Gauge theory for string algebroids

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Joint work with Rubio and Tipler arXiv:2004.11399

The mathematical study of the **Hull-Strominger system** was initiated by Fu, Li, Tseng, and Yau more than 15 years ago.

$$F \wedge \omega^2 = 0 \qquad F^{2,0} = F^{0,2} = 0$$
$$d(\|\Omega\|\omega^2) = 0 \qquad dd^c\omega - \operatorname{tr} R \wedge R + \operatorname{tr} F \wedge F = 0$$

- Li, Yau, JDG 70 (2005).
- Fu, Yau, JDG 78 (2008).
- Fu, Tseng, Yau, CMP 289 (2009).

Basic ingredients:

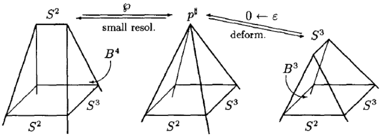
- A hermitian form ω on a Calabi-Yau threefold (X, Ω) (possibly non-Kähler).
- A unitary connection A on a bundle over X, with curvature $F = F_A$.
- A connection ∇ on $T\underline{X}$, with curvature $R = R_\nabla$

Due to its origins in heterotic string theory, ∇ is often required to be Hermite-Yang-Mills:

$$R \wedge \omega^2 = 0, \qquad R^{2,0} = R^{0,2} = 0.$$

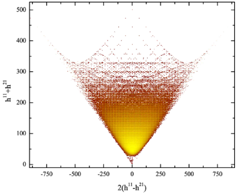
- Strominger, Nucl. Phys. B 274 (1986).
- Hull, Turin 1985 Proceedings (1986).

These equations provide a promising approach to the geometrization of *transitions* and *flops* in the passage from Kähler to non-Kähler Calabi-Yau three-folds ($\sim$ *Reid's Fantasy*) ...



- M. Reid, Math. Ann. **278** (1987) 329--334

... and relate to a conjectural generalization of *mirror symmetry* and *GW* theory, where the Calabi-Yau is endowed with a bundle E such that $c_2(E) = c_2(X)$.



- Melnikov, Plesser, *A* $(0,2)$*-mirror map*, JHEP 1102 (2011) • Donagi, Guffin, Katz, Sharpe, Asian J. Math. 18 (2014) • Garcia-Fernandez, Crelle's J. (2000)

The Hull-Strominger system has been an active topic of research in differential geometry and mathematical physics

• DG: Yau, Li, Fu, Tseng, Fernandez, Ivanov, Ugarte, Villacampa, Grantcharov, Fino, Vezzoni, Andreas, GF, Rubio, Tipler, Fei, Phong, Picard, Zhang, Shahbazi, ... • Hep-th: De la Ossa, Svanes, Anderson, Gray, Sharpe, Ashmore, Minasian, Strickland-Constable, Waldram, Tennyson, Candelas, McOrist, Larfors, ...

To the present day, there are two alternative approaches to the existence and uniqueness problem on a (non-Kähler) Calabi-Yau three-fold (X, Ω):

- Anomaly flow

$$\partial_t(\|\Omega\|_\omega \omega^2) = dd^c\omega - \operatorname{tr} R^2 + \operatorname{tr} F^2$$

  - Phong, Picard, Zhang, Math. Z. (2017)

- Dilaton functional

$$\int_X \|\Omega\|_\omega \, \omega^3$$

  - Garcia-Fernandez, Rubio, Shahbazi, Tipler, arXiv:1803.01873, under review
  - Garcia-Fernandez, Rubio, Tipler, TAMS (2020), to appear

- Anomaly flow: fix the *balanced class* $\mathfrak{b} = [\|\Omega\|_\omega \omega^2] \in H^{2,2}_{BC}(X, \mathbb{R})$ and look for solutions of the *Bianchi identity* via:

$$\partial_t(\|\Omega\|_\omega \omega^2) = dd^c\omega - \operatorname{tr} R^2 + \operatorname{tr} F^2$$

$$F \wedge \omega^2 = 0 \qquad F^{2,0} = F^{0,2} = 0$$

$$d(\|\Omega\|\omega^2) = 0 \qquad dd^c\omega - \operatorname{tr} R \wedge R + \operatorname{tr} F \wedge F = 0$$

- Dilaton functional: define Aeppli classes for solutions of the Bianchi identity via Bott-Chern secondary classes BCh_2

$$\mathfrak{a}_1 - \mathfrak{a}_0 = [\omega_1 - \omega_0 - BCh_2] \in H^{1,1}_A(X, \mathbb{R}).$$

'Fix $\mathfrak{a}$', and look for conformally balanced and Hermite-Einstein via

$$\int_X \|\Omega\|_\omega \, \omega^3$$

- Tseng, Yau, Generalized cohomologies and supersymmetry, Comm. Math. Phys. **326** (2014).

