Symplectic rational $G-$surfaces and the plane Cremona group

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Symplectic rational $G-$surfaces

We study symplectic 4-manifolds (X, ω) equipped with a finite symplectomorphism group G, where X is diffeomorphic to a rational surface. We shall call such a pair $((X, \omega), G)$ a **symplectic rational G-surface**.

They are the symplectic analog of **(complex) rational G-surfaces** studied in algebraic geometry, which are rational surfaces equipped with a holomorphic G-action. These rational G-surfaces played a central role in the classification of finite subgroups of the plane Cremona group, a problem dating back to the early 1880s.
The plane Cremon group is the group of birational transformations of the complex projective plane.

Dolgachev-Iskovskikh ([DI]) , *Finite subgroups of the plane Cremona group*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, Vol. I, 443-548, Progr. Math. 269, 2009.

Perspectives in algebraic geometry

Note that any rational G-surface can be regarded as a symplectic rational G-surface – simply endowing it with a G-invariant Kähler form which always exists.

We show that a large part of the story regarding the classification of rational G-surfaces can be recovered by techniques from 4-manifold theory and symplectic topology.

Furthermore, we add some new, interesting symplectic geometry aspect to the study of rational G-surfaces; in particular, in regard to the equivariant symplectic minimality and equivariant symplectic cone of the underlying smooth action of a rational G-surface.

In addition, we also obtain some result which does not seem previously known in the algebraic geometry literature.

Symplectic $G-$minimality

We first introduce symplectic G-minimality. Let (X, ω) be a symplectic 4-manifold with a finite symplectomorphism group G.

Suppose there exists a G-invariant set of **disjoint** union of symplectic (-1)-spheres in X. Then blowing down X along these (-1)-spheres gives rise to a symplectic 4-manifold (X', ω'), which can be arranged so that G is natually isomorphic to a finite symplectomorphism group of (X', ω').

The symplectic G-manifold X is called *minimal* if no such set of (-1)-spheres exists.

Comparing the minimality assumptions

Chen: When X is neither rational nor ruled,
the symplectic G-manifold is $(G-)$minimal if and only if the underlying smooth manifold is minimal.

However, the underlying (smooth) rational surface is often not minimal even though the corresponding symplectic rational G-surface is $(G-)$minimal.

Furthermore, it is not known whether the notion of G-minimality is the same for the various different categories, i.e., the holomorphic, symplectic, or smooth categories.

More on this later.

The objectives

The most fundamental problem in our study is to classify symplectic rational G-surfaces up to equivariant symplectomorphisms.

However, Chen showed that even in the simple case where X is $\mathbb{CP}^2$ or a Hirzebruch surface and G is a cyclic or meta-cyclic group, such a classification is already quite involved.
In fact, in one circumstance where G is meta-cyclic, a weaker classification, i.e., classification up to equivariant diffeomorphisms, still remains open.

For symplectic rational G-surfaces X in general, we would like to

(1) classify the possible symplectic structures;

(2) describe the induced action of G on $H_2(X)$;

(3) give a list of possible finite groups for G;

(4) understand the equivariant minimality and equivariant symplectic cones.

These problems are still highly non-trivial and not completely settled.

The scope

Part of the determination of G and the induced action on $H_2(X)$ relies on
the Dolgachev-Iskovskikh's solution of the corresponding problems in algebraic geometry,
with new inputs from Gromov-Witten theory and a detailed analysis of the symplectic structures.

We focus on the case where the rational surface X is $\mathbb{CP}^2$ blown up at 2 or more points.
More concretely, we consider **minimal** symplectic rational G-surfaces (X, ω) where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, for $N \geq 2$.

Note that the minimality assumption implies in particular that G is a nontrivial group.

Notations and terminology

We denote by $H, E_1, E_2, \cdots, E_N$ a basis of $H^2(X, \mathbb{Z})$, under which the intersection matrix takes its standard form,
$H^2 = 1,\ E_i^2 = -1,\ H \cdot E_i = 0,\ \forall i$, and $E_i \cdot E_j = 0, \forall i \neq j$.

