Joint work w/ Bernd Siebert (2019)

Intrinsic Mirror Symmetry

History: 1989-90 Candelas et al
Greene, Plesser

Calabi-Yau 3-folds tend to come in pairs $X, \check{X}$

$$h^{1,1}(X) = h^{1,2}(\check{X})$$

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Example: Quintic 3-fold

$$X = Z(x_0^5 + \cdots + x_4^5) \subseteq \mathbb{CP}^4$$

$$h^{1,1}(X) = 1, \quad h^{1,2}(X) = 101.$$

Greene-Plesser construction

$\mathbb{Z}_5^5 \circlearrowright \mathbb{CP}^4$ $(a_0, \ldots, a_4) \in \mathbb{Z}_5^5$

acts by

$$(x_0, \ldots, x_4) \longmapsto (\xi^{a_0} x_0, \ldots, \xi^{a_4} x_4)$$

where $\xi = e^{2\pi i/5}$

$G = \mathbb{Z}_5^4 \subseteq \mathbb{Z}_5^5$

$\shortparallel$

$\{ (a_0, \dots, a_4) \mid \sum a_i = 0 \}$

G acts on X, and $\exists$ a resolution of singularities

$$\check{X} \to X / G$$

with $\check{X}$ Calabi-Yau, and $h^{1,1}(\check{X}) = 101$, $h^{1,2}(\check{X}) = 1$.

Candelas, de la Ossa, Green, Parkes, 1990

Proposed can calculate the number of rational curves of degree d on X by performing certain period integrals on $\check{X}$

i.e, integrals $\int_{\alpha} \Omega_{\check{X}}$ where $\alpha \in H_3(\check{X}, \mathbb{Z})$

and $\Omega_{\check{X}}$ a holomorphic 3-form on $\check{X}$.

$\rightsquigarrow$ (a lot of work)

get a generating function for

$N_d :=$ # of holomorphic maps

$$f : \mathbb{CP}^1 \longrightarrow X$$ representing

a homology class of degree d

$(H_2(X, \mathbb{Z}) = \mathbb{Z})$

Correct way to define these numbers

is via Gromov-Witten theory.

e.g. $N_1 = 2875$ (19th century)

$N_2 = 609250$ 1986, S. Katz

$\vdots$

<u>Constructions of mirror pairs?</u>

<u>Batyrev</u>: (1992) Mirror pairs could be constructed as hypersurfaces in toric varieties. ($\sim 4 \times 10^8$)

<u>Question</u>: Is there a general mirror construction?

<u>Answer</u>: Yes!

Our context:

We fix a <u>log Calabi-Yau pair</u> (X, D) where

- X is a non-singular ~~proj.~~ variety.
- D is a reduced normal crossings divisor with $K_X + D = 0$.

In this case, $\exists$ a nowhere vanishing holomorphic n-form on $X \setminus D$.

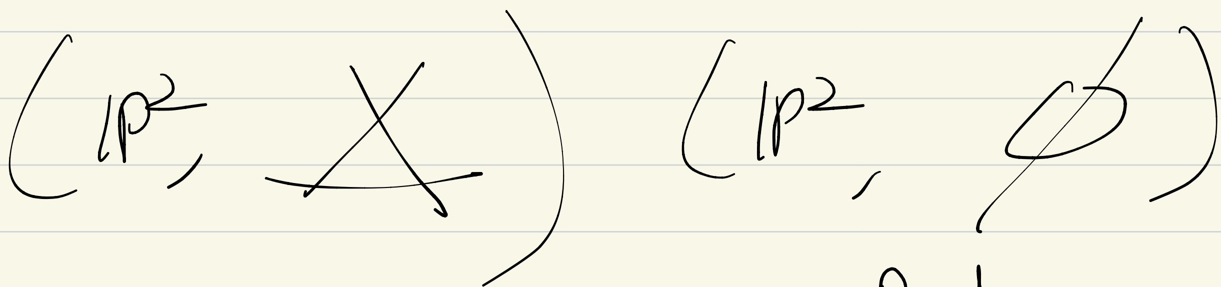
(Actually two cases: ① X projective.
② $X \to S$ a projective degeneration with S a germ of a smooth curve.)

Write $D = D_1 + \cdots + D_s$ be the irreducible decomposition.

<u>Assume</u>: For $I \subseteq \{1, \dots, s\}$,

$$D_I := \bigcap_{i \in I} D_i \quad \text{is connected.}$$

e.g.