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The aim of this talk is to reconcile to some extent these two alternative approaches, via the duality pairing

$$H_A^{1,1}(X) \otimes H_{BC}^{2,2}(X) \to \mathbb{C}$$

and a moment map picture for the moduli space metric.

- GF, Rubio, Tipler, Gauge theory for string algebroids, arXiv:2004.11399

The Calabi problem

(through the eyes of the heterotic string)

In the 1950s, E. Calabi asked the question of whether one can prescribe the volume form μ of a Kähler metric on a compact complex manifold X.

For metrics on a fixed Kähler class $[\omega_0] \in H^{1,1}(X, \mathbb{R})$, the *Calabi Problem* reduces to solve the Complex Monge-Ampère equation for a smooth function φ on X:

$$(\omega_0 + 2i\partial\bar{\partial}\varphi)^n = n!\mu.$$

Theorem (Yau 1977)

Let X be a compact Kähler manifold with smooth volume μ. Then there exists a unique Kähler metric with $\omega^n = n!\mu$ in any Kähler class.

Provided that X admits a holomorphic volume form Ω, taking μ as below reduces the holonomy of the metric to $SU(n)$ (Calabi-Yau metric)

$$\mu = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}.$$

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In 2011, Joel Fine gave a moment map interpretation of the Calabi problem using a gauge theory framework.

> *'Despite the fact that Yau has long since resolved the Calabi conjecture, this moment-map picture does raise interesting questions ...'*

- `Fine, J. Symp. Geom.` **12**`, 2011.`

In this talk, we shall give a different moment map picture, *through the eyes of the heterotic string*.

- `GF, Rubio, Tipler, arXiv:2004.11399`

X compact complex manifold of dimension n, possibly non-Kähler. Define

$$\Omega^{1,1}_{>0} = \{\omega \mid \omega(,J) > 0\} \subset \Omega^{1,1}_{\mathbb{R}}$$

and consider the tangent bundle

$$T\Omega^{1,1}_{>0} = \{(\omega, b)\} \cong \Omega^{1,1}_{>0} \times \Omega^{1,1}_{\mathbb{R}},$$

endowed with the complex structure

$$\mathbf{J}(\dot{\omega}, \dot{b}) = (-\dot{b}, \dot{\omega}).$$

Consider the partial action of the additive group of complex two-forms

$$\begin{aligned} \Omega^2_{\mathbb{C}} \times T\Omega^{1,1}_{>0} &\to T\Omega^{1,1}_{\mathbb{R}} \\ (B, (\omega, b)) &\mapsto (\omega + Re\ B^{1,1}, b + Im\ B^{1,1}). \end{aligned} \tag{1}$$

We study a Hamiltonian action of the subgroup of $i\Omega^2 \subset \Omega^2_{\mathbb{C}}$ for a natural family of Kähler structures on $T\Omega^{1,1}_{>0}$.

To define our family of Kähler structures, we fix a smooth volume form μ on X. For any $\omega \in \Omega^{1,1}_{>0}$, we define the *dilaton function* f_ω by

$$\omega^n = n! e^{2f_\omega} \mu.$$

Definition: Given $\ell \in \mathbb{R}$, the ℓ-*dilaton functional* on $T\Omega^{1,1}_{>0}$ is

$$M_\ell(\omega, b) := \int_X e^{-\ell f_\omega} \frac{\omega^n}{n!}.$$

For any $\ell \neq 2$, there is a pseudo-Kähler structure $\Omega_\ell := -d\mathbf{J}d \log M_\ell$. The associated metric is (the subscript 0 means primitive):