The canonical class of (X, ω) is denoted by $K_\omega \in H^2(X)$, in order to emphasize its dependence on the symplectic form ω.

$H^2(X)^G$ is used to denote the subset of $H^2(X, \mathbb{Z})$ consisting of elements fixed under the induced action of G, and is called the *invariant lattice*.

Monotone manifolds

A symplectic rational surface (X, ω) is called *monotone* if $K_\omega = \lambda[\omega]$ is satisfied in $H^2(X; \mathbb{R})$ for some $\lambda \in \mathbb{R}$.
In this case, we have $\lambda < 0$, and N must be in the range $N \leq 8$.
Such a symplectic rational surface is the symplectic analog of Del Pezzo surface in algebraic geometry.

Another important notion, given in the following definition and called a *symplectic G-conic bundle*, corresponds to a conic bundle structure on a rational G-surface in algebraic geometry.

Conic bundles

Let (X, ω) be a symplectic 4-manifold equipped with a finite symplectomorphism group G.

Definition

A symplectic G-conic bundle structure on (X, ω) is a genus-0 smooth Lefschetz fibration $\pi : X \to B$ which obeys the following conditions:

- each singular fiber of π contains exactly one critical point;
- there exists a G-invariant, ω-compatible almost complex structure J such that the fibers of π are J-holomorphic;
- the group action of G preserves the Lefschetz fibration.

A symplectic G-conic bundle is called *minimal* if for any singular fiber there is an element of G whose action switches the two components of the singular fiber.

Although the above definition looks more rigid than it should be (in particular, it is always an almost complex fibration), the Symplectic structure Theorem shows that this is a purely symplectic notion in the case of minimal symplectic rational G-surfaces.

Here are some immediate consequences from the definition:

- X is a rational surface if and only if $B = \mathbb{S}^2$; in this case, note that the number of singular fibers of π equals $N-1$, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$;
- the Lefschetz fibration is symplectic with respect to ω;
- the fiber class lies in the invariant lattice $H^2(X)^G$ as G preserves the Lefschetz fibration;
- if the underlying symplectic G-manifold is minimal, then the symplectic G-conic bundle must be also minimal.

Symplectic structure Theorem

The first theorem is concerned with the symplectic structure of a minimal symplectic rational G-surface.

Symplectic Structure Theorem:
Let (X, ω) be a minimal symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ for some $N \geq 2$. Then $N \neq 2$, and one of the following holds true:

(1) The invariant lattice $H^2(X)^G$ has rank 1.
In this case, $3 \leq N \leq 8$ and (X, ω) must be monotone.

(2) The invariant lattice $H^2(X)^G$ has rank 2.
In this case, $N = 5$ or $N \geq 7$, and there exists a symplectic G-conic bundle structure on $((X, \omega), G)$.

Analogue in algebraic geometry

An analog of the Symplectic Structure Theorem for minimal rational G-surfaces is a classical theorem in algebraic geometry:
Monotone $\Longleftrightarrow$ Del Pezzo
symplectic G-conic bundle $\Longleftrightarrow$ holomoprhic $G-$conic bundle
In algebraic Geometry, the proof can be given using the equivariant Mori theory.

The proof of the Symplectic Structure Theorem gives an independent proof of the corresponding result in algebraic geometry by taking ω to be a G-invariant Kähler form. Consequently, a significant portion of the theory of rational G-surfaces can be recovered.

Adapted homology basis

Let (X, ω) be a symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, and let $\pi : X \to \mathbb{S}^2$ be a symplectic G-conic bundle on (X, ω).

Each singular fiber of π consists of a pair of (-1)-spheres and there are $N-1$ singular fibers.

We pick a (-1)-sphere from each singular fiber and name the homology classes by $E_2, \cdots, E_N$. Then there is a unique pair of line class and exceptional class H and E_1 such that

(i) $H - E_1$ is the fiber class of π,

(ii) $H, E_1, E_2, \cdots, E_N$ form a basis of $H_2(X)$ with standard intersection matrix.

For the (-1)-sphere with class E_j, $j = 2, \cdots, N$, the other (-1)-sphere lying in the same singular fiber has class $H - E_1 - E_j$.

