Partial orders on contactomorphism groups and their Lie algebras

joint work with Y. Eliashberg and L. Polterovich

Kai Cieliebak

University of Augsburg

Motivation

Hofer 1990, . . . : The group $\mathrm{Ham}(W, \omega)$ of compactly supported Hamiltonian diffeomorphisms carries a natural bi-invariant metric (Hofer's metric) for every symplectic manifold (W, ω).

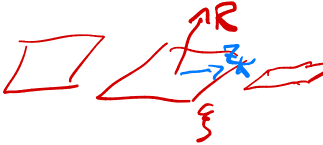
Eliashberg–Polterovich 1999, Eliashberg–Kim–Polterovich 2006: The (universal cover of the) group $\mathrm{Cont}(M, \xi)$ of compactly supported contactomorphisms carries a natural bi-invariant partial order for some contact manifolds (M, ξ), e.g. for $\mathbb{R}P^{2n-1}$ but **not** for its 2-1 cover S^{2n-1}.

Goal of this talk: Study the remnants of this partial order on the Lie algebra of $\mathrm{Cont}(M, \xi)$ modulo the adjoint action.

Setup

contact structure

- $(M^{2n-1}, \xi = \ker\alpha)$ contact manifold (not necessarily compact) with contact 1-form α ($\alpha \wedge (d\alpha)^{n-1} > 0$) and Reeb vector field R ($\alpha(R) = 1$ and $i_R d\alpha = 0$)
- $G := \mathrm{Cont}_0(M, \xi)$ identity component of the group of compactly supported contactomorphisms g (i.e. $g^*\xi = \xi$)
- $\mathfrak{g}$ its Lie algebra, consisting of compactly supported contact vector fields Y (i.e. $L_Y\xi = 0$)



We will use the canonical identification

$$\mathfrak{g} \cong C_0^\infty(M)$$
$$Y \mapsto K_Y := \alpha(Y)$$
$$Y_K := KR + Z_K \leftarrow K,$$

where $\overset{Z}{\cancel{Y}}_K \in \xi$ is defined by $(dK + i_{Z_K} d\alpha)|_\xi = 0$.

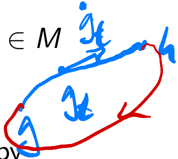
Setup

The convex cone

$$\mathfrak{g}^{\geq 0} := \{K \in \mathfrak{g} \mid K(x) \geq 0 \ \forall x \in M\}$$

defines a natural **partial order on** $\mathfrak{g}$ by

$$H \leq K :\Longleftrightarrow K - H \in \mathfrak{g}^{\geq 0} \Longleftrightarrow H(x) \leq K(x) \ \forall x \in M$$



From here 2 ways to proceed:

1) The partial order on $\mathfrak{g}$ gives rise to a preorder on G by
$g \leq h :\Longleftrightarrow \exists$ smooth path $(g_t)_{t\in[0,1]}$ in G with $g_0 = g$, $g_1 = h$, and $g_t^{-1}\dot{g}_t \in \mathfrak{g}^{\geq 0} \ \forall t$.
When is this preorder nondegenerate (i.e., $g \leq h$ and $h \leq g$ implies $g = h$) and thus defines a partial order on G (or on its univeral cover $\widetilde{G}$)? $\rightsquigarrow$ not today

Setup

2) $g \in G$ acts on $Y \in \mathfrak{g}$ by the **adjoint action**

$$\mathrm{Ad}_g Y = g_* Y.$$

On $K \in C_0^\infty(M)$ such that $Y = Y_K$ this becomes

$$\begin{aligned}\mathrm{Ad}_g K &= \alpha(g_* Y_K) \\ &= \underbrace{(g^*\alpha)}_{=c_g\alpha}(Y_K) \circ g^{-1} \\ &= \left(c_g \underbrace{\alpha(Y_K)}_{=K}\right) \circ g^{-1}\end{aligned}$$

with a smooth function $c_g : M \to \mathbb{R}_{>0}$, so

$$\boxed{\mathrm{Ad}_g K = (c_g K) \circ g^{-1}}$$

The multiplication by the positive function c_g (which is not present in the symplectic case) might change the order on $\mathfrak{g}$ a lot.

Setup

Question. What is left of the partial order on $\mathfrak{g}$ up to the adjoint action? More precisely:

- The adjoint action does not change signs of K, so it preserves $\mathfrak{g}^{\geq 0}$ and
$$H \leq K \implies \mathrm{Ad}_g H \leq \mathrm{Ad}_g K \;\forall g \in G$$
- If $K \in \mathfrak{g}^{\geq 0}$ and $H(x) < 0$ for some $x \in M$, then $\mathrm{Ad}_g K \not\leq H$ for any $g \in G$. So we will restrict to $K, H \in \mathfrak{g}^{\geq 0}$.