$$\begin{aligned} g_\ell = {} & \frac{2-\ell}{2M_\ell} \int_X (|\dot\omega_0|^2 + |\dot b_0|^2) e^{-\ell f_\omega} \frac{\omega^n}{n!} \\ & + \frac{2-\ell}{2M_\ell} \left(\frac{\ell}{2} - \frac{n-1}{n} \right) \int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega \dot\omega|^2) e^{-\ell f_\omega} \frac{\omega^n}{n!} \\ & + \left(\frac{2-\ell}{2M_\ell} \right)^2 \left(\left(\int_X \Lambda_\omega \dot\omega e^{-\ell f_\omega} \frac{\omega^n}{n!} \right)^2 + \left(\int_X \Lambda_\omega \dot b e^{-\ell f_\omega} \frac{\omega^n}{n!} \right)^2 \right). \end{aligned}$$

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Proposition (GF, Rubio, Tipler)

The $i\Omega^2$-action on $(T\Omega^{1,1}_{>0}, \Omega_\ell)$ is Hamiltonian, with equivariant moment map μ_ℓ.

$$\langle \mu_\ell(\omega, b), iB \rangle = \frac{2-\ell}{2M_\ell} \int_X B \wedge e^{-\ell f_\omega} \frac{\omega^{n-1}}{(n-1)!}.$$

Upon restriction to the subgroup $i\Omega^2_{ex} \subset i\Omega^2$ of imaginary exact 2-forms, zeros of the moment map are given by <u>conformally balanced</u> metrics $d(e^{-\ell f_\omega}\omega^{n-1}) = 0$.

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An interesting upshot of our picture is that the balanced property for a compact complex manifold arises as a balancing condition, analogue to the *zero center of mass*.

Assume X pluriclosed. Fix a real closed three-form $H_{\mathbb{R}} \in \Omega^3$ ($\sim$ NS flux in string theory), $dH_{\mathbb{R}} = 0$, and consider the complex subspace

$$\mathcal{W} = \{(\omega, b) \mid d^c\omega - db = H_{\mathbb{R}}\} \subset T\Omega^{1,1}_{>0}.$$

Observe: $i\Omega^2_{ex}$ are *symmetries* for $H_{\mathbb{R}}$.

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$$d(e^{-\ell f_\omega}\omega^{n-1}) = 0, \qquad dd^c(\omega + ib) = 0. \tag{2}$$

Observe: Equation (2) implies ω Kähler, $d\omega = 0$, and $f_\omega = const$.

Thus, the symplectic reduction $\mathcal{M}_\ell = \mu_\ell^{-1}(0)/i\Omega^2_{ex}$ can be identified with the moduli space of (complexified) solutions of the Calabi problem on X (for varying $c \in \mathbb{R}$)

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Assume for a moment that we do not know '$\mu_\ell = 0 \Rightarrow$ Kähler': then, any solution of our moment map equations

$$\mu_\ell = 0 \qquad \Leftrightarrow \qquad d(e^{-\ell f_\omega}\omega^{n-1}) = 0 \qquad dd^c(\omega + ib) = 0 \qquad (3)$$

has attached natural Bott-Chern and Aeppli classes:

$$\mathfrak{b} := [e^{-\ell f_\omega}\omega^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R}), \qquad \mathfrak{a} := [\omega + ib] \in H^{1,1}_A(X).$$

Recall:

$$H^{p,q}_{BC}(X) = \frac{ker\ d}{Im\ \partial\bar{\partial}} \qquad H^{p,q}_A(X) = \frac{ker\ \partial\bar{\partial}}{Im\ \partial \oplus \bar{\partial}}$$

and there is a duality pairing

$$H^{1,1}_A(X) \otimes H^{n-1,n-1}_{BC}(X) \to \mathbb{C}$$

Theorem (GF, Rubio, Tipler)