The canonical class $K_\omega = -3H + E_1 + \cdots + E_N$.

Minimal symplectic $G-$conic bundle

There are further consequences on the symplectic structure if the symplectic G-conic bundle is minimal.
In this case, for each E_j, $j = 2, \cdots, N$, there exists a $g \in G$ such that $g \cdot E_j = H - E_1 - E_j$. This implies

$$\omega(E_j) = \frac{1}{2}\omega(H - E_1), \ \ j = 2, \cdots, N.$$

Thus a minimal symplectic G-conic bundle falls into three cases:

$$(i)\ \omega(E_1) = \omega(E_j),\ (ii)\ \omega(E_1) > \omega(E_j),\ (iii)\ \omega(E_1) < \omega(E_j).$$

Case (i) occurs if and only if (X, ω) is monotone.
Since E_1 is a section class, we shall call case (ii) (resp. case (iii)) a symplectic G-conic bundle with *small fiber area* (resp. *large fiber area*).
Interestingly, the Lagrangian root system is D_{N-1} in case (ii).

Technical lemmas

Suppose J is an ω-compatible almost complex structure.
The following is the J-holomorphic analog of Lemma 5.1 in [DI].
($N \geq 5$ **Lemma**) Suppose E, E' are two distinct J-holomorphic sections of self-intersection $-m, -m'$, and let r be the number of singular fibers where E, E' intersect the same component. Then

$$N - 1 = r + m + m' + 2E \cdot E'.$$

Consequently, if the symplectic G-manifold (X, ω) is minimal, then $N \geq 5$ must be true.

The case $N = 6$ is special.
($N \neq 6$ **Conic Lemma**) Let $\pi : X \to \mathbb{S}^2$ be a minimal symplectic G-conic bundle where $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$. Then there is a G-invariant J-holomorphic (-1)-sphere.

Exceptional class with the smallest symplectic area

Let $\mathcal{E}_\omega$ be the set of homology classes of embedded $\omega-$symplectic spheres with self-intersection -1. The following fact is crucial in our considerations.

Pinsonnault: Assume $N \geq 2$. Then for any ω-compatible almost complex structure J on X, each class $E \in \mathcal{E}_\omega$ with minimal area, i.e., $\omega(E) = \min_{e \in \mathcal{E}_\omega} \omega(e)$, is represented by an *embedded* J-holomorphic sphere.

The key ingredient for the Symplectic Structure Theorem is a reduction procedure which involves a certain type of standard basis of $H^2(X)$, called a symplectically *reduced basis*.

Symplectically reduced basis

A *reduced basis* is a basis $H, E_1, \cdots, E_N$ of $H^2(X)$ with standard intersection matrix, where $E_i \in \mathcal{E}_\omega$, such that

$$\omega(E_N) = \min_{e \in \mathcal{E}_\omega} \omega(e),$$

and for any $i < N$, E_i satisfies the following inductive condition: let $\mathcal{E}_i = \{e \in \mathcal{E}_\omega | e \cdot E_j = 0 \ \ \forall i < j\}$, then $\omega(E_i) = \min_{e \in \mathcal{E}_i} \omega(e)$. Furthermore, the canonical class $K_\omega = -3H + E_1 \cdots + E_N$.

Reduced basis always exists. Note that this is a symplectic concept.

If $N = 2$ and $\{H, E_1, E_2\}$ is a reduced basis, then $\mathcal{E}_\omega = \{E_1, E_2, H - E_1 - E_2\}$.
For $N \geq 3$, a reduction procedure can be introduced.

Note that E_N has an embedded $J-$holomorphic sphere representative for any $\omega-$compatible almost complex structure.

$N = 2$

($N \neq 2$ **Lemma**) No minimal symp rat $G-$surface with $N = 2$.

Proof. Suppose $((X, \omega), G)$ is such a surface. Let $\{H, E_1, E_2\}$ be a reduced basis. Fix a G-invariant J, and let C be the J-holomorphic (-1)-sphere representing E_2. We set $\Lambda := \cup_{g \in G} \; g \cdot C$. Then Λ is a union of finitely many J-holomorphic (-1)-spheres, containing at least two distinct (-1)-spheres **intersecting** each other because of the minimality assumption.