More precise question. Given $K, H \in \mathfrak{g}^{\geq 0} \setminus \{0\}$, does there exist $g \in G$ such that $\mathrm{Ad}_g K \geq H$?

Remark. The motivation for this came from Borman–Eliashberg–Murphy's existence proof for contact structures in higher dimensions, which could be considerably simplified if this question had a positive answer for suitable (M, ξ) (which it does not).

Example 1

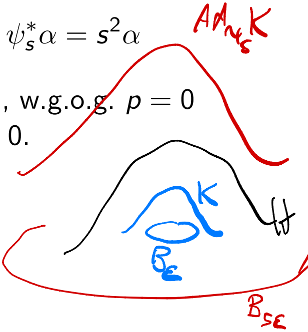
$$\left(M = \mathbb{R}^{2n-1},\ \alpha = dz - \sum_{i=1}^{n-1} y_i dx_i\right)$$

$$\psi_s(x,y,z) := (sx, sy, s^2 z), \quad s > 0, \qquad \psi_s^* \alpha = s^2 \alpha$$

$K \in \mathfrak{g}^{\geq 0} \setminus \{0\} \rightsquigarrow K(p) > 0$ for some $p \in \mathbb{R}^{2n-1}$, w.g.o.g. $p = 0$
$\rightsquigarrow K \geq \varepsilon > 0$ on ball B_ε around 0 for some $\varepsilon > 0$.

Then $K_s := \mathrm{Ad}_{\psi_s} K = s^2 K \circ \psi_s^{-1}$ satisfies

$$K_s \geq s^2 \varepsilon \text{ on } \psi_s(B_\varepsilon) \underset{s \geq 1}{\supset} B_{s\varepsilon}$$



$\Longrightarrow \forall H \in \mathfrak{g}^{\geq 0}\ \exists s \geq 1 : \mathrm{Ad}_{\psi_s} K \geq H.$

So the partial order becomes a totally degenerate preorder ($a \leq b\ \forall a, b$) modulo the adjoint action (flexibility)!

Example 2

$$(M^{2n-1}, \xi = \ker \alpha) \text{ with } M \text{ closed}$$

$$\mathfrak{g}^{>0} := \{K \in C^\infty(M) \mid K(x) > 0 \ \forall x \in M\}$$

Consider $K, H \in \mathfrak{g}^{>0}$ with $K \leq H \leq \mathrm{Ad}_g K$ for some $g \in G$. Then:

$$\mathrm{vol} := \alpha \wedge (d\alpha)^{n-1}, \quad g^*\alpha = c_g \alpha \Longrightarrow g^*\mathrm{vol} = c_g^n \mathrm{vol}\,,$$

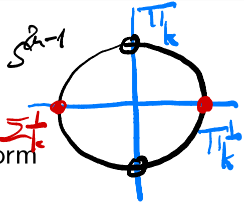
$$H \leq \mathrm{Ad}_g K = (c_g K) \circ g^{-1} \Leftrightarrow c_g K \geq H \circ g \Leftrightarrow H^{-n} \circ g \cdot c_g^n \geq K^{-n}$$

$$\Longrightarrow \int_M K^{-n}\mathrm{vol} \leq \int_M H^{-n} \circ g \cdot \underbrace{c_g^n \mathrm{vol}}_{=g^*\mathrm{vol}} = \int_M H^{-n}\mathrm{vol} \underbrace{\leq}_{K \leq H} \int_M K^{-n}\mathrm{vol}$$

$$\Longrightarrow K = H$$

So the partial order remains nondegenerate modulo the adjoint action (rigidity)!

Main class of examples



- $\mathbb{C}^n$, $\lambda := \frac{1}{2}\sum_{j=1}^n (x_j dy_j - y_j dx_j)$
- $S^{2n-1} \subset \mathbb{C}^n$ unit sphere, $\alpha := \lambda|_{S^{2n-1}}$ contact form
- $\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}$ standard basis of $\mathbb{C}^n$
- $\Pi_k < \mathbb{C}^n$ spanned by last k basis vectors, $1 \leq k \leq 2n-1$
- $\Pi_k^\perp < \mathbb{C}^n$ spanned by first $2n-k$ basis vectors
- $(M_k := S^{2n-1} \setminus \Pi_k, \xi = \ker \alpha)$ contact manifold (noncompact but convex at infinity)
- $\mathfrak{g}_k^+ := \{K \in \mathfrak{g}^{\geq 0} \mid K|_{g(\Sigma_k^\perp)} > 0 \text{ for some } g \in G\}$ Ad-invariant cone

$\Sigma_k^\perp = \Pi_k^\perp \cap S^{2n-1}$

Note. Π_k is isotropic for $1 \leq k \leq n$, and coisotropic for $n+1 \leq k \leq 2n-1$.