The moduli space $\mathcal{M}_\ell := \mu_\ell^{-1}(0)/i\Omega^2_{ex}$ of (complexified) solutions of the Calabi problem on X inherits a Kähler structure with metric

$$g_\ell = \frac{2-\ell}{2M_\ell}\left(\frac{2-\ell}{2M_\ell}(Re\, \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Re\, \dot{\mathfrak{a}} \cdot Re\, \dot{\mathfrak{b}} + \frac{2-\ell}{2M_\ell}(Im\, \dot{\mathfrak{a}} \cdot \mathfrak{b})^2 - Im\, \dot{\mathfrak{a}} \cdot Im\, \dot{\mathfrak{b}}\right)$$

and Kähler potential $-\log M_\ell$. Here, $\dot{\mathfrak{b}} \in H^{n-1,n-1}_{BC}(X)$ and $\dot{\mathfrak{a}} \in H^{1,1}_A(X)$ are 'complexified variations' of $\mathfrak{a}, \mathfrak{b}$ obtained via *gauge fixing* and $\cdot$ is the pairing

$$H^{1,1}_A(X) \otimes H^{n-1,n-1}_{BC}(X) \to \mathbb{C}.$$

Remark: by Yau's Theorem $\mathcal{M}_\ell \subset H^{1,1}_A(X) \cong H^{1,1}(X)$.

Remark: When (X, Ω) is a Calabi-Yau three-fold and we take

$$\mu = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}$$

we recover the Weil-Petersson metric on the complexified Kähler moduli of X.

- Candelas, De la Ossa, Moduli space of Calabi-Yau manifolds, Nuclear Phys. **B 355** (1991)

String algebroids, moment maps, and the Hull-Strominger system

M smooth manifold. K compact Lie group.

Definition: A string algebroid with structure group K is given by data (plus axioms):

- a principal K-bundle $P_K \to M$,
- a sequence $0 \to T^*M \to E_K \to A_{P_K} \to 0$ of smooth vector bundles, where $A_{P_K} := TP_K/K$ is the Atiyah algebroid,
- smooth metric $\langle \cdot, \cdot \rangle$ on E_K,
- bracket $[\cdot, \cdot]$ on E_K.

Remark: special class of Courant algebroids $\sim$ Generalized geometry (*à la Hitchin*).

Motivation: for $K = Spin(r)$, a string algebroid can be understood as the Atiyah Lie algebroid of a $String(r)$-principal bundle

$$String(r) \longrightarrow Spin(r) \longrightarrow SO(r) \longrightarrow O(r),$$

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Idea: string algebroids are constructed via a gluing procedure using gauge transformations of 'string principal bundles'.

M smooth manifold, K compact Lie group with quadratic Lie algebra $(\mathfrak{k}, \langle , \rangle_{\mathfrak{k}})$, and Cartan 3-form on K

$$\sigma = -\frac{1}{6}\langle \cdot, [\cdot, \cdot]\rangle_{\mathfrak{k}}.$$

Consider the smooth sheaf $\mathcal{S}_K$ of non-abelian groups ($U \subset M$ open)

$$\mathcal{S}_K(U) = \{(B, g) \in \Omega^2(U) \times C^\infty(U, K) \textit{ satisfying } dB = g^*\sigma\}.$$

- Sěvera, Letters to Alan Weinstein, 1998-2000, arXiv:1707.00265

A 1-cocycle for the sheaf $\mathcal{S}_K$ defines a string algebroid E_K by gluing, via its action on $TU \oplus \mathfrak{k} \oplus T^*U$ with Courant structure

$$\begin{aligned}\langle X + r + \xi, Y + r + \xi\rangle &= i_X\xi + \langle r, r\rangle_{\mathfrak{k}}\\ [X + r + \xi, Y + t + \eta] &= [X, Y] + i_X dt - i_Y dr + [r, t]_{\mathfrak{k}}\\ &\quad + L_X\eta - i_Y d\xi + 2\langle dr, t\rangle_{\mathfrak{k}}.\end{aligned}$$

Example: ($\sim$ Hull-Strominger) $K = \mathrm{SU}(3) \times \mathrm{SU}(k)$, and

$$\langle , \rangle_{\mathfrak{k}} = \mathrm{tr}_{\mathfrak{su}(3)} - \mathrm{tr}_{\mathfrak{su}(k)}.$$

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We aim for an Atiyah-Bott-Donaldson moment map picture for string algebroids ... look for *Chern correspondence.*

Let X be a compact complex manifold with underlying smooth manifold $\underline{X}$ and let E_K a string algebroid over X.

$$E_K \cong T\underline{X} \oplus \mathrm{ad}\, P_K \oplus T^*\underline{X}.$$

There is a *complexification* $E := E_K \otimes \mathbb{C}$.