Since $\mathcal{E}_\omega = \{E_1, E_2, H - E_1 - E_2\}$, there are only two possibilities:
(1) Λ has three (-1)-spheres, rep. E_1, E_2 and $H - E_1 - E_2$, and
(2) Λ is a union of two (-1)-spheres rep. E_2 and $H - E_1 - E_2$.
In either case, $H - E_1 - E_2$ is represented by a $J-$holomorphic (-1)-sphere C'. Since $H - E_1 - E_2$ is the only characteristic element in $\mathcal{E}_\omega$, C' must be fixed by the G-action, which contradicts the minimality of the symplectic G-manifold (X, ω).

$N \geq 3$

Key Lemma

Suppose (X, ω) is a minimal symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ with $N \geq 3$. Then one of the following must be true.

(i) (X, ω) is monotone.

(ii) The reduced basis of X satisfies $\omega(E_1) > \omega(E_2) = \cdots = \omega(E_N)$, $\omega(H - E_1) = 2\omega(E_j)$. Moreover, if $E \in \mathcal{E}_\omega$ has minimal area, then either $E = E_j$ or $E = H_{1j} = H - E_1 - E_j$ for some $j > 1$.

The Symplectic Structure Theorem is based on the $N \neq 2$ **Lemma,** $N \geq 5$ **Lemma,** $N \neq 6$ **Conic Lemma, Key Lemma**.

Three cases

Symplectic Structure Theorem:
Let (X, ω) be a minimal symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ for some $N \geq 2$. Then $N \neq 2$, and one of the following holds true:

(1) The invariant lattice $H^2(X)^G$ has rank 1.
In this case, $3 \leq N \leq 8$ and (X, ω) must be monotone.

(2) The invariant lattice $H^2(X)^G$ has rank 2.
In this case, $N = 5$ or $N \geq 7$, and there exists a symplectic G-conic bundle structure on $((X, \omega), G)$.

Sketch of proof. We divide into three cases.
(i) (X, ω) is not monotone.
In this case we apply the Key Lemma and the Gromov-Witten theory to construct a conic bundle structure.
Note that $N \geq 5$ and $N \neq 6$ in this case.

The two monotone cases

ii) (X, ω) *is monotone and* $H^2(X)^G$ *has rank* 1.
In this case, note that $K_\omega \in H^2(X)^G$ and is a primitive class, hence $H^2(X)^G$ is spanned by K_ω.
Note that $3 \leq N \leq 8$ in this case.

iii) (X, ω) *is monotone and rank* $H^2(X)^G > 1$.
We perturb ω to a $G-$invariant ω' which is not monotone, and show that (X, ω') is minimal as a symplectic G-manifold for a sufficiently small perturbation.

$N = 6$

$N = 6$ **Theorem** Let (X, ω) be a minimal symplectic rational G-surface where $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$. Then the invariant lattice $H^2(X)^G$ has rank 1.

In particular, a minimal complex rational G-surface X which is $\mathbb{CP}^2$ blown up at 6 points must be Del Pezzo with $\mathrm{Pic}(X)^G = \mathbb{Z}$.

Proof.

Under the above assumptions, the $N \neq 6$ Conic Lemma implies (X, ω) is not a G-conic bundle, hence $rank(H^2(X)^G) = 1$ by the Symplectic Structure Theorem.

For the second statement, notice that a conic bundle on a minimal complex rational G-surface defines a minimal symplectic G-conic bundle with respect to any G-invariant Kähler form. Theorem 3.8 of [DI] asserts then X must be a del Pezzo surface if $rank H^2(X)^G = 1$.

The invariant lattice

In case (1) of the Symplectic Structure Theorem, the invariant lattice $H^2(X)^G$ is spanned by K_ω.
In complex geometry, it was known that a minimal (complex) rational G-surface which is diffeomorphic to $\mathbb{CP}^2$ blown up at 6 points must be Del Pezzo. However, it seemed new that the invariant Picard group $\mathrm{Pic}(X)^G$ must be of rank 1.