Main Theorem. (a) For $k \geq n$ the partial order on $\mathfrak{g}_k^+$ becomes totally degenerate modulo the adjoint action, i.e.

$$\forall H, K \in \mathfrak{g}_k^+ \exists g \in G : \mathrm{Ad}_g H \leq K. \qquad \text{(flexibility)}$$

(b) For $k < n$ the partial order on $\mathfrak{g}_k^+$ does **not** become totally degenerate modulo the adjoint action, i.e.

weak

$$\exists H, K \in \mathfrak{g}_k^+ : \nexists g \in G : \mathrm{Ad}_g H \leq K. \qquad \text{(rigidity)}$$

Special cases

$\underline{n = 2, k = 3}$: $M_3 = S^3 \setminus \Pi_3 \cong \mathbb{R}^3 \amalg \mathbb{R}^3$ is flexible by Example 1.

~~$\underline{n = 2, k = 2}$: $M_3 = S^3 \setminus \Pi_2 \cong S^1 \times \mathbb{R}^3 \simeq J^1 S^1$ (1-jet space)~~

$\underline{n = 2, k = 2}$: $M_3 = S^3 \setminus \Pi_2 \cong S^1 \times \mathbb{R}^{\cancel{3}\,2} \cong J^1 S^1$ (1-jet space of S^1) is flexible

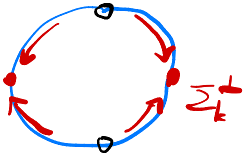
$\underline{\text{general } n \leq k}$: $M_k = S^{2n-1} \setminus \Pi_k \cong J^1(S^{2n-k-1}) \times \mathbb{R}^{2k-2n}$ is flexible

$\underline{n = 2, k = 1}$: $M_1 = S^3 \setminus \Pi_1 \cong S^2 \times \mathbb{R}$ is rigid

The last case appears in Borman–Eliashberg–Murphy's existence proof for contact structures in dimension 3.

Proof of (a)

For $k \geq n$ consider the vector field on $\mathbb{C}^n$



$$\widehat{Y} := \sum_{j=1}^{2n-k} (x_j \partial_{x_j} - y_j \partial_{y_j})$$

Explicit computations show:

- $L_{\widehat{Y}} \lambda = 0$ and $[\widehat{Y}, Z] = 0$ for the Liouville vector field $Z = \frac{1}{2} \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j})$;
- $\widehat{Y}$ induces a contact vector field $Y = \widehat{Y} - fZ$ on M_k;
- the flow g_t of Y contracts M_k into arbitrarily small neighbourhoods of $\Sigma_k^\perp$ as $t \to \infty$;
- for $H, K \in \mathfrak{g}_k^+$, $\mathrm{Ad}_{g_t} H$ becomes arbitrarily small (both its support and its values) as $t \to \infty$, so eventually $\mathrm{Ad}_{g_t} H \leq K$.

Proof of (b)

Idea (following Eliashberg–Kim–Polterovich): Translate the question into **symplectic non-squeezing** for certain unbounded domains in $\mathbb{C}^n$.

We will identify

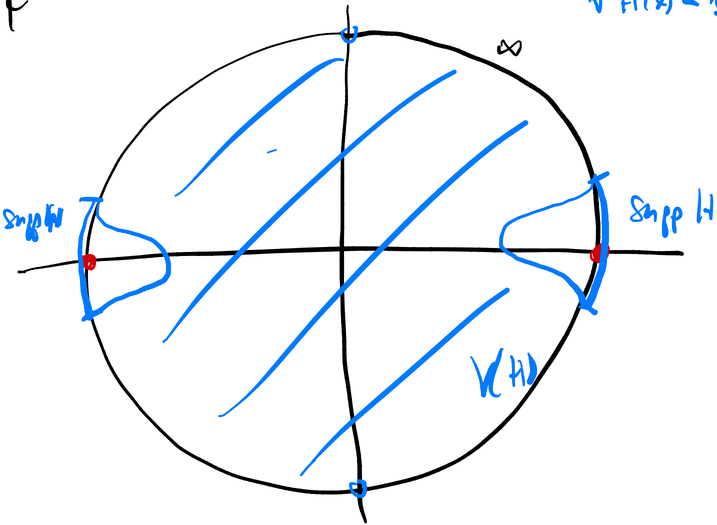
$$(\mathbb{C}^n \setminus \{0\}, \lambda) \cong (\underset{r}{\mathbb{R}_+} \times \underset{x}{S^{2n-1}}, r\alpha) \quad \text{symplectization of } (S^{2n-1}, \xi).$$

Main construction: To $H \in \mathfrak{g}_k^+$ we associate the unbounded domain

$$V(H) := \{0\} \cup \{(r, x) \in \mathbb{R}_+ \times S^{2n-1} \mid rH(x) < 1\} \subset \mathbb{C}^n.$$

$\mathbb{P}^n$

$\forall\ H(x) < 1.$



Properties of $V(H)$

1. $g \in G$ gives rise to $\widehat{g} \in \mathrm{Symp}(\mathbb{R}_+ \times S^{2n-1}, r\alpha)$ via

$$\widehat{g}(r,x) := \Big(\frac{r}{c_g(x)}, g(x)\Big).$$

   Proof: $\widehat{g}^*(r\alpha) = \frac{r}{c_g(x)} g^*\alpha = r\alpha$.

