QUASI-PARABOLIC HIGGS BUNDLES AND NULL HYPERPOLYGONS

ALESSIA MANDINI (IST & UFF)
joint with L. GODINHO

Geometria em Lisboa
16/06/2020

Plan of the talk:

- Polygon spaces (Haussmann-Knutson, Kapovich-Millson, Godinho-Agapito, M., etc)
- Hyperpolygon spaces (Konno, Harada-Proudfoot, Godinho-M., Fisher-Rayan, Rayan-Schaposnik...)
- Quasi-parabolic & parabolic Higgs bundles (Hitchin, Simpson, Konno, García Prada, Gothen, Oliveira, Biswas, Florentino, Godinho, M, ...)
- Involutions (Hitchin, Baraglia-Schaposnik, Jardim, Marchesi, Franco, BFGM, ...)

MODULI SPACE OF POLYGONS in $\mathbb{R}^3$

$\boxed{n \geq 3}$

volume form

Fix $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{R}^n_{>0}$, let $(S^2_{\alpha_i}, \omega_i)$ be the sphere

$$SO(3) \overset{\text{Ham}}{\circlearrowright} \left(\prod_{i=1}^{n} S^2_{\alpha_i}, \omega \right) \xrightarrow{\mu} so(3)^* \simeq \mathbb{R}^3$$

$$(v_1, \dots, v_n) \longmapsto \sum_{i=1}^{n} v_i$$

MODULI SPACE OF POLYGONS in $\mathbb{R}^3$

Fix $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{R}^n_{>0}$, let $(S^2_{\alpha_i}, \omega_i)$ be the sphere

volume form

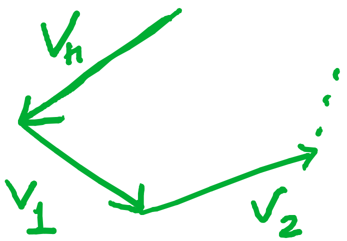
$$SO(3) \curvearrowright \left(\prod_{i=1}^{n} S^2_{\alpha_i}, \omega \right) \xrightarrow{\ \mu\ } so(3)^* \simeq \mathbb{R}^3$$

$$(v_1, \dots, v_n) \longmapsto \sum_{i=1}^{n} v_i$$

$$M(\alpha) = \prod S^2_{\alpha_i} /\!/_{0}\, SO(3) = \mu^{-1}_{SO(3)}(0) / SO(3)$$

POLYGON SPACE

$(v_1, \dots, v_n) \in \mu^{-1}_{SO(3)}(0) \iff$



$v_i \in \mathbb{R}^3$

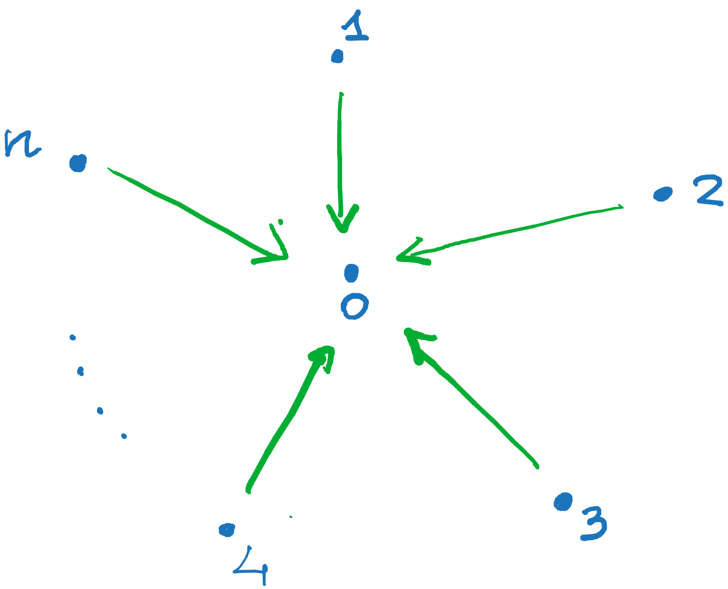
$\|v_i\| = \alpha_i$

$M(\alpha)$ smooth kähler $\iff \varepsilon_I(\alpha) := \sum_{i \in I} \alpha_i - \sum_{j \in I^c} \alpha_j \neq 0 \quad \forall I$

α GENERIC

POLYGON SPACES AS QUIVER VARIETIES

Star-shaped quiver



$V_0 = \mathbb{C}^2$, $V_i = \mathbb{C}$ $\forall i = 1, \dots, n$

$$\mathrm{Rep}\, Q = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathbb{C}^{2n}$$