In case (2), $H^2(X)^G$ is spanned by K_ω and the fiber class of the symplectic G-conic bundle.

The analysis on the type of the group G, the induced action on $H^2(X)$, as well as the structure of the equivariant symplectic cone, depends on the rank of the invariant lattice $H^2(X)^G$.

The three minimality for conic bundles

A minimal symplectic G-conic bundle is different from *a minimal symplectic G-surface with G-conic bundle structure*: a minimal symplectic G-conic bundle may still contain a G-invariant disjoint union of symplectic (-1)-spheres. However, this cannot happen when $N \geq 7$ and $N = 5$.

The minimal G-conic bundles is an intermediate notion between minimal symplectic G-surfaces and minimal complex G-surfaces.

There is always a G-invariant symplectic form compatible with the given complex structure on a minimal complex G-surface. Although we do not know whether this symplectic form is always G-minimal, if we assume a symplectic G-conic bundle structure underlies this action, this conic bundle structure is always minimal.

minimal G complex surface with conic bundle structure

$$\Downarrow$$

minimal G conic bundle

$$\Uparrow$$

minimal G symplectic surface with conic bundle structure

Therefore, proving results in the more general minimal G-conic bundle context plays an important role in bridging the complex and symplectic G-surfaces as well as in the study of the equivariant symplectic cones.

We now describe possible candidates for G.
We begin with case (1) in the Symplectic Structure Theorem where (X,ω) is a minimal symplectic rational G-surface such that the invariant lattice $H^2(X)^G$ has rank 1.
In this case (X,ω) is monotone, $H^2(X)^G$ is spanned by K_ω, and $3\leq N\leq 8$.
Note that the orthogonal complement of K_ω in $H^2(X)$ (with respect to the intersection product), denoted by R_N, is a G-invariant root lattice of type

$$E_N\,(N=6,7,8),\quad D_5\,(N=5),\quad A_4\,(N=4),\quad A_2+A_1\,(N=3)$$

respectively.
We denote by W_N the corresponding Weyl group.

Monotone Theorem

Let (X, ω) be a minimal symplectic rational G-surface such that $H^2(X)^G$ has rank 1. There are two cases:

(1) Suppose $4 \leq N \leq 8$. Then the induced action of G on $H^2(X)$ is faithful, which gives rise to a monomorphism $\rho : G \to W_N$. Moreover, the image $\rho(G)$ in W_N satisfies

$$\sum_{g \in G} trace\{\rho(g) : R_N \to R_N\} = 0.$$

(2) Suppose $N = 3$. Let Γ be the subgroup of G which acts trivially on $H^2(X)$ and let $K := G/\Gamma$ be the quotient group. Then Γ is isomorphic to a subgroup of the 2-dimensional torus and K is isomorphic to $\mathbb{Z}_6$ or the dihedral group D_{12}. Furthermore, G is a semi-direct product of Γ by K. As a corollary, G can be written as a semi-direct product of an imprimitive finite subgroup of $PGL(3)$ by $\mathbb{Z}_2$.

Remark

1. For $N = 4, 5$, the subgroups of the Weyl group W_N which satisfy the condition $\sum_{g \in G} trace\{\rho(g) : R_N \to R_N\} = 0$ are determined. All such groups can be realized by a minimal G-Del Pezzo surface, which is also minimal as a symplectic rational G-surface with respect to any G-invariant Kähler form.

2. For $N = 3$, the Symplectic Structure Theorem (2) and the Monotone Theorem completely determined all possible G that acts minimally on X. The statement in the Monotone Theorem implies the corresponding statement in algebraic Geometry, and the semi-direct product structure of G is an improvement.

Dolgachev-Iskovskikh: Finite subgroups of the plane Cremona group, 2009.

Now we consider case (2) of the Symplectic Structure Theorem.