Good

Bad

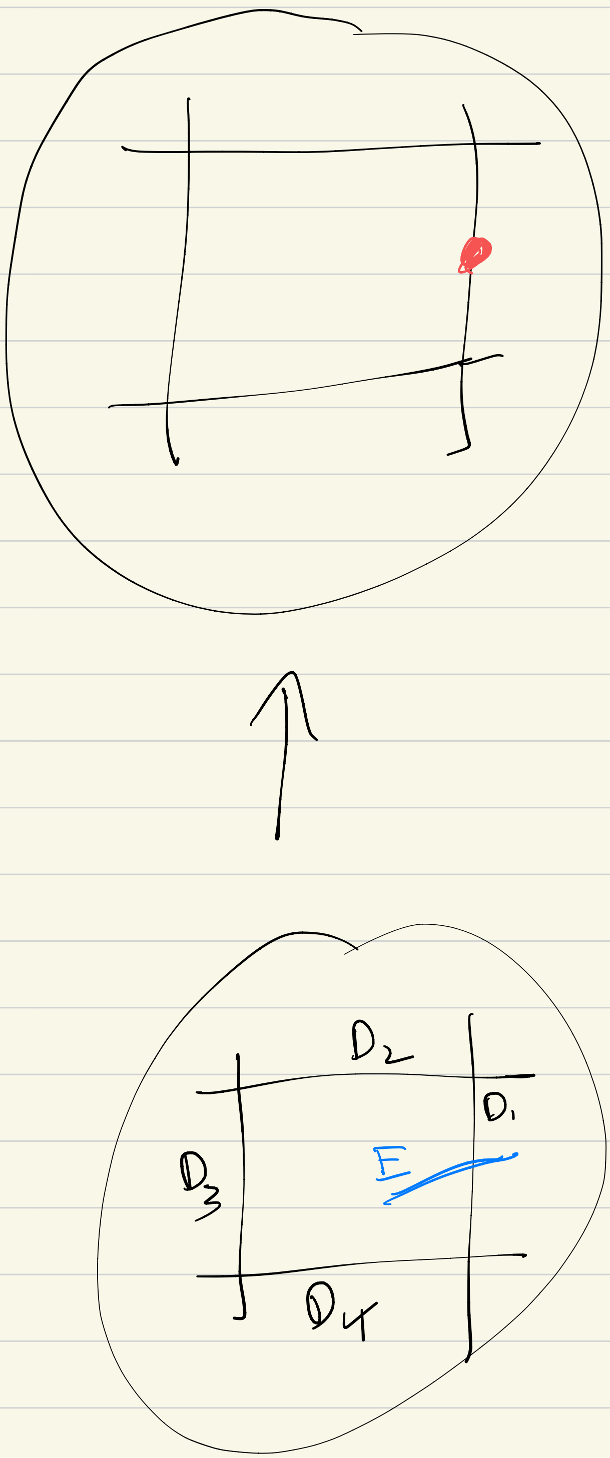
We will build the <u>dual complex</u> of <u>D</u> as cone complex in the $\mathbb{R}$-vector space with basis $D_1, \dots, D_s$.

$$\mathcal{P} := \left\{ \sum_{i \in I} \mathbb{R}_{\geq 0} D_i \;\middle|\; I \subseteq \{1, \dots, s\},\ D_I \neq \emptyset \right\}$$

$$\mathcal{B} = \bigcup_{\sigma \in \mathcal{P}} \sigma.$$

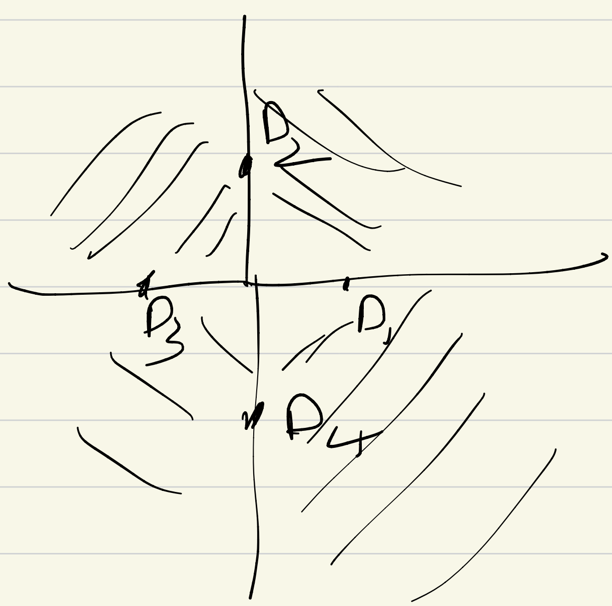
Example: Start with

$$(\mathbb{P}^1 \times \mathbb{P}^1, \ \overline{D} = (\{0,\infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{0,\infty\}))$$



Blow up one point on the boundary, and take D to be the strict transform of $\overline{D}$.