$q_1, \dots, q_n$

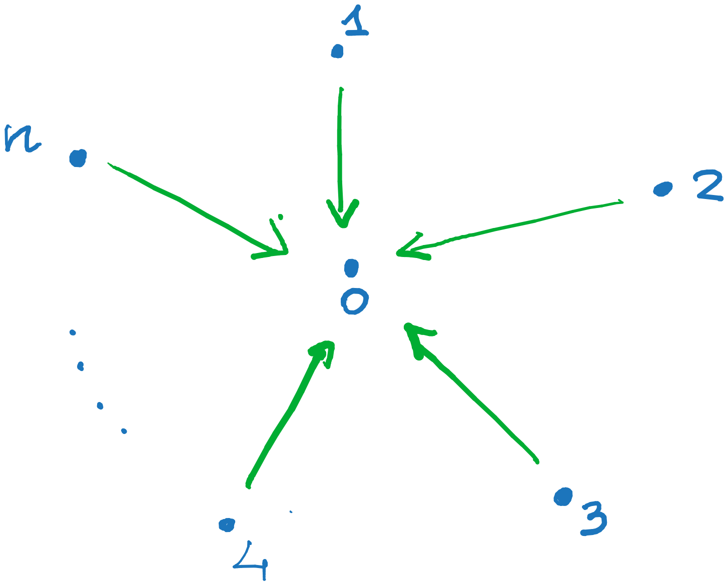
$q_i : \mathbb{C} \longrightarrow \mathbb{C}^2$

$$K = U(2) \times U(1)^n \big/ \Delta$$

diagonal S^1 in $U(2) \times U(1)^n$

POLYGON SPACES AS QUIVER VARIETIES

Star-shaped quiver



$V_0 = \mathbb{C}^2 \;,\quad V_i = \mathbb{C} \quad \forall i = 1, \dots, n$

$$\operatorname{Rep} Q = \bigoplus_{i=1}^{n} \operatorname{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathbb{C}^{2n}$$

$q_1, \dots, q_n$

$q_i : \mathbb{C} \longrightarrow \mathbb{C}^2$

$$K = U(2) \times U(1)^n \big/ \Delta$$

diagonal S^1 in $U(2) \times U(1)^n$

$K \curvearrowright \mathbb{C}^{2n}$ by $[q](A, \lambda_1, \dots \lambda_n) = [\dots, A^{-1} q_i \lambda_i, \dots]$ with moment map

$$\mu(q_1, \dots, q_n) = \left(\frac{\sqrt{-1}}{2} \sum_{i=1}^{n} (q_i q_i^*)_0 \,, \dots, \frac{1}{2}|q_i|^2, \dots \right)$$

$$\mathbb{C}^{2n} /\!\!/_{(0,\alpha)} K = M(\alpha) = \mu^{-1}(0,\alpha) \big/ K$$

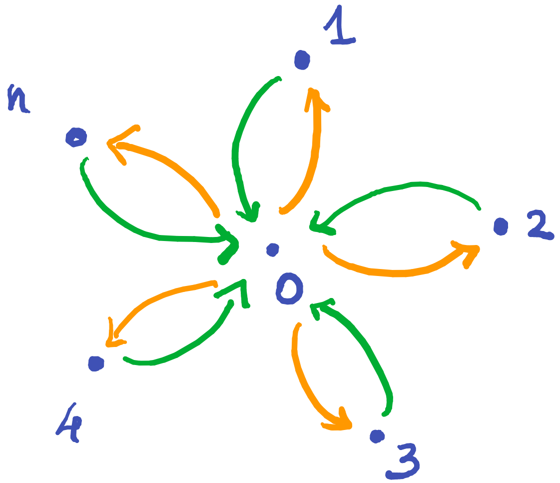
$\mathbb{R}^3 \cong \mathfrak{su}(2)^*$

CLOSING CONDITION $\sum v_i = 0$

LENGTH OF v_i $= \alpha_i$

$[q_1, \dots q_n] \longrightarrow [v_1, \dots v_n]$ where $v_i = (q_i q_i^*)_0$

HYPERPOLYGON SPACES



$$V_0 = \mathbb{C}^2 \qquad V_i = \mathbb{C} \qquad \forall\ i = 1, \dots n$$

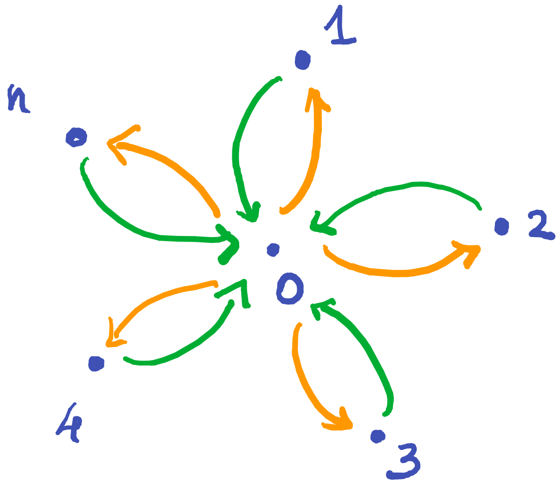
$$\operatorname{Rep} Q = \bigoplus_{i=1}^{n} \operatorname{Hom}(\mathbb{C}, \mathbb{C}^2) \bigoplus_{i=1}^{n} \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}) = T^*\mathbb{C}^{2n}$$

$$q_i : \mathbb{C} \hookrightarrow \mathbb{C}^2 \qquad p_i : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