Definition: A lifting $L \subset E$ of $T^{0,1}X$ is an isotropic, involutive subbundle $[L, L] \subset L$, mapping isomorphically to $T^{0,1}X$ under $\pi \colon E \to T\underline{X} \otimes \mathbb{C}$.

Remark: L is the analogue of a *Dolbeault operator*.

Lemma (GF-Rubio-Tipler): A lifting $L \subset E$ of $T^{0,1}X$ determines a *holomorphic string algebroid* $Q_L = L^\perp / L$, where $L^\perp$ is the orthogonal complement of $L \subset E$.

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X complex manifold of dimension n, E_K string algebroid.

Proposition (Chern correspondence) (GF-Rubio-Tipler)

Given a lifting $L \subset E = E_K \otimes \mathbb{C}$, there exists a unique rank $2n$ (real) subbundle $W \subset E_K$ such that

$$L_W := \{e \in W \otimes \mathbb{C} \mid \pi(e) \in T^{0,1}X\} = L.$$

W determines $\omega \in \Omega^{1,1}_{\mathbb{R}}$ and a classical Chern connection θ^h, such that

$$dd^c\omega + \langle F_h \wedge F_h \rangle = 0.$$

Remark: W is the analogue of the Chern connection in our context.

Motivation: the data (ω, θ^h) seems to be in agreement with structural properties of connections in *higher gauge theory*.

• Sämann, Wolf, Lett. Math. Phys. 104 (2014)

X complex manifold of dimension n, E_K string algebroid.

Proposition (Chern correspondence) (GF-Rubio-Tipler)

Given a lifting $L \subset E = E_K \otimes \mathbb{C}$, there exists a unique rank $2n$ (real) subbundle $W \subset E_K$ such that

$$L_W := \{e \in W \otimes \mathbb{C} \mid \pi(e) \in T^{0,1}X\} = L.$$

W determines $\omega \in \Omega^{1,1}_{\mathbb{R}}$ and a classical Chern connection θ^h, such that

$$dd^c\omega + \langle F_h \wedge F_h \rangle = 0.$$

Remark: W is the analogue of the Chern connection in our context.

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X compact complex manifold of dimension n, possibly non-Kähler. E_K string algebroid over X. Define

$$\mathcal{W} = \{W \subset E_K \mid [L_W, L_W] \subset L_W, \quad \omega(,J) > 0\}.$$

Remark: Upon fixing $W \in \mathcal{W}$, we can regard $\mathcal{W} \subset \Omega^2_{\mathbb{C}} \times \mathcal{A}$ and there is a fibration structure

$$\mathcal{W} \to \mathcal{A}^0 = \{\theta \text{ connection on } P_K \mid F_\theta^{0,2} = 0\}.$$

From the *Chern correspondence*, $\mathcal{W}$ inherits a natural complex structure preserved by an action of $\mathrm{Aut}(E_K)$

$$\mathrm{Aut}(E_K) \times \mathcal{W} \to \mathcal{W}. \tag{4}$$

We study a Hamiltonian action of a subgroup of $\mathcal{H} \subset \mathrm{Aut}(E_K)$ for a natural family of Kähler structures on $\mathcal{W}$.

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Fix a smooth volume form μ on X. For any $\omega \in \Omega^{1,1}_{>0}$, define the *dilaton function* f_ω by $\omega^n = n! e^{2f_\omega}\mu$.

Definition: Given $\ell \in \mathbb{R}$, the ℓ-*dilaton functional* on $\mathcal{W}$ is

$$M_\ell(W) := \int_X e^{-\ell f_\omega} \frac{\omega^n}{n!}.$$

For any $\ell \neq 2$, there is a pseudo-Kähler structure $\Omega_\ell := -d\mathbf{J}d \log M_\ell$. The associated metric is (the subscript 0 means primitive):

$$\begin{aligned} g_\ell = {} & \frac{\ell-2}{M_\ell}\int_X \langle \dot\theta \wedge J\dot\theta \rangle \wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\ & + \frac{2-\ell}{2M_\ell}\int_X (|\dot\omega_0|^2 + |\dot b_0|^2) e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ & + \frac{2-\ell}{2M_\ell}\left(\frac{\ell}{2} - \frac{n-1}{n}\right)\int_X (|\Lambda_\omega \dot b|^2 + |\Lambda_\omega \dot\omega|^2) e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ & + \left(\frac{2-\ell}{2M_\ell}\right)^2 \left(\left(\int_X \Lambda_\omega \dot\omega e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2 + \left(\int_X \Lambda_\omega \dot b e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2\right). \end{aligned}$$

