Toric Kähler geometry and probability
Lisbon zoom seminar

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Probability measures arising in toric Kähler geometry

This talk is a survey of results on sequences $\{\mu_k^x\}_{k=1}^{\infty}$ of probability measures arising in toric Kähler geometry. The parameter k corresponds to the power L^k of a positive Hermitian line bundle $L \to M$ over a toric Kähler manifold of (complex) dimension m. The parameter x is in the Delzant polytope.

- The $\{\mu_k^x\}$ are lattice prob measures supported on $\mathbb{Z}^m \cap kP$, the lattice points in the kth dilate of a Delzant polytope P, where $x \in P$. They are generalization of multi-nomial distributions and satisfy many of the same properties.

- The measures are also closely related to the Wright-Fisher Markov chains in population genetics and their large k limits as diffusion processes on P (new and in progress).

Convolution and dilation of probability measures

The sequences μ_k^x should be compared with the sequence of convolution powers μ^{*k} of a probability measure μ on $\mathbb{R}^m$. The convolution $\mu * \nu$ of two probability measures is defined by

$$\mu * \nu(E) = \int_{\mathbb{R}^n} \mu(E - x)\nu(dx). \tag{1}$$

Convolution powers arise when one studies sums $\sum_{j=1}^k X_j$ of i.i.d. random variables with values in $\mathbb{R}^m$. Three (or four) classical results involve limits of dilates of μ^{*k}. By a dilate we mean $D_t\mu(E) = \mu(tE)$.

Classical results on sums of independent random variables = convolution powers of a probability measure μ on $\mathbb{R}^m$

- The weak LLN (law of large numbers): $D_{k*}\mu^{*k} \rightharpoonup \delta_m$, where $m = \int x d\mu$ is the mean;
- The CLT (central limit theorem): If μ is re-centered to have mean zero, and normalized to have variance 1, then $D_{\sqrt{k}*}\mu^{*k} \rightharpoonup N(0,1)$.
- The Cramer LDP (large deviations principle: measures exponential decay of $D_k\mu_k^*\{x : |x - m| \geq C\}$.
- Entropy asymptotics of μ^{*k}.
- Scaling limits as diffusion processes on P.

Convolution powers of Bernoulli distributions = Binomial distributions

We review convolution powers in the case of Bernoulli distributions.

Bernoulli measures are discrete measures on $\{0, 1\}$ (the lattice points in $P = [0, 1]$) defined for $p \in [0, 1]$ by

$$\mu_p = (1 - p)\delta_0 + p\delta_1.$$

The kth convolution power

$$\mu_p^{*k} = 2^{-k} \sum_{n=0}^{k} p^k (1-p)^{n-k} \binom{k}{n} \delta_n$$

has its support in $[0, k] \cap \mathbb{Z}$.

$[0, 1]$ is the Delzant polytope of $\mathbb{CP}^1$. As we will see, the measures involve the symplectic potential of the Fubini-Study metric.

Review of the convolution of binomial distributions

We illustrate the formula

$$\mu_p^{*n} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k$$

with $p = \frac{1}{2}$. Consider $\mu = \mu_{\frac{1}{2}} := \frac{1}{2}(\delta_0 + \delta_1)$. Then

$$\mu * \mu = \tfrac{1}{4}(\delta_0 + 2\delta_1 + \delta_2),$$

$$\mu * \mu * \mu = \tfrac{1}{8}(\delta_0 + 3\delta_1 + 3\delta_2 + \delta_3),$$

$$\mu^{*n} = \tfrac{1}{2^n}(\delta_0 + \tbinom{n}{1}\delta_1 + \tbinom{n}{2}\delta_2 + \cdots + \tbinom{n}{n}\delta_n).$$

Scaling for the LLN and CLT

Dilation of a measure μ on $\mathbb{R}$ at scale t is defined by $D_t\mu(E) = \mu(tE)$. The classical results involve weak limits of different scalings:

- In the LLN, one dilates μ^{*n} back to $[0,1]$ to get

$$D_k\mu^{*k} = \frac{1}{2^k}(\delta_0 + \binom{k}{1}\delta_{\frac{1}{k}} + \binom{k}{2}\delta_{\frac{2}{k}} + \cdots + \binom{k}{k}\delta_1).$$

  The measures peak when $n = \binom{k}{k/2}$ at the point $\frac{1}{2}$ and tend to $\delta_{\frac{1}{2}}$. For general p, the dilation is $2^{-k}\sum_{n=0}^{k} p^k(1-p)^{n-k}\binom{k}{n}\delta_{\frac{n}{k}}$, which tends weakly to δ_p.
- Suppose μ is a probability measure on $\mathbb{R}^n$ with mean 0 and $\int x_i x_j d\mu = A_{ij}$.
  Then

$$k^{n/2}D_{\sqrt{k}}\mu^{*k} \to \frac{1}{\sqrt{|\det A|}}e^{-\langle A^{-1}x,x\rangle}dx.$$

Compact toric manifolds

We now define probabilities measures analogous to binomial (or, multinomial) distributions associated to any Kähler metric on a toric Kähler manifold. First, some background.