$T^*\mathbb{C}^{2n}$ has a natural hyperkähler structure

$K \circlearrowright T^*\mathbb{C}^{2n}$, $[p,q](A, \lambda_1, \dots, \lambda_n) = [\dots, \lambda_i^{-1} p_i A, \dots, A^{-1} q_i \lambda_i, \dots]$

α-HYPERPOLYGON SPACES



$$V_0 = \mathbb{C}^2 \qquad V_i = \mathbb{C} \qquad \forall\ i = 1, \dots n$$

$$\mathrm{Rep}\,Q = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}) = T^*\mathbb{C}^{2n}$$

$$q_i : \mathbb{C} \hookrightarrow \mathbb{C}^2 \qquad p_i : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

$T^*\mathbb{C}^{2n}$ has a natural hyperkähler structure

$$K \circlearrowright T^*\mathbb{C}^{2n}, \quad [p, q](A, \lambda_1, \dots, \lambda_n) = [\dots, \lambda_i^{-1} p_i A, \dots, A^{-1} q_i \lambda_i, \dots]$$

hyperHamiltonian

$$\mu_{\mathbb{R}}(p,q) = \left(\frac{\sqrt{-1}}{2} \sum_{i=1}^{n} (q_i q_i^* - p_i^* p_i)_0, \dots, \frac{1}{2}\left(|q_i|^2 - |p_i|^2\right), \dots \right)$$

$$\mu_{\mathbb{C}}(p,q) = \left(\sum_{i=1}^{n} (q_i p_i)_0, \dots, p_i q_i, \dots \right)$$

$$\rightsquigarrow X(\alpha) := T^*\mathbb{C}^{2n} /\!/\!/_{(0,\alpha)(0,0)} K = \mu_{\mathbb{R}}^{-1}(0,\alpha) \cap \mu_{\mathbb{C}}^{-1}(0,0) / K$$

HYPERPOLYGON SPACE

- $X(\alpha)$ is a smooth hyperkähler manifold of $\dim = 4(n-3)$

$$\Updownarrow$$

$$\varepsilon_I(\alpha) \neq 0 \quad \forall\, I \subseteq \{1,\dots,n\}$$

- $M(\alpha) = \{[p,q] \in X(\alpha) \mid p = 0\}$

NULL HYPERPOLYGON SPACES

As before, consider

$$K = \left(U(2) \times U(1)^n\right)/\Delta \curvearrowright T^*\mathbb{C}^{2n} \xrightarrow{\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}} k^* \oplus k^*_{\mathbb{C}}$$

$$P_0^n := \left\{ (p,q) \in \mu_{\mathbb{R}}^{-1}(0,0) \cap \mu_{\mathbb{C}}^{-1}(0,0) : |p_i|^2 + |q_i|^2 \neq 0 \quad \forall\, i = 1, \dots n \right\}$$

<u>PROP. (Godinho-M., 2019)</u>: K acts freely on P_0^M and

$$X_0^n := P_0^n / K$$

is a smooth hyperKähler manifold of $\dim = 4(n-3)$

QUASI-PARABOLIC HIGGS BUNDLES

$E \longrightarrow \mathbb{CP}^1$ $\quad$ $\mathrm{rank}(E) = r = 2$

$D = \{x_1, \dots, x_n\}$ marked points on $\mathbb{CP}^1$, $x_i \neq x_j \ \forall i \neq j$

A quasi-parabolic structure on E is a full flag

$$E_{x_i} = E_{x_i,1} \supset E_{x_i,2} \supset \{0\} \qquad \forall\ x_i \in D$$

$$\text{q-par}\deg(E) = \deg(E) + rn$$

$$\text{q-par}\ \mu(E) = \frac{\text{q-par}\deg(E)}{r} = n + \frac{\deg(E)}{2}$$

QUASI-PARABOLIC HIGGS BUNDLES

$E \longrightarrow \mathbb{CP}^1$ $\quad$ $\mathrm{rank}(E) = r = 2$

$D = \{x_1, \dots, x_n\}$ marked points on $\mathbb{CP}^1$, $x_i \neq x_j$ $\forall\, i \neq j$