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Proposition (GF, Rubio, Tipler)

The $\mathrm{Aut}(E_K)$-action on $(\mathcal{W}, \Omega_\ell)$ is Hamiltonian. Upon restriction to a normal subgroup $\mathcal{H} \subset \mathrm{Aut}(E_K)$, zeros of the moment map μ_ℓ are given by solutions of the *Calabi system*

$$\begin{aligned} F_\theta \wedge \omega^{n-1} &= 0, & F_\theta^{0,2} &= 0, \\ d(e^{-\ell f_\omega}\omega^{n-1}) &= 0, & dd^c\omega + \langle F_\theta \wedge F_\theta \rangle &= 0. \end{aligned} \tag{5}$$

Observe: $W \in \mathcal{W}$ determines an isomorphism

$$\mathrm{Lie}\,\mathrm{Aut}(E_K) = \{(u, B) \mid d(B - 2\langle u, F_\theta \rangle) = 0\} \subset \Omega^0(\mathrm{ad}\, P_K) \times \Omega^2.$$

The normal Lie subalgebra $\mathrm{Lie}\,\mathcal{H}$ is given by $B^{1,1} - 2\langle u, F_\theta \rangle \in \ Im\, \partial \oplus \bar{\partial}$.

Remark: When (X, Ω) is a Calabi-Yau three-fold and we take $\ell = 1$ and

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An interesting upshot of our *gauge theory picture* is that the Hull-Strominger system is characterized as a moment map condition.

- GF, Rubio, Tipler, Gauge theory for string algebroids, arXiv:2004.11399

$$F \wedge \omega^2 = 0 \qquad F^{2,0} = F^{0,2} = 0$$

$$d(\|\Omega\|\omega^2) = 0 \qquad dd^c\omega - \operatorname{tr} R \wedge R + \operatorname{tr} F \wedge F = 0$$

Remark: An alternative moment map construction 'moving the complex structure' on X has been provided very recently in the physics literature.

- A. Ashmore, C. Strickland-Constable, D. Tennyson, D. Waldram, arXiv:1912.09981.

The moduli space metric

X compact complex manifold of dimension n, possibly non-Kähler, with fixed volume form μ. E_K string algebroid over X.

The symplectic reduction $\mathcal{M}_\ell := \mu_\ell^{-1}(0)/\mathcal{H}$ of the space of horizontal subbundles $W \subset E_K$ is the moduli space of solutions of the Calabi system:

$$\begin{aligned} F_\theta \wedge \omega^{n-1} &= 0, & F_\theta^{0,2} &= 0, \\ d(e^{-\ell f_\omega}\omega^{n-1}) &= 0, & dd^c\omega + \langle F_\theta \wedge F_\theta \rangle &= 0. \end{aligned} \tag{6}$$

Definition: We say that $[W] \in \mathcal{M}_\ell$ satisfies **Condition A** if the linearization of (6) at W along the 'Aeppli class' has kernel $\mathrm{Lie\,Aut}\, Q_{L_W}$.

Observe: fixing the *holomorphic string algebroid* Q_{L_W}, Aeppli classes for solutions of the Bianchi identity are defined via Bott-Chern secondary classes BCh_2

$$\mathfrak{a}_1 - \mathfrak{a}_0 = [\omega_1 - \omega_0 - BCh_2] \in H_A^{1,1}(X, \mathbb{R}).$$

Kähler analogue of Condition A: solutions of the linearized cscK equation in a fixed Kähler class are Hamiltonian Killing vector fields.

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Assume $h_A^{0,1}(X) = 0$, $h_{\bar\partial}^{0,2}(X) = 0$, and **Condition A**. The moduli space of solutions of the *Calabi system* on X inherits a pseudo-Kähler structure with metric g_ℓ and Kähler potential $-\log M_\ell$

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When (X, Ω) is a Calabi-Yau three-fold and we take $\ell = 1$ and $\mu = i\Omega \wedge \overline{\Omega}$, we obtain the moduli space metric for Hull-Strominger, with Kähler potential

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-\log \int_X \|\Omega\|_\omega \omega^3.
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As a consequence of our result, the Kähler potential on the moduli space of solutions of the Hull-Strominger system is given by

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Our formula for the Kähler potential shall be compared with Candelas-De la Ossa-McOrist formula, given by $K_{CDM} = -\log \int_X i\Omega \wedge \overline{\Omega} - \frac{4}{3}\log \int_X \omega^3$.

