Abel-Jacobi maps and the moduli of differentials

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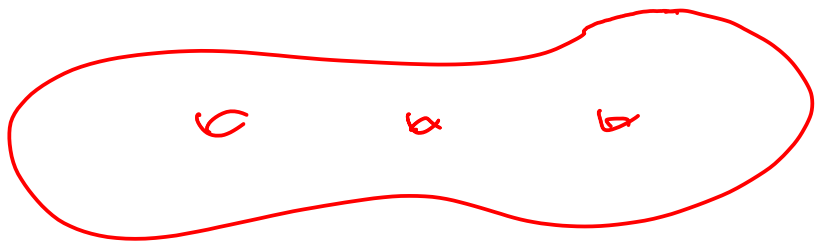
(ZOOM)

Let C be a nonsingular complex algebraic curve of genus g / $\mathbb{C}$

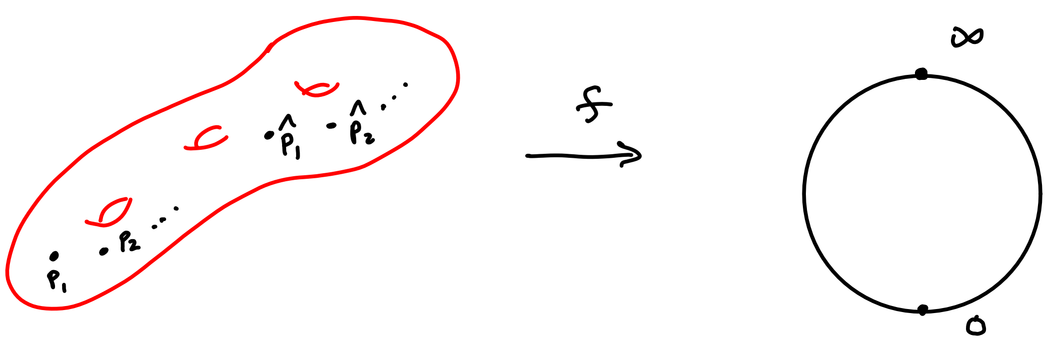


A function f can viewed

$f : C \to \mathbb{C}$ rational algebraic

$f : C \to \mathbb{C}$ meromorphic

$f : C \to \mathbb{CP}^1$ morphism



$$\operatorname{div}(f) = \sum_{p_i \in f^{-1}(0)} m_i p_i - \sum_{\hat{p}_j \in f^{-1}(\infty)} \hat{m}_j \hat{p}_j$$

The other way is to start with

$A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} a_i = 0$

and distinct points $x_1, \dots, x_n \in C$.

Question: Does there exist a rational function

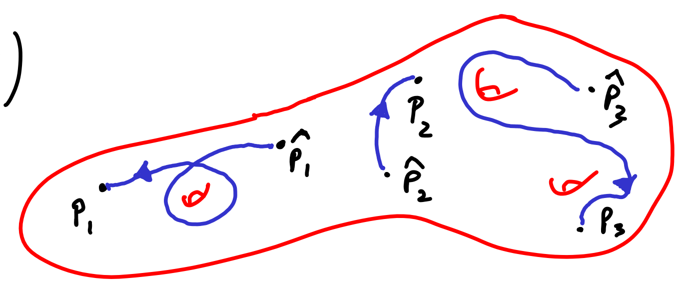
$$f : C \to \mathbb{C}$$

such that $\operatorname{div}(f) = \sum_{i=1}^{n} a_i x_i$?

Basic question in the history of algebraic geometry.
Solved by Abel and Jacobi in the 19th Century.

New ideas: divisor, Jacobian, analytic methods

Solution $A = (1,1,1,-1,-1,-1)$



- Choose γ with

$$\partial\gamma = P_1 + P_2 + P_3 - \hat{P}_1 - \hat{P}_2 - \hat{P}_3$$

- Choose $\omega_1, \dots, \omega_g$ basis of holomorphic differentials on C
- Integrate : $\left(\int_\gamma \omega_1 , \dots , \int_\gamma \omega_g \right) \in \mathbb{C}^g / H_1(C, \mathbb{Z})$

$$= \text{Complex torus } \operatorname{Jac}(C)$$

ambiguity in choice of γ

Answer (Abel-Jacobi)

$$\exists f : C \to \mathbb{C} \quad \text{with} \quad \operatorname{div}(f) = P_1 + P_2 + P_3 - \hat{P}_1 - \hat{P}_2 - \hat{P}_3$$

$$\Updownarrow$$

$$\left(\int_\gamma \omega_1 , \dots , \int_\gamma \omega_g \right) = 0 \in \operatorname{Jac}(C)$$

Perfect gem from the 19th century.