A quasi-parabolic structure on E is a full flag

$$E_x = E_{x_i,1} \supset E_{x_i,2} \supset \{0\} \qquad \forall\ x_i \in D$$

q-par $\deg(E) = \deg(E) + 2n$; $\quad$ q-par $\mu(E) = n + \dfrac{\deg(E)}{2}$

A quasi-parabolic Higgs bundle over $\mathbb{CP}^1$, rk 2, is a pair (E, ϕ) with E as above and

$$\phi \in H^0(\mathbb{CP}^1, SQPEnd(E) \otimes K_{\mathbb{CP}^1}(D))$$

where $f : E \longrightarrow E$ is a strongly quasi-parabolic end. iff $f(E_{x_i,1}) \subset E_{x_i,2}$ and $f(E_{x_i,2}) \subset 0$ $\forall x_i \in D$

DEF. A quasi-parabolic $SL(2,\mathbb{C})$-Higgs bundle over $\mathbb{CP}^1$ at D is a rank-2 q-parabolic Higgs bundle (E,ϕ), where E has trivial determinant and ϕ is traceless

DEF (stability)

A quasi-parabolic Higgs bundle (E,ϕ) is stable if

① $\mathrm{Res}_{x_i}\phi \neq 0 \quad \forall\, x_i \in D$

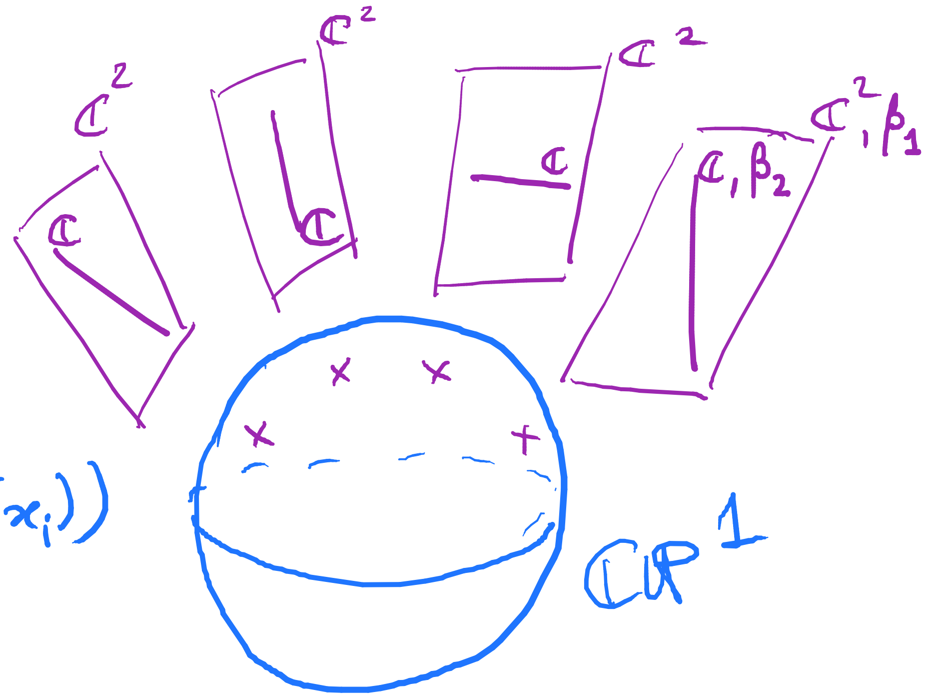
② $\text{q-par}\,\mu(E) > \text{q-par}\,\mu(L) \quad \forall\, L \subset E$ quasi-parab. subb. that is preserved by ϕ

$\mathcal{M}_0^n$ = moduli space of stable q.p. $SL(2,\mathbb{C})$ Higgs bundles

PARABOLIC HIGGS BUNDLES

A parabolic structure on a quasi-parabolic bundle E.
is the choice of parabolic weights

$$0 \leq \beta_1(x_i) < \beta_2(x_i) < 1 \quad \forall \; x_i \in D$$



$$\text{par}\deg(E) = \deg(E) + \sum_{i=1}^{M} \left(\beta_2(x_i) + \beta_1(x_i)\right)$$

$$\text{par}\,\mu(E) = \frac{\text{par}\deg(E)}{2}$$

E is semi-stable $\Longleftrightarrow$ $\mu(E) \geq \mu(L)$ $\forall\, L \subset E$ parab. subbundle

$\rightsquigarrow$ $\mathcal{M}(\beta)$ = moduli space of PBs over $\mathbb{P}^1$, rk 2, deg 0