A compact toric variety M is a complex manifold of dimension m on which $(\mathbb{C}^*)^m$ acts holomorphically with an open dense orbit. Thus, M is a compactification of $(\mathbb{C}^*)^m \simeq M^o$ (the open orbit).

Let $\mathbf{T}^m \subset (\mathbb{C}^*)^m$ be the totally real subgroup (the m-torus). We pick $z_0 \in M$ such that its orbit $(\mathbb{C}^*) \cdot z_0 = M^o$ and use coordinate $z = e^{\rho/2 + i\theta}$ if $e^{\rho/2+i\theta} z_0 = z$.

Open orbit Kähler potential and the symplectic potential

We endow M with a $\mathbf{T}^m$-invariant Kähler metric ω. In coordinates $z = e^{\rho/2+i\theta}$ in the open orbit, there is a $\mathbf{T}^m$-invariant Kähler potential $\varphi(\rho)$, such that $\omega = \omega_\varphi = i\partial\bar{\partial}\varphi$ on the open orbit. It defines a convex function on $\mathbb{R}^n$. Its Legendre transform,

$$u(x) = \sup_\rho(\langle \rho, x\rangle - \varphi(\rho))$$

is known as the symplectic potential. It is a convex function on the polytope P.
The moment map

$$\nu_h = \nabla_\rho \varphi(\rho) : M \to P \subset \mathbb{R}^m, \tag{2}$$

defines a singular $\mathbf{T}^m$ (torus) bundle on the open orbit over a convex lattice (Delzant) polytope P.

Polarized toric Kähler manifolds

We further assume M is a complex projective manifold with a toric line bundle $L \to M$. We consider Hermitian metrics h on L with positive $(1,1)$ curvature $\omega_h = i\partial\bar{\partial}\log h$ (the toric Kähler metrics).

We denote the space of holomorphic sections of L^k by $H^0(M, L^k)$. There is a natural basis $\{s_\alpha\}_{\alpha \in kP}$ of the space $H^0(M, L^k)$ of holomorphic sections of the k-th power of L by eigensections s_α of the $\mathbf{T}^m$ action. In a standard frame e_L of L over M^o, they correspond to monomials z^α on $(\mathbb{C}^*)^m$ where

$$\alpha \in k\overline{P} \cap \mathbb{Z}^m.$$

Monomial basis and L^2 theory

The Hermitian metric induces an L^2 inner product on each $H^0(M, L^k)$ and

$$\langle s_\alpha, s_\beta \rangle = \int_M (s_\alpha(z), s_\beta(z))_{h^k} dV_h(z).$$

They are orthogonal if $\alpha \neq \beta$. In an invariant frame, $s_\alpha(z) = z^\alpha e_L^k$ and

$$Q_{h^k}(\alpha) = ||s_\alpha||_{L^2}^2 = \int_M |z^\alpha|^2 e^{-k\varphi(z)} dV_h(z).$$

We use both notations below: $||s_\alpha||_{L^2}^2$ is simpler, but we use $Q_{h^k}(\alpha)$ to emphasize that the L^2-norm square depends on h^k and α only. These norming constants determine h and ω.

Bergman kernels, partial Bergman kernels, spectral projections kernels

The probability measures in the Kähler setting are constructed from Bergman kernels. The kth Bergman kernel is the orthogonal projection:

$$\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k) := \text{ holomorphic sections of } \mathrm{L}^\mathrm{k}.$$

Its kernel w.r.t the Kähler volume form is denoted $\Pi_{h^k}(x, y)$. For any such kernel, the metric contraction (density of states) is denoted (in terms of an ONB),

$$\Pi_{h^k}(x) := \sum_{j=1}^{N_k} |s_{k,j}(z)|^2_{h^k}, \quad N_k = \dim H^0(M, L^k)$$

The probability measures

For any $z \in M^o$ and $k \in \mathbb{N}$, we define the probability measure,

$$\mu_k^z = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \, \delta_{\frac{\alpha}{k}} \in \mathcal{M}_1(\mathbb{R}^m), \qquad (3)$$

on $\mathbb{R}^m$. Here, $\Pi_{h^k}(z,z)$ is the contracted Szegö kernel on the diagonal (or density of states). The measures are discrete measures supported on $P \cap \frac{1}{k}\mathbb{Z}^m$.