(B, P) the tropicalization of (X, D).



$$B(\mathbb{Z}) = \left\{ \sum a_i D_i \in B \;\middle|\; a_i \in \mathbb{Z}_{\geq 0} \right\}$$

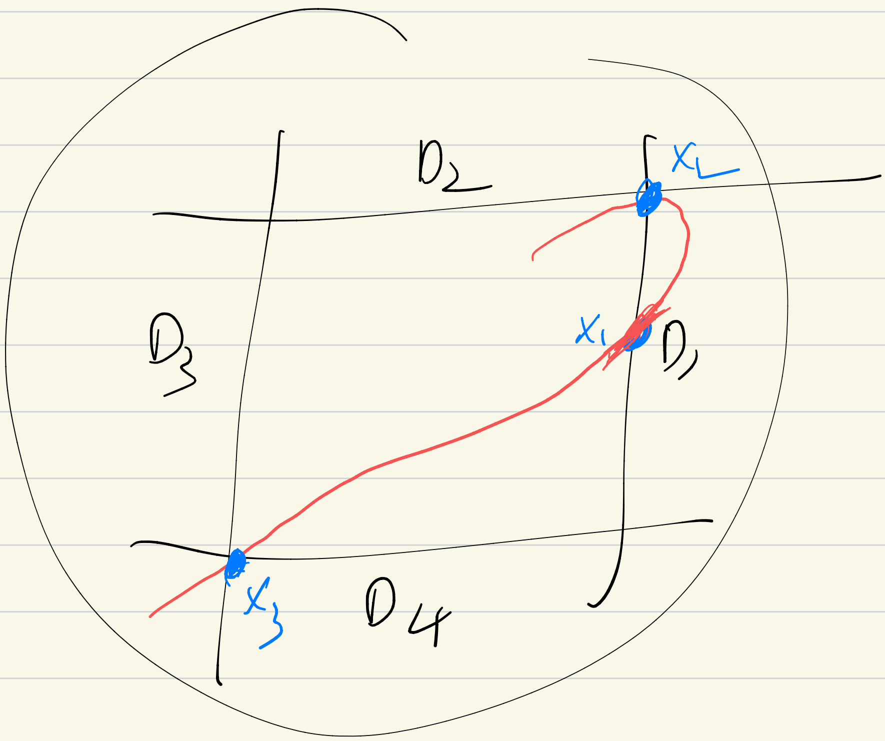
A point of $B(\mathbb{Z})$ records orders of tangency (or contact orders) of maps $f : C \longrightarrow X$, C a curve.

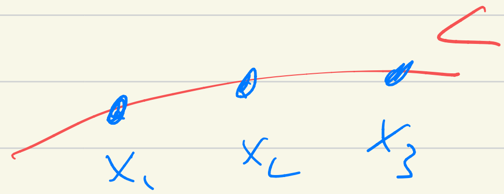
Might want to count maps $f : (C, x_1, \ldots, x_n) \longrightarrow X$

$x_1, \ldots, x_n \in C$ distinct points

such that we specify the contact order of f at a point x_i with each divisor D_j.

$p \in B(\mathbb{Z})$. $p = \sum a_i D_i$ is interpreted as such a tangency condition with contact order a_i with D_i.





Tangency condition

at x_1 : D_1

x_2 : $D_1 + 2D_2$

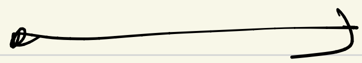
x_3 : $D_3 + D_4$

$\exists$ a good theory of Gromov-Witten invariants which encode these kind of tangency conditions. (Log Gromov-Witten theory, Abramovich-Chen, G-Siebert 2010)

Assume: $\dim_{\mathbb{R}} B = \dim_{\mathbb{C}} X$.

In this case we say D is a maximal boundary.

e.g. $(\mathbb{P}^2$, smooth elliptic curve$)$ Bad

B 

$(\mathbb{P}^2, X)$  Good

In the case of a degeneration $X \to S$, we take D to ~~be~~ the singular fibre, and then this condition is equivalent to maximal unipotency of the degeneration.