~ 200 years later, we return to this question from a different point of view.

As before, $A = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$

with $\sum_{i=1}^{n} a_i = 0$.

$\mathcal{M}_{g,n} \longleftarrow$ moduli space of genus g curves with n distinct markings

Define an algebraic locus

$$Z_{g,A} \subset \mathcal{M}_{g,n} = \left\{ (C, x_1, \ldots, x_n) \;\middle|\; \sum_{i}^{n} a_i x_i \sim 0 \right\}$$

Expect $\mathrm{codim}_{\mathbb{C}} = g$ true (except in degenerate case)

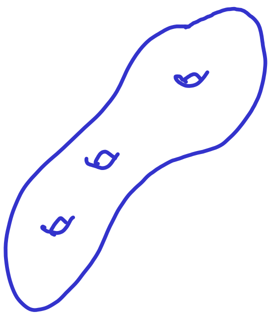
$$\dim Z_{g,A} = 3g - 3 + n - g = 2g - 3 + n$$

We can approach $Z_{g,A}$ via Abel-Jacobi:

$AJ(C, p_1, \dots p_n) = \mathcal{O}_C(\Sigma a_i p_i)$

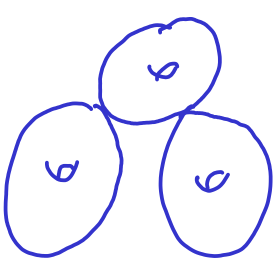
$$\begin{tikzcd} & \mathrm{Jac} \arrow[d] \\ \mathcal{M}_{g,n} \arrow[ur, "AJ"] \arrow[r, "\text{forget markings}"'] & \mathcal{M}_g \arrow[u, bend right, "O_{sect}"'] \end{tikzcd}$$

Smooth Curves



$$Z_{g,A} = AJ^{-1}(O_{sect}) \subset \mathcal{M}_{g,n}$$

Stable curve of compact type



$$\begin{tikzcd} & \mathrm{Jac} \arrow[d] \\ \mathcal{M}^{ct}_{g,n} \arrow[ur, "AJ"] \arrow[r, "\text{forget markings}"'] & \mathcal{M}^{ct}_g \arrow[u, bend right, "O_{sect}"'] \end{tikzcd}$$

$$[Z^{ct}_{g,A}] = AJ^*([O_{sect}]) = AJ^*\left(\frac{1}{g!}\Theta^g\right)$$

$[Z^{ct}_{g,A}] \in H^{2g}(\mathcal{M}^{ct}_{g,n})$

Hain

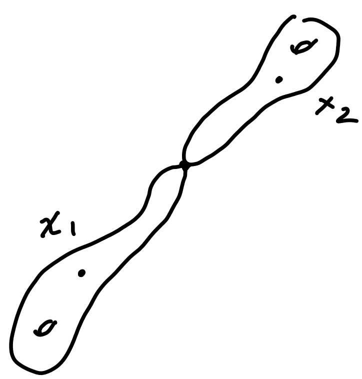
Grushevsky - Zakharov

The departure from Abel-Jacobi theory occurs when we ask the question for $\overline{\mathcal{M}}_{g,n}$

"What is $[Z_{g,A}] \in H^{2g}(\overline{\mathcal{M}}_{g,n})$"

Not well defined since the definition of $Z_{g,A}$ is not clear.

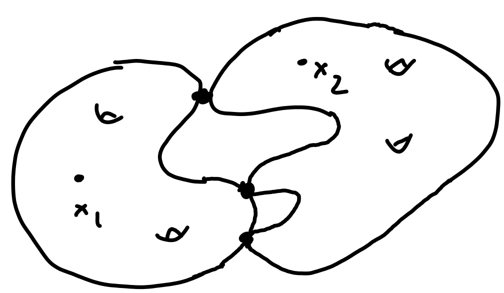
Why? $A = (2,-2)$



what does $2x_1 - 2x_2 \sim 0$ mean?