2.

$$\begin{aligned} V(\mathrm{Ad}_g H) &= \{(r,x) \mid r \cdot (c_g H) \circ g^{-1}(x) < 1\} \\ &= \Big\{\Big(\frac{r}{c_g(x)}, g(x)\Big)\Big) \;\Big|\; \underbrace{\frac{r}{c_g(x)} \cdot (c_g H)(x)}_{=rH(x)} < 1\Big\} \\ &= \widehat{g}\big(V(H)\big) \end{aligned}$$

3. $H \le K \iff rH(x) \le rK(x) \forall (r,x) \iff V(K) \subset V(H)$
4. $V(sH) = \frac{1}{s} V(H)$ $\quad \forall s > 0$

Proof of (b)

Properties (iii) and (iv) translate the relation $\mathrm{Ad}_g H \leq K$ into the symplectic embedding $V(K) \subset \widehat{g}\big(V(H)\big)$. An obstruction to such an embedding is provided by

Proposition. There exists a **"symplectic capacity"**

$$c : \mathcal{C}_k \to [0, \infty],$$

defined on a suitable class of unbounded open subsets of $\mathbb{C}^n$ containing the domains $V(H)$ for $H \in \mathfrak{g}_k^+$, with the following properties:

1. $U \subset V \Longrightarrow c(U) \leq c(V)$;
2. $c\big(\widehat{g}(U)\big) = c(U) \ \forall g \in G$;
3. $c(sU) = s^2 c(U) \ \forall s > 0$;
4. $0 < c\big(V(H)\big) < \infty \ \forall H \in \mathfrak{g}_k^+$.

Proof of (b)

Proof of the Main Theorem assuming the Proposition.

This follows directly with $K := sH$ from the following

Claim: For $H \in \mathfrak{g}_k^+$ and $0 < s < 1$ $\nexists g \in G : \mathrm{Ad}_g H \leq sH$.

Proof of Claim:

$$
\begin{aligned}
& Ad_g H \leq sH && \Big|\ V(\cdot) \\
\Longrightarrow\ & \frac{1}{s} V(H) = V(sH) \subset V(\mathrm{Ad}_g H) = \widehat{g}\big(V(H)\big) && \Big|\ c(\cdot) \\
\Longrightarrow\ & \frac{1}{s^2} c\big(V(H)\big) = c\Big(\frac{1}{s} V(H)\Big) \leq c\Big(\widehat{g}\big(V(H)\big)\Big) = \underbrace{c\big(V(H)\big)}_{\in (0,\infty)} \\
\Longrightarrow\ & s \geq 1.
\end{aligned}
$$

This proves the Claim and thus the Main Theorem $\square$

Proof of (b)

Construction of the symplectic capacity c

$$u^2 := \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-k} y_i^2, \qquad v^2 := \sum_{i=n-k+1}^{n} y_i^2$$

(Recall $k < n$). For $a, b > 0$ consider the hyperboloids

$$V_k^{a,b} := \{z \in \mathbb{C}^n \mid \frac{u^2}{a^2} - \frac{v^2}{b^2} < 1\}$$

Their **symplectic homology** in Conley-Zehnder index $n-k$ is

$$SH_{n-k}^{(0,c)}(V_k^{a,b}) = \begin{cases} \mathbb{Z}_2 & c > \pi a^2, \\ 0 & c \leq \pi a^2. \end{cases}$$

Proof of (b)

For an open subset $U \subset \mathbb{C}^n$ with $V_k^{a,b} \subset U$ for some a, b set

$$c(U) := \inf\{c > 0 \mid SH_{n-k}^{(0,c)}(U) \xrightarrow{surj} SH_{n-k}^{(0,c)}(V_k^{a,b}) = \mathbb{Z}_2\},$$

where $a > 0$ is chosen very small. This definition does not depend on a, b and defines a symplectic capacity satisfying

$$0 < \pi a^2 = c(V_k^{a,b}) \leq c\big(V(H)\big) \leq c(V_k^{a',b'}) = \pi a'^2 < \infty,$$

where a, b, a', b' are chosen such that $V_k^{a,b} \subset V(H) \subset V_k^{a',b'}$.