$\mathcal{M}(\beta)$ is smooth $\Longleftrightarrow$ (stability $\Longleftrightarrow$ semistability)

HIGGS FIELD $\quad \phi \in H^0(\mathrm{SPEnd}(E) \otimes K(D))$

$f : E \to E$ is a strongly parabolic endomorphism iff $f(E_{x_{i,j}}) \subseteq E_{x_{i,j+1}}$

$$\boxed{(E, \phi) \quad \text{PARABOLIC HIGGS BUNDLE}}$$

(E, ϕ) is semi-stable $\iff$ $\mu(E) \geq \mu(L)$ $\forall\, L \subset E$ parabolic Higgs subbundle

$\rightsquigarrow \mathcal{H}(\beta) =$ moduli space of β-stable, rk 2, hdom. trivial PHBs over $\mathbb{CP}^1$ w/ fixed determinant and trace-free Higgs field.

Theorem [Godinho-M. 2013 + 2019]

(i) $X(\alpha) \simeq \mathcal{H}(\beta)$ whenever $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ generic

(ii) $X_0^n \simeq \mathcal{H}_0^n$

$[p,q]$ α-hyperpolygon $\longrightarrow$ $E = \mathbb{CP}^1 \times \mathbb{C}^2$

$$\begin{tikzcd} E_{x_i} = E_{x_i,1} \arrow[d, phantom, "\cong"] & \supset & E_{x_i,2} \arrow[d, phantom, "\cong"] & \supset 0 \\ \mathbb{C}^2 & & \mathbb{C} \arrow[ll, "q_i"] \end{tikzcd}$$

remember

$\sum_i (q_i p_i)_0 = 0$

"RESIDUE THEOREM"

$\exists!\, \phi$ s.t. $\mathrm{Res}_{x_i} \phi = (q_i p_i)_0$

Theorem [Godinho–M. 2013 + 2019]

(i) $X(\alpha) \simeq \mathcal{H}(\beta)$ whenever $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ generic

(ii) $X_0^n \simeq \mathcal{H}_0^n$

$[p, q]$ α-hyperpolygon $\longrightarrow E = \mathbb{CP}^1 \times \mathbb{C}^2$

0-hyperpolygon

remember

$\sum (q_i p_i)_0 = 0$

$$\begin{tikzcd} E_{x_i} = E_{x_{i,1}} \arrow[d, phantom, "\cong" rotate=-90] & \supset & E_{x_{i,2}} \arrow[d, phantom, "\cong" rotate=-90] & \supset 0 \\ \mathbb{C}^2 & & \mathbb{C} \arrow[ll, hook', "q_i"] & \end{tikzcd}$$

" RESIDUE THEOREM "

$\exists!\, \phi$ s.t. $\mathrm{Res}_{x_i} \phi = (q_i p_i)_0$

INVOLUTIONS

[Biswas, Florentino, Godinho, M.]

$$i_1 : X(\alpha) \longrightarrow X(\alpha)$$
$$[p, q] \longmapsto [-p, q]$$

The components of the fixed locus $X(\alpha)^{i_1}$ are:

- $M(\alpha) = \{[q, 0] \in X(\alpha)\}$ polygon space

- Z_S, for all $S \subseteq \{1, \dots n\}$ s.t. $|S| \geq 2$ and $\varepsilon_S(\alpha) < 0$:

$$Z_S = \{[p, q] \in X(\alpha) \mid S \ \& \ S^c \text{ are straight at } [p, q],\ p_j = 0 \ \forall j \in S^c\}$$

where S straight at $[p, q] \iff q_i = \lambda_j q_j \ \forall i, j \in S$

INVOLUTIONS

[Biswas, Florentino, Godinho, M.]

$$i_1 : X(\alpha) \longrightarrow X(\alpha) \qquad \rightsquigarrow \qquad i_1 : \mathcal{H}(\beta) \longrightarrow \mathcal{H}(\beta)$$
$$[p,q] \longmapsto [-p,q] \qquad\qquad [E,\Phi] \longmapsto [E,-\Phi]$$

The components of the fixed locus $X(\alpha)^{i_1}$ are:

- $M(\alpha) = \{[p,0] \in X(\alpha)\}$ polygon space
- Z_S, for all $S \subseteq \{1,\dots,n\}$ s.t. $|S| \geq 2$ and $\varepsilon_S(\alpha) < 0$:

  $Z_S = \{[p,q] \in X(\alpha) \mid S \,\&\, S^c$ are straight at $[p,q]$, $p_j = 0 \ \forall j \in S^c\}$

  where S straight at $[p,q] \iff q_i = \lambda_j q_j \ \forall i,j \in S$

The components of the fixed locus $\mathcal{H}(\beta)^{i_1}$ are:

- $\mathcal{M}(\beta) = \{(E, \Phi \equiv 0)\}$ m.s. of parabolic bundles
- Z_S, for all $S \subseteq \{1,\dots,n\}$ s.t. $|S| \geq 2$ and $\varepsilon_S(\alpha) < 0$:

$$Z_S = \left\{ (E,\Phi) : E = L_0 \oplus L_1 \ ; \ \underset{\forall i \in S}{\mathrm{Res}_{x_i} \Phi = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}}, \ \underset{\forall i \in S^c}{\mathrm{Res}_{x_i} \Phi = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}} \right\}$$

INVOLUTIONS

[Biswas, Florentino, Godinho, M.]

$$i_1 : X(\alpha) \longrightarrow X(\alpha) \qquad [p,q] \longmapsto [-p,q]$$

$$\rightsquigarrow$$

$$i_1 : \mathcal{H}(\beta) \longrightarrow \mathcal{H}(\beta) \qquad [E,\phi] \longmapsto [E,-\phi]$$

The components of the fixed locus $\mathcal{H}(\beta)^{i_1}$ are:

- $\mathcal{M}(\beta) = \{(E, \phi \equiv 0)\}$ m.s. of parabolic bundles
- Z_S, for all $S \subseteq \{1, \dots n\}$ s.t. $|S| \geq 2$ and $\varepsilon_S(\alpha) < 0$: $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$

$$Z_S = \left\{ (E,\Phi) : \underbrace{E = L_0 \oplus L_1}_{\substack{(L_0)_{x_i} = E_{x_i,2},\ i \in S \\ (L_1)_{x_i} = E_{x_i,2},\ i \in S^c}} ;\ \underset{\forall i \in S}{\mathrm{Res}_{x_i} \phi = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}},\ \underset{\forall i \in S^c}{\mathrm{Res}_{x_i} \phi = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}} \right\}$$

INVOLUTIONS

[Godinho, M 2019]

$$i_0 : X_0^n \longrightarrow X_0^n$$
$$[p,q] \longmapsto [q^t, p^t]$$

The fixed point set of i_0 has $2^{n-1} - (n+1)$ components Z_S

$Z_S = \{[p,q] :$ S & S^c are straight at $[p,q]\}$

$\forall S \subseteq \{1, \dots n\}$ s.t. $2 \leq |S| \leq n-2$ and $1 \in S$.

INVOLUTIONS

[Godinho, M 2019]

$i_0 : X_0^n \longrightarrow X_0^n$ $\rightsquigarrow$ $i_0 : \mathcal{H}_0^n \longrightarrow \mathcal{H}_0^n$

$[p,q] \longmapsto [q^t, p^t]$ $\qquad$ $(E,\phi) \longrightarrow (E^*, \phi^t)$

The fixed point set $(\mathcal{H}_0^n)^{i_0}$ has $2^{n-1} - (n+1)$ components Z_S indexed by $S \subseteq \{1, \dots m\}$ s.t. $2 \leq |S| \leq m-2$ and $1 \in S$:

$(E,\phi) \in Z_S \Leftrightarrow (E,\phi)$ is a QPH $SL(2,\mathbb{R})$-bundle s.t.:

(i) $E = L_0 \oplus L_1$: $E_{x_i,2} = (L_0)_{x_i}$ $\forall i \in S$ & $E_{x_i,2} = (L_0)_{x_i}$ $\forall i \in S$

(ii) $\mathrm{Res}_{x_i} \phi = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$; $\mathrm{Res}_{x_j} \phi = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$

$\forall i \in S$ $\qquad$ $\forall j \in S^c$

G-POLYGON SPACE

G Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^*$ its dual

Choose $\xi_i \in \mathfrak{g}^*$, $\xi_i \neq 0$ $\forall$ $i=1,\dots,n$ $\rightsquigarrow$ $\mathcal{O}_{\xi_i}$ = coadjoint orbit

$$G \circlearrowright \prod_{i=1}^{n} \mathcal{O}_{\xi_i} \xrightarrow{\mu_G} \mathfrak{g}^* \Rightarrow \prod \mathcal{O}_{\xi_i} /\!\!/_{0} G$$

Ham.

$$(A_1,\dots A_n) \longmapsto \sum_{i=1}^{n} A_i$$

G - polygon space

$G = SU(2) \rightsquigarrow$ moduli space of polygons in $\mathbb{R}^3$

$G = SL(2,\mathbb{C}) \rightsquigarrow$ HYPERP.