Note: μ_k^z depends only on the moment map image $\nu_h(z) \in P$. These are generalizations of multi-nomial measures.

Another formula for the measures

Define

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|z^\alpha|^2 e^{-k\varphi(z)}}{Q_{h^k}(\alpha)}, \tag{4}$$

where $Q_{h^k}(\alpha) = ||z^\alpha||^2_{L^2}$.
Then,

$$\mu_k^z :== \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{P}_{h^k}(\alpha, z) \delta_{\frac{\alpha}{k}}$$

Note that $\frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP} \mathcal{P}_{h^k}(\alpha, z) = 1$.

Mean and variance of the toric measures

The mean is defined by

$$\vec{m}_k(z) = \int_P \vec{x} d\mu_k^z(x),$$

resp. the covariance matrix is defined by

$$[\Sigma_k]_{ij}(z) = \int_P (x_i - m_{k,i}(z))(x_j - m_{k,j}(z)) d\mu_k^z.$$

Lemma

Let $\mu_h : M \to P$ be the moment map. Then,

$$\vec{m}_k(z) = \nu_h(z) + O(1/k), \quad \Sigma_k(z) = \frac{1}{k} Hess\ \varphi(z) + O(\frac{1}{k^2}),$$

Note that $Hess \varphi = \omega_\varphi$.

Weak LLN for toric measures

Proposition

Let $\mu : M \to P$ be the moment map with respect to the symplectic form ω. Then for any $z \in M$,

$$\mu_k^z \rightharpoonup \delta_{\nu_h(z)}.$$

Thus, the measures concentrate at the moment map image of z.

Normalizing the measures to have mean zero and variance one

We re-center the measures at $\mu(z)$, i.e. put

$$\tilde{\mu}_k^z = \mu_k^z(x - \nu_h(z)),$$

and then dilate by $\sqrt{k}$ to define the normalized sequence,

$$D_{\sqrt{k}}\tilde{\mu}_k^z = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \, \delta_{\sqrt{k}(\frac{\alpha}{k} - \nu_h(z))}. \tag{5}$$

Equivalently, if $f \in C_b(\mathbb{R}^m)$. Then,

$$\langle f, D_{\sqrt{k}}\tilde{\mu}_k^z \rangle = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \, f(\sqrt{k}(\frac{\alpha}{k} - \nu_h(z)), \tag{6}$$

Here, $C_b(\mathbb{R}^m)$ denotes the space of bounded continuous functions on $\mathbb{R}^m$.

CLT for toric Kähler manifolds

THEOREM

In the topology of weak convergence on $C_b(\mathbb{R}^m)$,*

$$D_{\sqrt{k}}\widetilde{\mu}_k^z \stackrel{w*}{\to} \gamma_{0,Hess\ \varphi(z)}.$$

That is, for any $f \in C_b(\mathbb{R}^m)$,

$$\int_{\mathbb{R}^m} f(x) D_{\sqrt{k}} d\widetilde{\mu}_k^z(x) \to \int_{\mathbb{R}^m} f(x) d\gamma_{0,Hess\ \varphi(z)}(x).$$

The role of the parameter z is similar to that of the parameter p in the Bernoulli measures $\mu_p = p\delta_0 + (1-p)\delta_1$ and their convolution powers on the unit interval $[0,1]$. In very special cases, such as the Fubini-Study metric h of $M = \mathbb{CP}^m$, μ_k^z is itself a sequence of dilated convolution powers, $\mu_k^z = (\mu_1^z)^{*k} = \mu_1^z * \mu_1^z \cdots * \mu_1^z$ (k times).

Entropy of the toric measures μ_k^z

The main result is an asymptotic formula for the entropy $H(\mu_k^z)$ as $k \to \infty$. There are very few results, even classical, on asymptotic entropy.

For a finite probability distribution $\{p_\alpha\}$, the entropy of the distribution is

$$H = -\sum_\alpha p_\alpha \ln p_\alpha.$$

Thus, the entropy of μ_k^z is

$$H(\mu_k^z) = -\sum_{\alpha \in kP} \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} \ln \frac{|s_\alpha(z)|^2_{h^k}}{\|s_\alpha\|^2_{h^k}}.$$

Entropy $H(\mu)$ of a discrete probability measure μ is a measure of the degree to which μ is uniform. The larger the entropy, the more uniform the measure. Thus, entropy of μ_k^z is a measure of its uniformity as a measure on $kP \cap \mathbb{Z}^m$.