(Large complex structure limit)

We will construct a ring $R(X,D)$ which can be used to construct the mirror as either ① $\mathrm{Spec}\, R(X,D)$

or ② $\mathrm{Proj}\, R(X,D)$

i.e., $R(X,D)$ is the affine or homogeneous coordinate ring of the mirror.

Actually, will construct a family.

Fix $P \subseteq H_2(X,\mathbb{Z})$ a submonoid, such that

- P contains the class of every effective curve.
- $P \cap (-P) = H_2(X,\mathbb{Z})_{tors}$.
- P saturated, i.e., $nP \in P \Rightarrow p \in P$.

$$A = \mathbb{C}[P] = \bigoplus_{p \in P} \mathbb{C} \cdot t^p \quad \text{the monoid ring}$$

$$t^p \cdot t^{p'} = t^{p+p'}.$$

Fix a monomial ideal $I \subseteq A$ such that $A_I := A/I$ is Artinian.

<u>Actually</u>: For each such I, we will construct a flat A_I-algebra $R_I(X,D)$, giving families

$$\begin{tikzcd} \operatorname{Spec} R_I(X,D) \arrow[d] \arrow[r, phantom, "\subset"] & \operatorname{Proj} R_I(X,D) \arrow[d] \\ \operatorname{Spec} A_I & \operatorname{Spec} A_I \end{tikzcd}$$

(Can take limits to get formal versions. $\operatorname{Spec} \hat{A}$ should be viewed as the <u>Kähler moduli space</u> of X.)

\varTheta

$$R_I(X,D) := \bigoplus_{p \in B(\mathbb{Z})} A_I \cdot \vartheta_p$$

theta functions.

(Generalization of classical theta functions.)

This is a free A_I-module.

Need to define an algebra structure:

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr} \vartheta_r$$

with $\alpha_{pqr} \in A_I$.

$$\alpha_{pqr} = \sum_{\beta \in P \setminus I} N_{pqr}^{\beta} \cdot t^{\beta}.$$

Key point: def'n of the N_{pqr}^{β}.

Def: For $r \in B(\mathbb{Z})$, $r = \sum_{i \in I} a_i D_i$, $a_i > 0$

gives a stratum D_I of D

Fix a point $z \in D_I$.

Let $N^{\beta}_{pqr} = \#$ of maps

$$f : (C, x_1, x_2, x_{out}) \longrightarrow X$$

with

- C genus 0
- $f_*[C] = \beta$
- The tangency condition at x_1 is given by p.
- The tangency condition at x_2 is given by q.
- The tangency condition at x_{out} is given by $-r$, and $f(x_{out}) = z$.

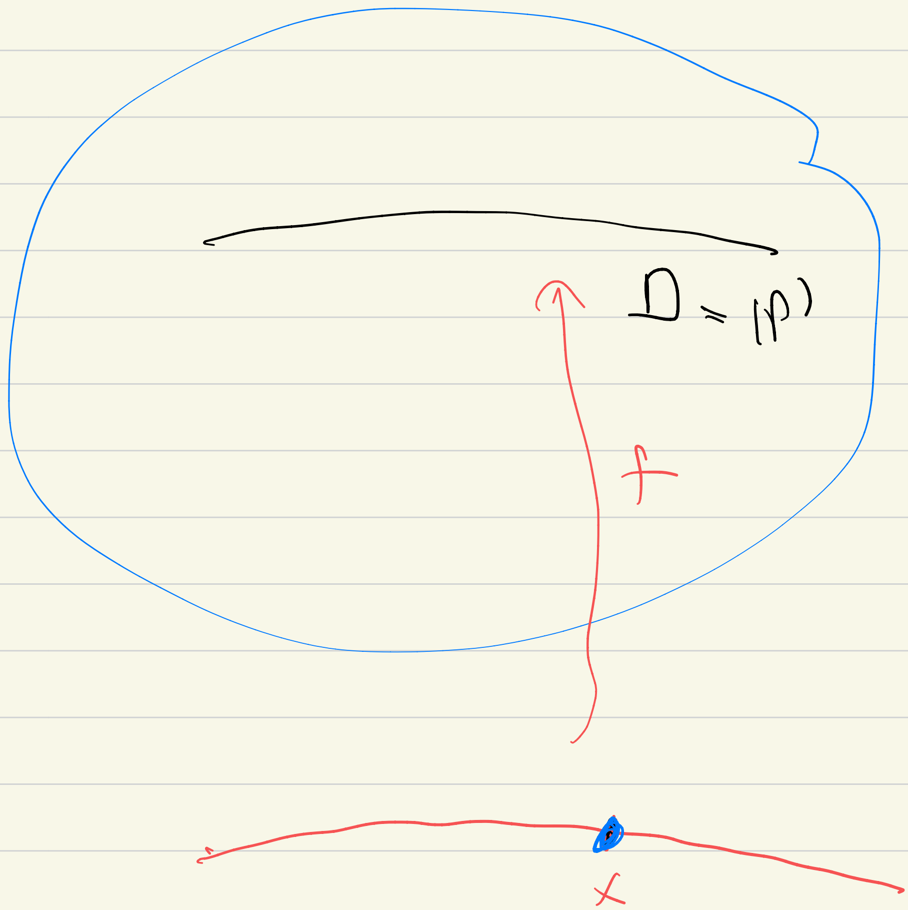
This involves negative orders of tangency.