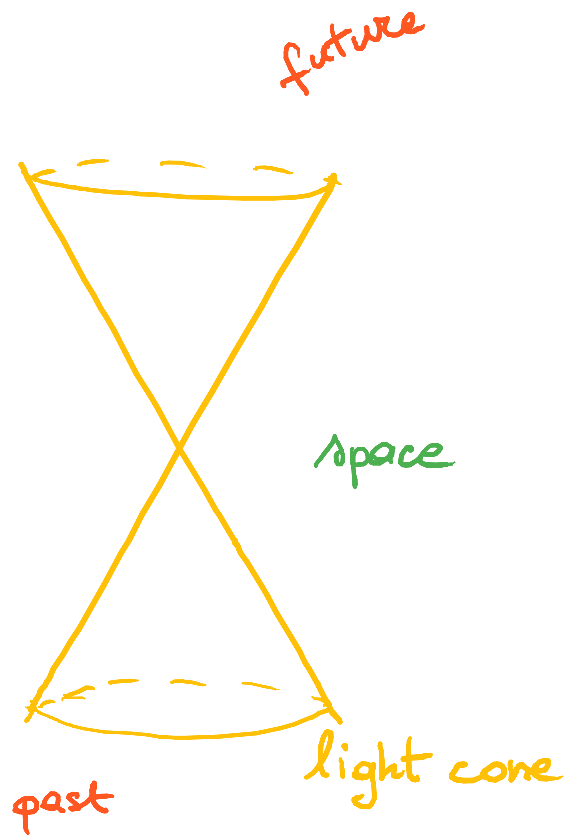
$SL(2,\mathbb{R})$- POLYGON SPACES

$\mathrm{Lie}\,(SL(2,\mathbb{R})) \simeq \mathbb{R}^{2,1} = \left(\mathbb{R}^3, (-,-,+)\right)$ MINKOWSKI 3-SPACE

i.e $u=(x,y,t) \in \mathbb{R}^{2,1}$, $u \cdot u = -x^2 - y^2 + t^2$

$u \in \mathbb{R}^{2,1}$ is
- SPACE-LIKE, if $u \circ u < 0$
- LIGHT-LIKE, if $u \circ u = 0$
- TIME-LIKE, if $u \circ u > 0$

$u = (x,y,t) \in \mathbb{R}^{2,1}$ is
- future if $t > 0$
- past if $t < 0$



Coadjoint orbits of $SL(2,\mathbb{R}) \simeq \mathbb{R}^{2,1}$

$$S_R := \{(x,y,t) \in \mathbb{R}^{2,1} \mid -x^2-y^2+t^2 = R^2\} \begin{cases} S_R^+ & \text{if } t>0 \quad \text{FUTURE PSEUDOSPHERE} \\ S_R^- & \text{if } t<0 \quad \text{PAST PSEUDOSPHERE} \end{cases}$$

$$H_R := \{(x,y,t) \in \mathbb{R}^{2,1} \mid +x^2+y^2-t^2 = R^2\}$$ SPACE PSEUDOSPHERE

$$C^- = \{(x,y,t) \in \mathbb{R}^{2,1} : x^2+y^2-t^2 = 0,\ t<0\}$$ ← the past light cone

$$C^+ = \{(x,y,t) \in \mathbb{R}^{2,1} : x^2+y^2-t^2 = 0,\ t>0\}$$ ← the future light cone

$$\{0\}$$

TIME-LIKE CASE

$$S_R := \{(x,y,t) \in \mathbb{R}^{2,1} \mid -x^2 - y^2 + t^2 = R^2\}$$

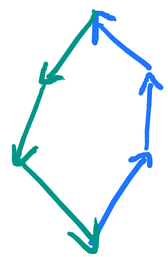
S_R^+ if $t>0$ FUTURE PSEUDOSPHERE

S_R^- if $t<0$ PAST PSEUDOSPHERE

$$SL(2,\mathbb{R}) \circlearrowright \prod_{i \in S} S^+_{\alpha_i} \times \prod_{i \in S^c} S^-_{\alpha_i} \xrightarrow{\mu_{SL(2,\mathbb{R})}} \mathrm{Lie}\, SL(2,\mathbb{R})^* \simeq \mathbb{R}^{2,1}$$

Ham.

$$(u_1, \dots, u_n) \longmapsto \sum_{i=1}^{n} u_i$$



$$M^S(\alpha) := \prod_{i \in S} S^+_{\alpha_i} \times \prod S^-_{\alpha_i} \Big/\!\!\Big/_0 SL(2,\mathbb{R}) = \mu^{-1}_{SL(2,\mathbb{R})}(0) \Big/ SL(2,\mathbb{R})$$

[Foth] $M^S(\alpha)$ is smooth $\Longleftrightarrow \varepsilon_I(\alpha) \neq 0 \quad \forall\, I \subseteq \{1, \dots, n\}$