(a good answer in terms of twists)

Harder



(really not clear)

Some guiding questions:

Q1) Define and calculate $[Z_{g,A}] \in H^{2g}(\overline{\mathcal{M}}_{g,n})$.

Different approaches to $Z_{g,A}$ have led to the same class

Li-Ruan
J. Li
Graber-Vakil

log curves
Gross-Siebert
Marcus-Wise

Gromov Witten theory $\longrightarrow$ $[Z_{g,A}]$

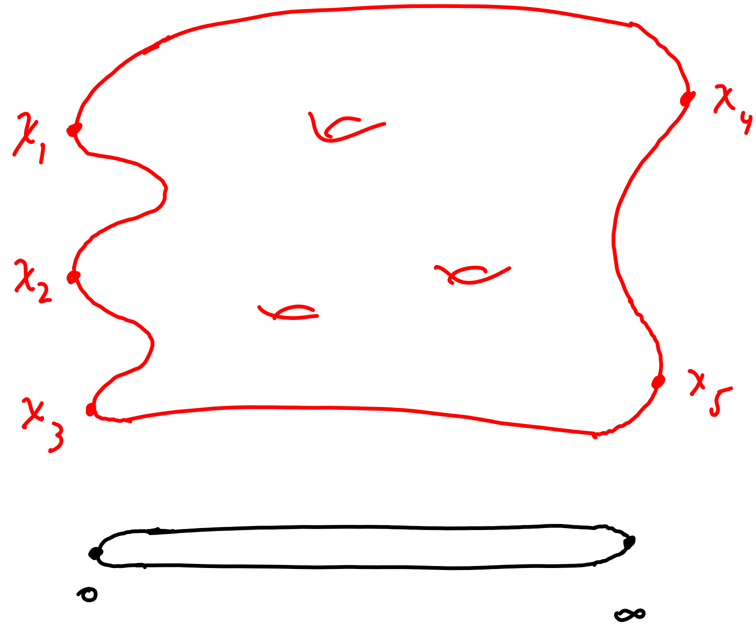
Holmes
Classical resolution of the Abel-Jacobi map

To calculate $[Z_{g,A}]$ is sometimes called Eliashberg's question ~ 2000 SFT

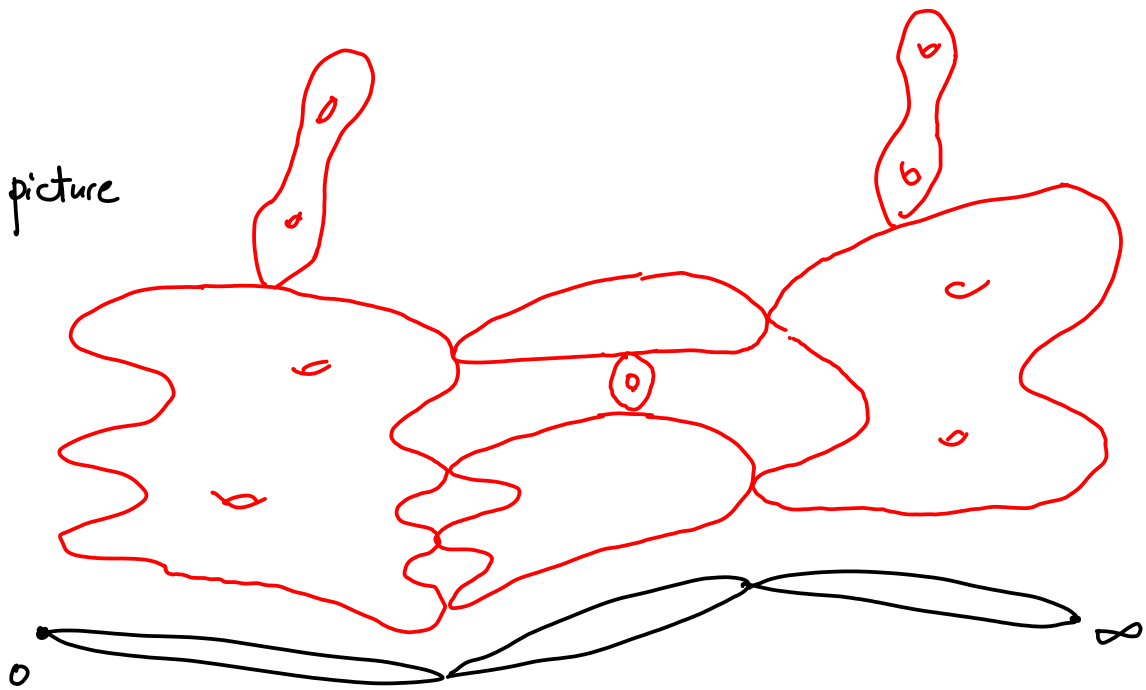
GW theory definition

$\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^{\sim}$ moduli of stable maps to $\mathbb{P}^1$ with ramifications over $0, \infty \in \mathbb{P}^1$ given by A.

