subspace which is Lagrangian for the symplectic form

$$\Omega(\alpha, \beta) = \int \alpha \cup \beta.$$

<u>Pf</u>: $f_{\mathbb{L}^-}(\alpha^-) = f_{\mathbb{L}^+}(\alpha^+) \Rightarrow \exists$ cobordism V; $\underset{T Z_V}{\text{deformations of } V} = H^1(V) \longrightarrow \underset{\text{def. of ends}}{H^1(\partial V)}$

im is Lagrangian by std argument; finally $\exists$ countably many homotopy classes of V. $\square$

<u>Prop</u>: If $\underset{g \geq 1}{L} \subset \underset{\dim_{\mathbb{R}} \geq 4}{X}$ then $Cob^{or}(X)$ is "∞-dim'l": can't be covered by a countable union of images of $f_{\mathbb{L}}$ with $\dim \mathcal{U}_{\mathbb{L}}$ bounded.