Ricci curvature and measures of maximal entropy

The entropy $H(\mu)$ of a discrete probability measure μ is a measure of the degree to which μ is uniform. The larger the entropy, the more uniform the measure, so that the measure of maximal entropy in a given family of probability measures is the most uniform measure. This measure of maximal entropy is often considered the most important. Hence it is natural to ask for which z does μ_k^z have maximal entropy in the family μ_k^z, at least asymptotically as $k \to \infty$. For instance, in the case of binomial measures μ_p^{*k}, $p = \frac{1}{2}$.

Asymptotics of entropy of μ_k^z (joint with Pierre Flurin)

Recall that $H = -\sum_\alpha p_\alpha \ln p_\alpha$. Also,

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|z^\alpha|^2 e^{-k\varphi(z)}}{Q_{h^k}(\alpha)}, \tag{7}$$

where $Q_{h^k}(\alpha) = ||z^\alpha||^2_{L^2}$. Thus, the entropy of μ_k^z is

$$H(\mu_k^z) = -\sum_{\alpha \in kP} \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z)} \ln \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z)}. \tag{8}$$

The asymptotic entropy result is:

Theorem

Let $h = e^{-\varphi}$ be a toric Hermitian metric on $L \to M$ and let $\omega_\varphi = i\partial\bar{\partial}\varphi$ be the corresponding Kähler metric. Then, as $k \to \infty$,

$$H(\mu_k^z) = \frac{1}{2} \log(\det\left((2\pi e k)(i\partial\bar{\partial}\varphi|_z)\right) + o(1)$$

Entropy asymptotics in terms of the symplectic potential

The entropy depends only on the image $\mu_h(z) = x_0$ of z under the moment map. We rewrite $\log\det i\partial\bar{\partial}\varphi$ in terms of the symplectic potential and its Hessian in action-angle variables, with action variables $x \in P$ and angle variables θ on $\mu_h^{-1}(x)$. Then set, $H_{ij} = (\mathrm{Hess(u)})_{\mathrm{ij}}^{-1} = \mathrm{u}^{,\mathrm{ij}}$ and

$$L(x) = \frac{1}{2}\log\det\nabla^2 u(x) = -\frac{1}{2}\log\det i\partial\bar{\partial}\varphi, \tag{9}$$

and Theorem 4 may be reformulated as follows.

THEOREM

Let $h = e^{-\varphi}$ be a toric Hermitian metric on $L \to M$ and let u be the open orbit symplectic potential. Then, as $k \to \infty$,

$$H(\mu_k^z) = \frac{1}{2}\log(\det\frac{(2\pi ek)}{\nabla^2 u|_{\mu_h(z)}}) + o(1) = \frac{m}{2}\log(2\pi ek) - L(x) + o(1).$$

Intuition for the entropy asymptotics

Note that the entropy of uniform measure $\mu_{kP\cap\mathbb{Z}^m}$ on a set of r element is $\log r$. The number $\#(kP \cap \mathbb{Z}^m)$ of such lattice points is $\simeq k^m \#(P \cap \mathbb{Z}^m)$, so that uniform measure on these lattice points has entropy $m \log k + \log \#(P \cap \mathbb{Z}^m)$. μ_k^z is not uniform, but rather is approximately a discretized Gaussian distribution centered at $\mu(z)$ and of width $k^{-\frac{1}{2}}$. A discretized Gaussian of width $k^{-\frac{1}{2}}$ and of height k^m is concentrated in the Ball $B(z, k^{-\frac{1}{2}})$ and is similar to uniform measure on that ball of the same height. This approximation accurately predicts the leading order term $\log k^{m/2}$.

Binomial distributions and Fubini-Study metrics

In dimension $m = 1$, the binomial distributions are convolution powers $\mu_k^p = (\mu_p)^{*k}$ of the Bernoulli measure μ_p defined by $\mu_p(\{1\}) = p, \mu_p(\{0\}) = 1 - p$. The entropy of μ_p is

$$p \log p + (1 - p) \log(1 - p).$$

This entropy is also the Fubini-Study symplectic potential $u_{FS}(p)$. The parameter $p \in [0, 1]$ is the image of the parameter $z \in \mathbb{CP}^1$ under the Fubini-Study moment map. The kth convolution power

μ_k^p is the binomial measure, for which $p_{k,\ell} = \binom{k}{\ell} p^\ell (1 - p)^{k-\ell}$. Its Shannon entropy has the asymptotics,

$$H(\mu_k^p) = \frac{1}{2} \log k + \frac{1}{2}(1 + \log(2\pi p(1 - p)) + O(k^{-\frac{1}{2}} + \epsilon).$$

To compare with the Theorem, we note that in the Fubini-Study case, $u''_{FS}(x) = \frac{1}{x(1-x)}$, $\log(u''_{FS}(x))^{-1} = \log x(1 - x)$.