ideal element



general picture



$\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^\sim$ has 2 term obstruction theory

$\Rightarrow$ virtual fundamental class

$[\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^\sim]^{vir}$ of dim $2g-3+n$ (no degenerate cases)

$$\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^{\sim} \xrightarrow{\varepsilon} \overline{\mathcal{M}}_{g,n}$$ forgetful map

Definition : $[Z_{g,A}] = \varepsilon_* [\overline{\mathcal{M}}_g(\mathbb{P}^1, A)]^{\sim}$

Double ramification cycle

View $Z_{g,A}$ "=" $\mathrm{Im}(\varepsilon)$ $\quad DR_g(A)$

Now we have a precise mathematical question.

Q2) Let $\begin{array}{c}\mathbb{E}_g \\ \downarrow \\ \overline{\mathcal{M}}_g\end{array}$ be the Hodge bundle with fiber $H^0(C, \omega_C)$ over $[C] \in \overline{\mathcal{M}}_g$

Over $\mathcal{M}_g^{ct}$ we have the universal Jac

$$\mathcal{M}_g^{ct} \xrightarrow{\tau} \mathcal{A}_g \quad \text{moduli of PPAV}$$

$$\mathbb{E}_g = \tau^*(\mathbb{E}_g)$$

GRR calculation $\Rightarrow$ $c_g(\mathbb{E}_g) = 0$ on $\mathcal{A}_g$

Van der Geer, Cycles on the moduli space of Abelian varieties

Conclusion : $\lambda_g = c_g(\mathbb{E}_g) \in CH^g(\overline{M}_g)$ is

supported on $\overline{M}_g \setminus M_g^{ct} = \Delta_0$

Curves with a non disconnecting node

Question: find a formula for λ_g supported on $\Delta_0 \subset \overline{M}_g$.

Q3) Instead of functions, we can consider differentials (or k-differentials)

Let $A = (a_1, \dots, a_n)$ $\sum_{i=1}^{n} a_i = 2g-2$

$$\mathcal{H}_{g,A} \subset M_{g,n} = \left\{ (C, x_1, \dots, x_n) \,\middle|\, \sum a_i x_i \sim \omega_C \right\}$$

Question: Calculate $[\overline{H}_{g,A}] \in H^{2g}(\overline{M}_{g,n})$

$\overline{H}_{g,A}$ is the Zariski closure of $H_{g,A}$

Bainbridge - Chen - Gendron - Grushevsky - Möller
Farkas - P, Schmitt, Sauvaget

The point of this talk is to explain the solution to all 3 questions.

Farkas - P 1508.07940

JPPZ DR 1602.04705

Holmes-Schmitt 1909.11981

Schmitt 1607.08429

JPPZ2 DR(x) 1812.10136

BHPSS Pic 2004.08676

JPPZ = Janda P Pixton Zvonkine

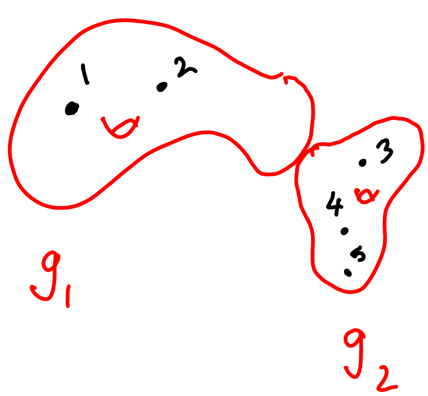
BHPSS = Bae Holmes P Schmitt Schwarz

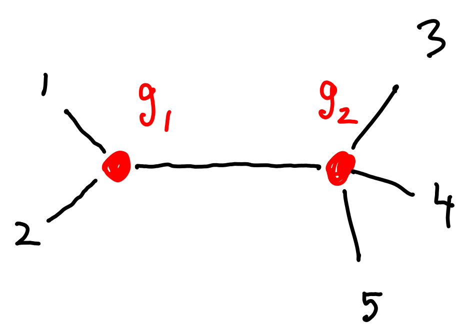
Q1) Eliashberg's question to calculate the double ramification cycle
$$DR_g(A) \in H^{2g}(\bar{M}_{g,n})$$

Answer: Formula conjectured by Pixton (and proven in JPPZ).