To define properly, we use punctured log Gromov–Witten theory

(Abramovich, Chen, G., Siebert)

e.g. X a surface

$D \subseteq X$ a smooth rational curve with $D^2 = -1$.



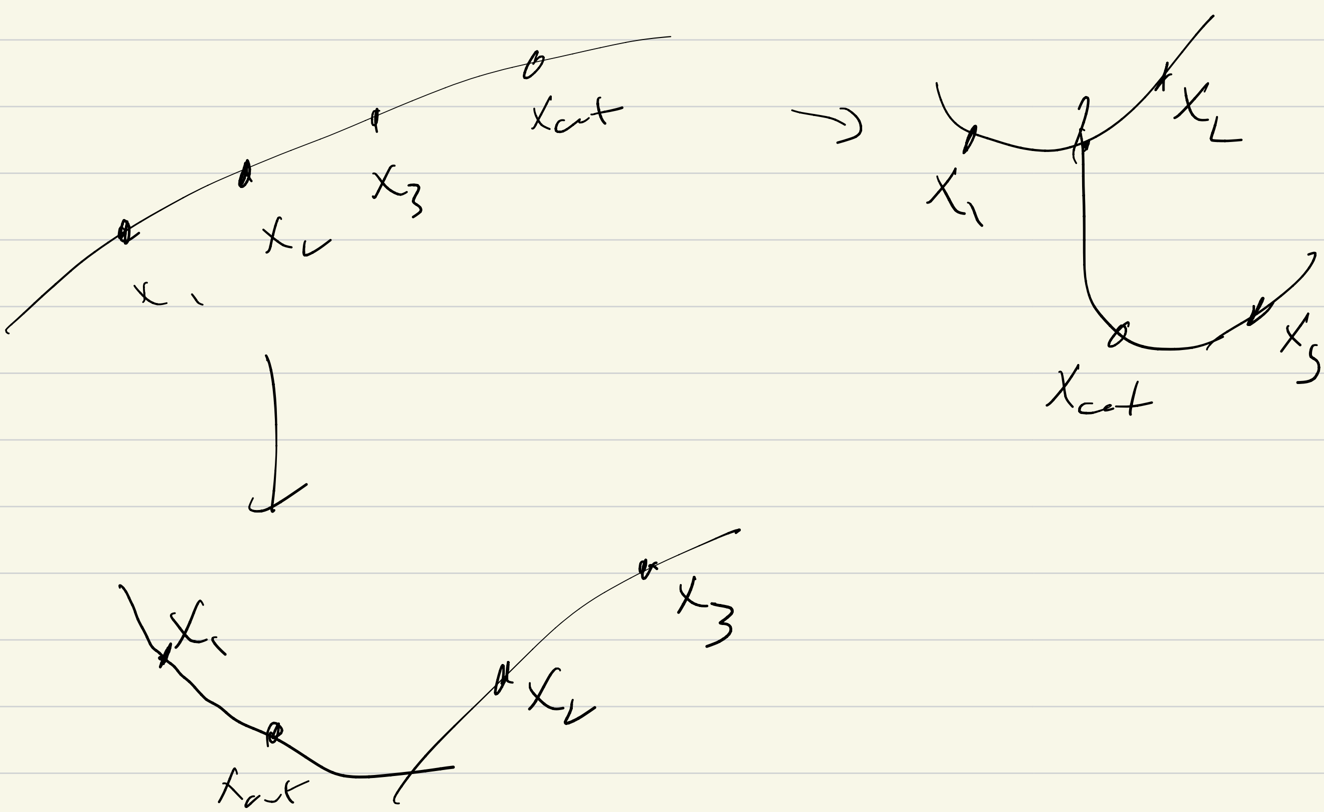
$\mathbb{P}^1$

$\shortparallel$

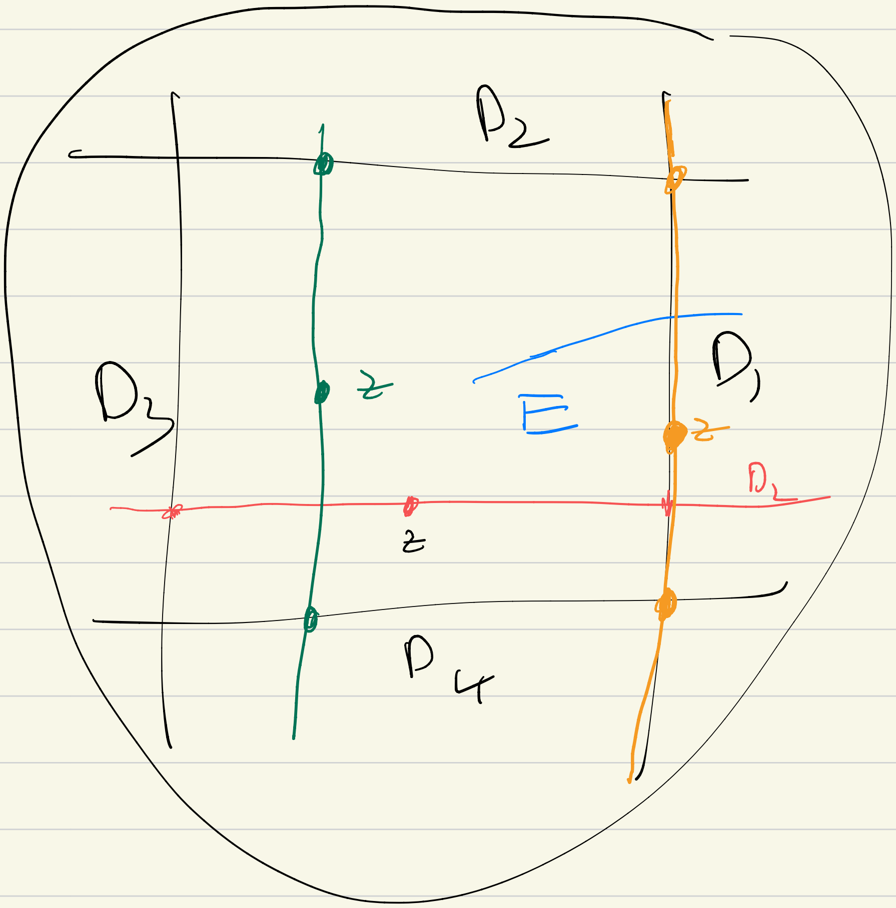
(C, x)

$f : C \to D$ identity

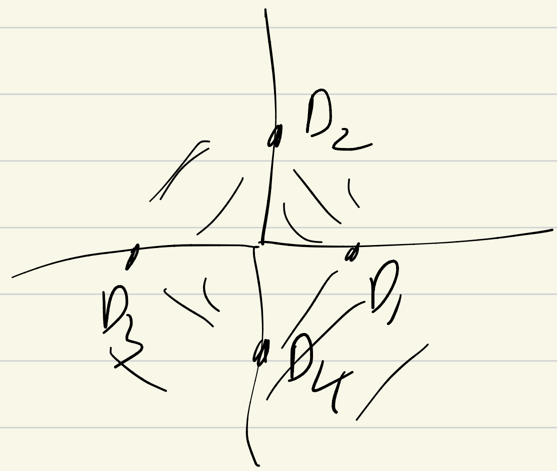
Theorem (G.–Siebert, '19) The numbers N^{β}_{pqr} can be defined rigorously, and they give $R_I(X,D)$ the structure of an associative, commutative A_I-algebra with unit $1 = \vartheta_0$.



Example: (X, D) the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ considered earlier.



(B, ρ)



$\vartheta_{D_1}, \dots, \vartheta_{D_4}$

$$\vartheta_{D_1} \cdot \vartheta_{D_3} = \underbrace{N^{D_2}_{D_1 D_3 0}}_{=1} \cdot t^{D_2} \cdot \vartheta_0$$

$$\boxed{\vartheta_{D_1} \vartheta_{D_3} = t^{D_2}}$$

$$\boxed{\vartheta_{D_2} \cdot \vartheta_{D_4} = t^{D_3} + t^{D_1} \vartheta_{D_1}}$$