- Candelas, De la Ossa, McOrist, A Metric for Heterotic Moduli, Comm. Math. Phys. **356** (2017)

Observation (McOrist): both formulas agree to order 0 in α' expansion.

$$K_{CDM} \sim -\log\left(\frac{2}{9}\int_X \|\Omega\|^2\omega^3 \cdot \int_X \omega^3\right) \sim -2\log\left(\int_X \|\Omega\|\omega^3\right) + O(\alpha').$$

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By construction, the moduli space $\mathcal{M}_\ell$ has a natural map to the classical moduli space of holomorphic principal G-bundles $\mathcal{M}_{bundles}$, with $G = K^c$,

$$\mathcal{M}_\ell \to \mathcal{M}_{bundles}.$$

Morally, a 'conformal submersion' for the Atiyah-Bott-Donaldson metric.

$$\begin{aligned} g_\ell = {} & \frac{\ell-2}{M_\ell}\int_X \langle\dot\theta\wedge J\dot\theta\rangle\wedge e^{-\ell f_\omega}\frac{\omega^{n-1}}{(n-1)!} \\ & + \frac{2-\ell}{2M_\ell}\int_X(|\dot\omega_0|^2+|\dot b_0|^2)e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ & + \frac{2-\ell}{2M_\ell}\left(\frac{\ell}{2}-\frac{n-1}{n}\right)\int_X(|\Lambda_\omega\dot b|^2+|\Lambda_\omega\dot\omega|^2)e^{-\ell f_\omega}\frac{\omega^n}{n!} \\ & + \left(\frac{2-\ell}{2M_\ell}\right)^2\left(\left(\int_X\Lambda_\omega\dot\omega e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2+\left(\int_X\Lambda_\omega\dot b e^{-\ell f_\omega}\frac{\omega^n}{n!}\right)^2\right). \end{aligned}$$

Observe: in our picture, ω varies in moduli.

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Theorem (GF, Rubio, Tipler)

Assume $h^{0,1}_A(X) = 0$, $h^{0,2}_{\bar\partial}(X) = 0$, and **Condition A**. Then, the metric g_ℓ along the fibres of (7) is given by

$$g_\ell = \frac{2-\ell}{2M_\ell}\left(\frac{2-\ell}{2M_\ell}(Re\ \dot{\mathfrak{a}}\cdot\mathfrak{b})^2 - Re\ \dot{\mathfrak{a}}\cdot Re\ \dot{\mathfrak{b}} + \frac{2-\ell}{2M_\ell}(Im\ \dot{\mathfrak{a}}\cdot\mathfrak{b})^2 - Im\ \dot{\mathfrak{a}}\cdot Im\ \dot{\mathfrak{b}}\right)$$

Here, $\dot{\mathfrak{b}} \in H^{n-1,n-1}_{BC}(X)$, $\dot{\mathfrak{a}} \in H^{1,1}_A(X)$ are 'complexified variations'–obtained via *gauge fixing*–of the balanced class $\mathfrak{b}$ and the Aeppli class $\mathfrak{a}$ of a solution and $\cdot$ is

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Note: Bott-Chern and Aeppli classes of solutions are defined by

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shows that g_ℓ is 'semi-topological', in the sense that fibre-wise it can be expressed in terms of classical cohomological quantities.

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When (X,Ω) is a Calabi-Yau three-fold and we take $\mu = i\Omega\wedge\overline{\Omega}$ we recover *Strominger's formula* for the metric on the *complexified Kähler moduli* of X.