Expressed in terms of tautological classes

Topological type $\longleftrightarrow$ Stable graph





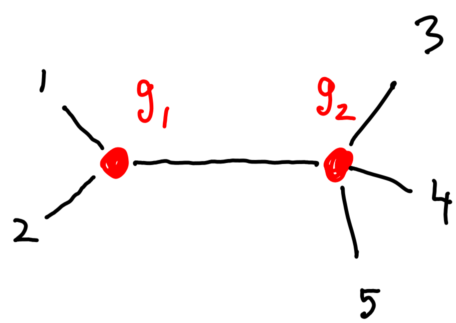
Let $G_{g,n}$ = set of stable graphs of genus g with n markings

finite set

for $\Gamma \in G_{g,n}$ $\rightsquigarrow$ $\overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$

product of moduli spaces determinded by the vertices

for Γ =



$$\overline{M}_\Gamma = \overline{M}_{g_1,3} \times \overline{M}_{g_2,4}$$

Tautological classes are given by

$$\xi_{\Gamma *}\Big(\prod_{\text{markings}} \psi_i^{m_i} \prod_{\substack{\text{halves}\\ \text{of edges}}} \psi_j^{n_j} \prod_{\text{vertices}} \text{kappas} \Big) \in H^*(\overline{M}_{g,n})$$

$$\overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$$

The linear span of all such classes defines the tautological ring

$$RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$$

We can also define the tautological ring

$$R^*(\overline{M}_{g,n}) \subset CH^*(\overline{M}_{g,n}).$$

Let $\Gamma \in G_{g,n}$ be a stable graph.

Let r be a positive integer

A weighting mod r of Γ is

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\}$$

half edges

Remember

$A = (a_1, \ldots, a_n)$

$\sum a_i = 0$

(I) $i \in$ Marking, $w(i) = a_i \mod r$

(II) $e = (h, h') \in$ Edge, $w(h) + w(h') = 0 \mod r$

(III) $v \in$ Vertex, $\sum_{h \vdash v} w(h) = 0 \mod r$

$W_{\Gamma, r}$ is set of weightings mod r of Γ

$|W_{\Gamma,r}| = r^{h^1(\Gamma)}$

Definition (Pixton)

Let $P_g^{d,r}(A) \in R^d(\overline{M}_{g,n})$ be the degree d component of

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h'(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\left[\prod_{1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i\right) \cdot \prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2} \cdot (\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}}\right]$$

Various motivations: Compact type restriction, Givental - Teleman theory

Claim (Pixton):

$P_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ is polynomial in r for all $r >> 0$.

Definition (Pixton):

$P_g^{d}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ is the constant term of $P_g^{d,r}(A)$

$r=0$

Theorem (Conjectured by Pixton, proven in JPPZ)

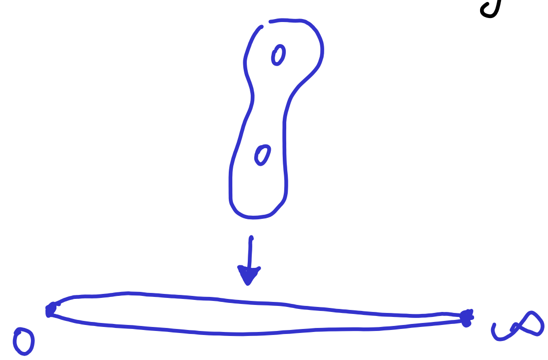
$$DR_g(A) = P_g^g(A) \in R^g(\overline{\mathcal{M}}_{g,n}).$$

Q2) formula for $\lambda_g = c_g(\mathbb{E}_g)$ supported on $\Delta_0 \subset \overline{M}_g$

Answer: We view $\overline{M}_g = \overline{M}_{g,0}$ $\leftarrow n=0$

Let $A = \phi$

Geometry $\Rightarrow$ $\overline{M}_g(\mathbb{P}^1, \phi)^{\sim} \cong \overline{M}_g$