We work in the slightly more general situation, where the symplectic rational G-surface (X, ω) is not assumed to be minimal, but only admits a minimal symplectic G-conic bundle $\pi : (X, \omega) \to \mathbb{S}^2$. Furthermore, we assume $N \geq 4$ (instead of the fact that $N \geq 5$ when (X, ω) is a minimal symplectic rational G-surface).

- $Q \lhd G$ is the subgroup that acts trivially on the base $\mathbb{S}^2$ of the G-conic bundle;
- $G_0 \lhd Q$ is the subgroup that acts trivially on $H^2(X)$ (in the case we consider, there are $N - 1 \geq 3$ critical values on the base that was fixed, hence $G_0 \lhd Q$);
- $P = G/Q$, so that G decomposes as

$$1 \to Q \to G \to P \to 1. \tag{1}$$

We denote by Σ the subset of $\mathbb{S}^2$ which parametrizes the singular fibers of the symplectic G-conic bundle π.
Note that $\#\Sigma = N - 1$, and the induced action of P on $\mathbb{S}^2$ leaves the subset Σ invariant. The action of P on $\mathbb{S}^2$ is effective, so P is isomorphic to a polyhedral group, i.e., a finite subgroup of $SO(3)$.

Therefore, up to an extension problem, the description of G boils down to the following theorem which describes the subgroups G_0 and Q.

Conic Bundle Theorem

Let (X, ω) be a symplectic rational G-surface equipped with a minimal symplectic G-conic bundle structure with at least three singular fibers (i.e, $N \geq 4$). Let G_0, Q be given as above. Then one of the following is true.

1. $G_0 = \mathbb{Z}_m$, $m > 1$, and Q is either the dihedral group D_{2m} or $Q = G_0$ and m is even.
   Moreover, N must be odd, and any element $\tau \in Q \setminus G_0$ switches the two (-1)-spheres in each singular fiber.
2. G_0 is trivial and $Q = \mathbb{Z}_2$ or $(\mathbb{Z}_2)^2$.
   In the latter case, let τ_1, τ_2, τ_3 be the distinct involutions in Q. Then Σ is partitioned into subsets Σ_1, Σ_2, Σ_3, where Σ_i parametrizes those singular fibers of which τ_i leaves each (-1)-sphere invariant, and $\#\Sigma_i \equiv N - 1 \pmod 2$, for $i = 1, 2, 3$.

We also investigate the underlying **smooth action** of a minimal symplectic rational G-surface. We begin with the setup of our study here.

Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, which is equipped with a smooth action of a finite group G.

Suppose there is a G-invariant symplectic form ω_0 on X such that the corresponding symplectic rational G-surface (X, ω_0) is minimal. Then the space

$$\Omega(X, G) := \{\omega : \omega \text{ is symplectic on } X, g^*\omega = \omega, \text{ for any } g \in G\}.$$

is non-empty as $\omega_0 \in \Omega(X, G)$.

The following theorem shows that the underlying smooth action of a minimal (complex) rational G-surface satisfies the above assumption, where we can take ω_0 to be any G-invariant Kähler form.

Symplectic Minimality Theorem

(1) Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, which is equipped with a smooth action of a finite group G. Suppose there is a G-invariant symplectic form ω_0 on X such that $((X, \omega_0), G)$ is minimal. Then for any $\omega \in \Omega(X, G)$, the canonical class $K_\omega = K_{\omega_0}$ or $-K_{\omega_0}$, and the symplectic rational G-surface (X, ω) is minimal.

(2) Let X be any minimal (complex) rational G-surface which is $\mathbb{CP}^2$ blown up at 2 or more points. Then for any symplectic form ω which is invariant under the underlying smooth action of G (e.g., any G-invariant Kähler form), the corresponding symplectic rational G-surface (X, ω) is minimal.

It is not known whether the symplectic minimality would imply the smooth minimality of the underlying group action.

The Symplectic Minimality Theorem is a weaker statement that symplectic minimality is determined by the underlying smooth action, but the proof is still quite non-trivial.
There are related (stronger) results in the case of G-Hirzebruch surfaces,

Chen: $G-$minimality and invariant negative spheres in $G-$Hirzebruch surfaces, Journal of Topology, 2015.