$M^S(\alpha)$ is non-compact unless $|S| = 1$ or $|S^c| = 1$

$i_1([p,q]) = [-p,q]$

Theorem [BFGM]

The components $Z_s \subseteq X(\alpha)$ and $\mathcal{Z}_s \subseteq \mathcal{H}(\beta)$ are isomorphic to $M^s(\alpha)$

$$(X(\alpha) \supseteq)\ Z_s \longrightarrow M^s(\alpha)$$

$$[p,q] \longmapsto [u_1, \dots u_n]$$

where

$$u_i = \frac{\sqrt{-1}}{2}\left(p_i^* p_i - q_i q_i^*\right)_0 + \frac{1}{2}\left(p_i^* q_i^* + q_i p_i\right)_0$$

LIGHT-LIKE CASE

[Godinho-M 2019]

$C^- = \{(x,y,t) \in \mathbb{R}^{2,1} : x^2+y^2-t^2=0,\ t<0\}$ ← the past light cone

$C^+ = \{(x,y,t) \in \mathbb{R}^{2,1} : x^2+y^2-t^2=0,\ t>0\}$ ← the future light cone

$$SL(2,\mathbb{R}) \circlearrowright \quad \prod_{i\in S} C^- \times \prod_{i\in S^c} C^+ \xrightarrow{\mu_{SL(2,\mathbb{R})}} \mathbb{R}^{2,1}$$

$$(u_1,\dots,u_n) \longmapsto \sum_{i=1}^{n} u_i$$

$$\mu^{-1}(0)_{reg} := \mu^{-1}_{SL(2,\mathbb{R})}(0) \cap \left(\prod_{i\in S} C^- \times \prod_{i\in S^c} C^+\right)^{reg}$$

$$\Rightarrow M_0^{|S|} = \mu^{-1}(0)_{reg} \Big/ SL(2,\mathbb{R})$$

MODULI SPACE OF CLOSED NULL POLYGONS in $\mathbb{R}^{2,1}$

is a symplectic non-cpct manifold, non-empty $\Leftrightarrow n \geq 4$.

$i_0([p,q]) = [q^t, q^t]$

Theorem [Godinho-M. 2019]

For any $S \subseteq \{1, \dots, n\}$ s.t. $1 \in S$ and $2 \le |S| \le n-2$, the components $Z_S \subseteq (X_0^n)^{i_0} \simeq (\mathcal{H}_0^n)^{i_0}$ are isomorphic to $M_0^{|S|}$.

$$\begin{array}{ccc} Z_S & \longrightarrow & M_0^{|S|} \\ [p,q] & \longmapsto & (u_1, \dots, u_n) \end{array}$$

where, if $p_i = (0, b_i)$ & $q_i = \binom{c_i}{0}$ $\forall i \in S$

$p_j = (a_j, 0)$ & $q_j = \binom{0}{d_j}$ $\forall j \in S^c$

$$\Rightarrow \quad u_i = \left(\mathrm{Re}(b_i c_i), \mathrm{Im}(b_i c_i), -|c_i|^2 \right) \quad \forall i \in S$$

$$u_j = \left(\mathrm{Re}(a_j d_j), \mathrm{Im}(a_j d_j), |a_i|^2 \right) \quad \forall i \in S^c$$

BASED ON :

* L. Godinho, A. Mandini, "Hyperpolygon spaces and moduli spaces of parabolic Higgs bundles", Adv. Math. 244, 2013

* I. Biswas, C. Florentino, L. Godinho, A. Mandini "Polygons in the Minkowski three space and parabolic Higgs bundles", Transf. Groups, 2013

* I. Biswas, C. Florentino, L. Godin , A. Mandini "Symplectic form on hyperpolygon spaces" Geom. Dedicata, 2015

→ L. Godinho, A. Mandini, "Quasi-parabolic Higgs bundles and null hyperpolygon spaces", arXiv: 1907.1937

THANKS!

INVOLUTIONS

[Godinho, M.] on going:

$$i_2 : X(\alpha) \longrightarrow X(\alpha) \qquad \rightsquigarrow \qquad i_2 : \mathcal{H}(\beta) \longrightarrow \mathcal{H}(\beta)$$

$$[p, q] \longmapsto [-\bar{p}, \bar{q}] \qquad\qquad (E, \Phi) \longmapsto (E^*, -\bar{\Phi})$$

$$X(\alpha)^{i_2} = \left\{ [p, q] : p_i = \left(\sqrt{-1}\, a_i, b_i \right), q_j = \begin{pmatrix} c_i \\ \sqrt{-1}\, d_i \end{pmatrix}, a_i, b_i, c_i, d_i \in \mathbb{R} \ \forall i \right\}$$

$$\mathcal{H}(\beta)^{i_2}$$