- Strominger, Phys. Rev. Lett. 55 (1985)

Observe: the formula for the holomorphic prepotential on the complexified Kähler moduli of X, given by the natural cubic form

$$H^{1,1}(X)\to\mathbb{C}\colon [\alpha]\mapsto \int_X \alpha^3 + \textit{quantum corrections}\ (\sim GW),$$

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Infinitesimal Donaldson-Uhlenbeck-Yau

To finish, we discuss the relation between $\mathcal{M}_\ell$ and the moduli space of holomorphic string algebroids over X. This is suggested by the correspondence between solutions of the Hermite-Yang-Mills equations and polystable bundles, given by the *Donaldson-Uhlenbeck-Yau* Theorem

$$F \wedge \omega^{n-1} = 0, \qquad F^{2,0} = F^{0,2} = 0.$$

Let X be a compact complex manifold with underlying smooth manifold $\underline{X}$ and let E_K a string algebroid over X with *complexification* $E := E_K \otimes \mathbb{C}$. By the *Chern correspondence*, there is a diagram of moduli spaces:

$$\begin{array}{ccc} \mathcal{M}_\ell := \mu_\ell^{-1}(0)/\mathcal{H} & \longrightarrow & \mathcal{L}/\mathcal{H}^c \\ & & \downarrow \\ & & \mathcal{L}/\operatorname{Aut}(E) \\ & & \downarrow \\ & & \mathcal{M}_{bundles}, \end{array} \tag{8}$$

where $\mathcal{G}_{P_G}$ is the (smooth) complex gauge group, $\mathcal{L}/\operatorname{Aut}(E)$ is the moduli space of *holomorphic string algebroids* and $\mathcal{L}/\mathcal{H}^c$ is a *Teichmüller space*.

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Consider the map which associates to a solution W of the Calabi system the holomorphic string algebroid $Q := Q_{L_W}$:

$$\mathcal{M}_\ell := \mu_\ell^{-1}(0)/\mathcal{H} \longleftrightarrow \mathcal{L}/\mathcal{H}^c \colon [W] \mapsto [Q].$$

Theorem (GF, Rubio, Tipler)

Assume $h_A^{0,1}(X) = 0$, $h_{\bar\partial}^{0,2}(X) = 0$, and **Condition A**. Then, the tangent to the moduli space $\mathcal{M}_\ell$ at $[W]$ is canonically isomorphic to the tangent to the Teichmüller space $\mathcal{L}/\mathcal{H}^c$ for holomorphic string algebroids at $[Q]$.

This strongly suggests that the existence of solutions should be related to a stability condition for holomorphic string algebroids Q. The precise relation is still unclear, as the balanced class $\mathfrak{b} \in H_{BC}^{n-1,n-1}(X,\mathbb{R})$ of solutions in our setup varies in moduli.

- The conjectural stability condition should be related to properties of the *integral of the moment map* along the ‘Aeppli class’, given by the dilaton functional M_ℓ.

- GF, Rubio, Shahbazi, Tipler, Canonical metrics on holomorphic Courant algebroids, arXiv:1803.01873

- Complex gauge symmetries for string algebroids can be regarded as (isomorphism classes) of Morita equivalences. This points towards a 2-category, which may play an important role in the stability condition

$$\begin{array}{ccccccc} & & & E & & & \\ & & \swarrow & & \searrow & & \\ Q & \longleftarrow & Q_L & & Q_{L'} & \longrightarrow & Q'. \end{array}$$

- GF, Rubio, Tipler, Gauge theory for string algebroids, arXiv:2004.11399

- We speculate that there is a relation between this new form of stability and the following conjectural inequality, motivated by a *Gukov's type formula* for the four-dimensional gravitino mass

$$M_{3/2} = c_0 e^{K/2} W \qquad M_{3/2} = \frac{\sqrt{8} e^{\phi_4} W}{4 \int_X \|\Omega\|_\omega \frac{\omega^3}{6}}.$$

- Gurrieri, Lukas, Micu, Heterotic on half-flat, Phys. Rev. **D70** (2004)

Conjecture (GF, Rubio, Tipler)

Let (X, Ω) be a Calabi-Yau three-fold with bundle P. If (X, Ω, P) admits a solution of the Hull-Strominger system, then the variations of the Aeppli and balanced classes of nearby solutions with fixed bundle must satisfy

$$Re\ \dot{\mathfrak{a}} \cdot Re\ \dot{\mathfrak{b}} < \frac{1}{2 \int_X \|\Omega\|_\omega \frac{\omega^3}{6}} (Re\ \dot{\mathfrak{a}} \cdot \mathfrak{b})^2. \tag{9}$$

Thank you!