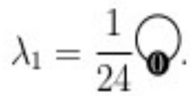
Moreover, $[\overline{M}_g(\mathbb{P}^1, \phi)^{\sim}]^{vir} = (-1)^g \lambda_g$

$$DR_g(\phi) = (-1)^g \lambda_g$$

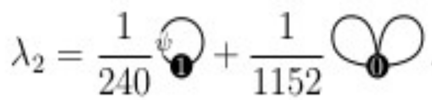
So we can apply the DR cycle Formula:

from JPPZ

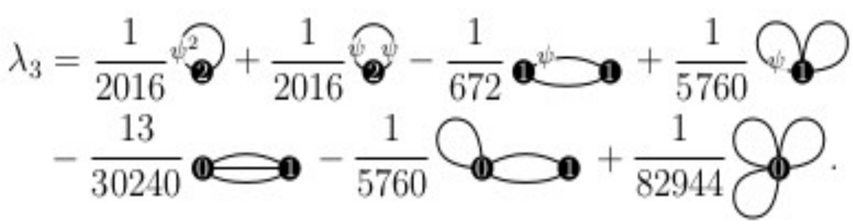
Genus 1.



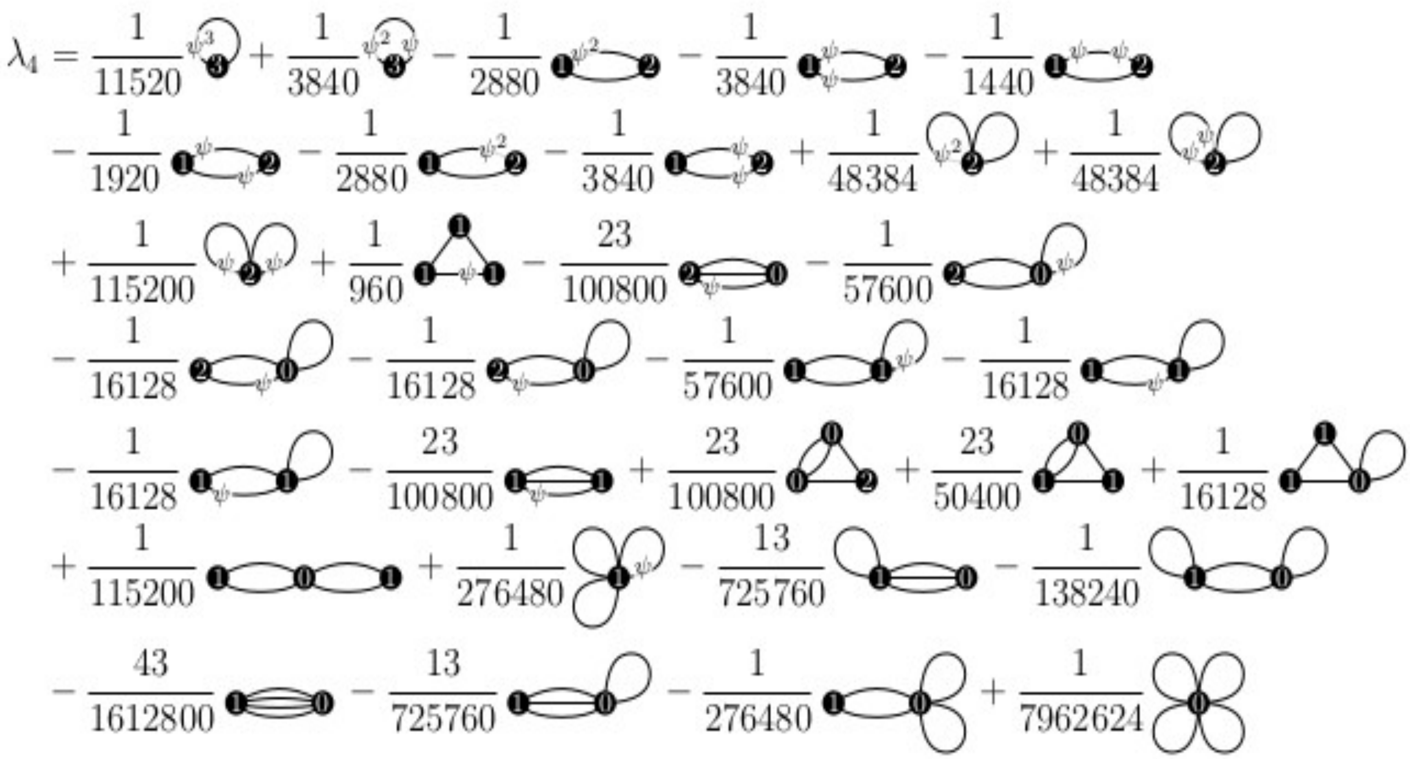
Genus 2.