The equivariant symplectic cone

The equivariant symplectic cone of (X, G) is defined as

$$\tilde{C}(X,G) = \{\alpha : \alpha = [\omega],\ \omega \textit{ is a } G\textit{-invariant symplectic form}\}.$$

Note that $K_{-\omega} = -K_\omega$. Therefore it suffices to consider the subset

$$C(X,G) := \{[\omega] | \omega \in \Omega(X,G), K_\omega = K_{\omega_0}\} \subset H^2(X;\mathbb{R})^G.$$

Furthermore, we observe that if $H^2(X)^G$ has rank 1,
$C(X,G) = \{\lambda K_{\omega_0} | \lambda \in \mathbb{R}, \lambda < 0\}$.

In what follows, we shall assume that $H^2(X)^G$ has rank 2. Note that under this assumption, $N \neq 6$ by part (2) of the Symplectic Structure Theorem.

The fiber class F

In order to describe $C(X,G)$, it is helpful to introduce the following terminology.
A class $F \in H^2(X)^G$ is called a **fiber class** if there exists an $\omega \in \Omega(X,G)$ such that F is the class of the regular fibers of a symplectic G-conic bundle on $((X,\omega),G)$.
Since we will focus on the set $C(X,G)$, we shall assume further that $[\omega] \in C(X,G)$, i.e., $K_\omega = K_{\omega_0}$. Since rank $H^2(X)^G = 2$, and

$$\omega(F) > 0, F \cdot F = 0, K_\omega(F) = -2,$$

we have

$$[\omega] = -aK_{\omega_0} + bF, a > 0.$$

Consider the following subset of $C(X,G)$ and its projective classes

$$C(X,G,F) = \{[\omega] \in C(X,G) : \omega = -aK_{\omega_0} + bF, a > 0, b \geq 0\},$$

$$\hat{C}(X,G,F) = \{[\omega] \in C(X,G,F) : \omega(F) = 2\} \text{ (equivalently, } a = 1).$$

The gap $\delta_{\omega,F}$

Now $[\omega] \in \hat{C}(X,G,F)$ can be written as $[\omega] = -K_{\omega_0} + \delta_{\omega,F} F$.
Then the one to one correspondence $[\omega] \mapsto \delta_{\omega,F}$ identifies $\hat{C}(X,G,F)$ with a subset of $\mathbb{R}$.
With this understood, we introduce

$$\delta_{X,G,F} := \inf_{[\omega]\in \hat{C}(X,G,F)} \delta_{\omega,F} \in [0,\infty).$$

Note that (X,ω) is monotone if and only if $\delta_{\omega,F} = 0$, so $\delta_{\omega,F}$ may be thought of as a sort of gap function which measures how far away (X,ω) is from being monotone.

Equivariant Cone Theorem

Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, be equipped with a smooth finite G-action which is symplectic and minimal with respect to some symplectic form ω_0. Furthermore, assume rank $H^2(X)^G = 2$.

(1) If $N \geq 9$ or G_0 is nontrivial, there is a unique fiber class F, and $C(X, G) = C(X, G, F)$.

(2) For $N = 5, 7, 8$, either there is a unique fiber class F, or there are two distinct fiber classes F, F'. In the former case, $C(X, G) = C(X, G, F)$, and in the latter case, $C(X, G) = C(X, G, F) \cup C(X, G, F')$, with $C(X, G, F) \cap C(X, G, F')$ being either empty or consisting of $[\omega]$ such that (X, ω) is monotone.

(3) For any fiber class F, $\hat{C}(X, G, F)$ is identified with either $[0, \infty)$ or $(\delta_{X,G,F}, \infty)$ under $[\omega] \mapsto \delta_{\omega,F}$. (In particular, $\delta_{X,G,F}$ can not be attained unless it equals 0.)

We conjecture that when there are two distinct fiber classes, the equivariant symplectic cone must contain the class of a monotone form.

Furthermore, it is an interesting problem to determine the gap functions $\delta_{X,G,F}$ for a given minimal (complex) rational G-surface X with $\text{Pic}(X)^G = \mathbb{Z}^2$.