Genus 3.



Genus 4.



Why only graphs with loops?

If Γ is an unpointed tree, then $w = 0$.

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{\mathrm{Aut}(\Gamma)} \overbrace{\frac{1}{r^{h^1(\Gamma)}}}^{1} \cdot \xi_{\Gamma *}\left[\prod_{e=(h,h')} \underbrace{\frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}}_{0} \right]$$

The formula has no contributions in degree ≥ 1!

$0 \quad 0$

Can have no nodes

Ideas in the proof of DR formula

(A) Pixton conjectured the complete formula.

(B) $P_g^{d,r}(A)$ is similar to a GRR calculation on $\overline{\mathcal{M}}_{g,A}(B\mathbb{Z}_r)$ for $r \gg 0$

orbifold GW theory (Ruan)

The expressions are not equal but differ by higher powers of r, so have the same constant term.

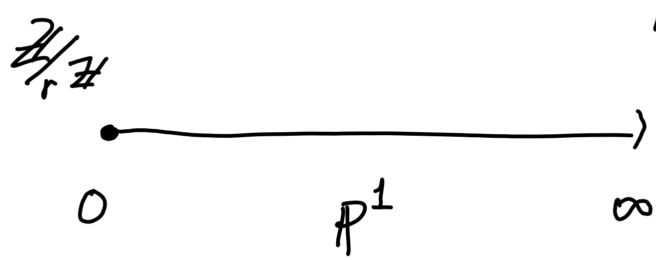
(C) Search for a target geometry in which both

$$\overline{\mathcal{M}}_{g,A}(B\mathbb{Z}_r) \quad \text{and} \quad \overline{\mathcal{M}}_g(\mathbb{P}^1, A)^{\sim}$$

play a role.

Gromov Witten theory of the target

relative / log



(D) Use virtual localization formula in the limit $r >> 0$ to prove the DR cycle formula $\square$

Graber–P

for more details see JPPZ 1602.04705

Q3) Differentials (or k-differentials)

So far we have discussed the cycle defined by the condition

$$\mathcal{O}_C\left(\sum a_i p_i\right) \cong \mathcal{O}_C .$$

We would like now to consider

$$\mathcal{O}_C\left(\sum a_i p_i\right) \cong \omega_C$$

Idea is to develop a much more general perspective:

Universal twisted DR cycle on the Picard stack

BHPSS 2004.08676

We return to (Q3).

Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$

$$\sum_{i=1}^{n} a_i = 2g-2$$

Let $\mathcal{H}_{g,A} \subset \mathcal{M}_{g,n} \ni [C, p_1, \dots, p_n]$

be the locus satisfying

$$\mathcal{O}_C\left(\sum a_i p_i\right) \cong \omega_C .$$

assume $\exists\, a_i < 0$ strictly meromorphic case

$\mathcal{H}_{g,A}$ is nonsingular of pure codimension $\underline{g}$

Polishchuk

For $k=s-P$: Define $\widetilde{\mathcal{H}}_{g,A} \subset \overline{\mathcal{M}}_{g,n}$

not closure but carries equivalent cycle data

Question: Calculate $[\widetilde{\mathcal{H}}_{g,A}] \in A^g(\overline{\mathcal{M}}_{g,n})$

Answer (Conj JPPZ 2016, Theorem BHPSS 2020)

Take $r=0$ limit of codim g part of:

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h'(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\Bigg[\prod_{v \in \mathrm{Vert}} \exp(-\kappa_1(v)) \cdot \prod_{i=1}^{n} \exp\left(\frac{(a_i+1)^2}{2}\psi_i\right)$$

$$\cdot \prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg]$$

The definitions are all the same

except:

(III) $v \in \text{Vertex}, \quad \sum_{h \vdash v} w(h) = 0 \mod r$

becomes:

(III) $v \in \text{Vertex}, \quad \sum_{h \vdash v} w(h) = 2g(v) - 2 + n(v) \mod r$

degree of
the restriction of
the dualizing sheaf

The End