Convolution powers?

In view of the resemblence of the entropy asymptotics of the toric Kähler probability measures μ_k^z to convolution powers, it is natural to characteristic the toric Hermitian line bundles $(L, h) \to (M, \omega)$ for which μ_k^z is a sequence of convolution powers.

Theorem

The sequence $\{\mu_k^z\}_{k=1}^{\infty}$ is a sequence of convolution powers for all z if and only if

- $\mathrm{Hilb_k(h)}$ *is balanced for all k, i.e. the density of states $\Pi_{h^k}(z) = C_k$ is constant for all k. Hence, ω is a Kähler metric of constant scalar curvature;*
- $\Pi_{h^k}(z, z) = C_k[\Pi_{h^1}(z, z)]^k$ *where*

$$C_k = \left(\frac{\#\{\alpha \in k\overline{P} \cap \mathbb{Z}^m\}}{(2\pi)^m Vol(P)} \right) \left(\frac{(2\pi)^m Vol(P)}{\#\{\alpha \in \overline{P} \cap \mathbb{Z}^m\}} \right)^k .$$

Ricci curvature and measures of maximal entropy

Locating the point $\mu(z) = x$ where μ_k^z has asymptotically maximal entropy is related to the Ricci curvature of (M, ω). We recall that the Ricci curvature of the Kähler metric ω_φ is given by $\mathrm{Ric}(\omega) = -i\partial\bar{\partial} \log \det(g_{i\bar{j}})$, i.e. $\mathrm{Ric}_{k\ell} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_\ell}(\log \det g_{i\bar{j}})$ where $\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. In the toric case,

$$\mathrm{Ric} = -\frac{1}{2} dd^c \log \det H = -\frac{1}{2} \sum_{i,j,k}^{m} H_{ij,jk} dx_k \wedge d\theta_j, \qquad (10)$$

Thus, the Ricci potential is the function $-L(x)$ where, $H_{ij} = (\mathrm{Hess}(u))^{-1}_{ij} = u^{,ij}$ and

$$L(x) = \frac{1}{2} \log \det \nabla^2 u(x) = -\frac{1}{2} \log \det i\partial\bar{\partial}\varphi, \qquad (11)$$

Ricci curvature and measures of maximal entropy

Due to the inverse relation of $i\partial\bar{\partial}\varphi$ and $\nabla^2 u$, points where the Ricci potential is maximal are points where (12) is minimal. In the simplest case of the Fubini-Study symplectic potential on $\mathbb{CP}^1$, in a standard gauge the symplectic potential satisfies, $\log u''_{FS}(x) = -\log x(1-x)$, and $\frac{d^2}{dx^2}\log u''_{FS}(x) = x^{-2} + (1-x)^{-2}$. The unique minimum point of $\log u''_{FS}$ occurs at $x = \frac{1}{2}$. In the case of multinomial distributions and Fubini-Study potentials in higher dimensions, the maximum occurs at the center of mass of the simplex. These are model cases of toric Fano Kähler -Einstein manifolds. It turns out that related statements are true for compact toric Kähler manifolds with positive Ricci curvature.

Ricci curvature and measures of maximal entropy

We recall that $\mathrm{Ric}(\omega)$ represents the first Chern class $c_1(M)$ and $\mathrm{Ric} > 0$ implies that (M, ω) is a toric Fano manifold. That is, if $\mathrm{Ric}(\omega) > 0$, then ω is a positively curved metric on the anti-canonical bundle $-K_X$, hence $-K_X$ is ample. A toric Fano manifold has a distinguished center, namely the center of mass of polytope.

Theorem

For fixed (L, h, M, ω), the points $x = \mu(z)$ for which the measures μ_k^z have asymptotically maximal entropy as $k \to \infty$ occur at the minimum points of

$$L(x) = \frac{1}{2} \log \det \nabla^2 u(x) = -\frac{1}{2} \log \det i\partial\bar{\partial}\varphi. \qquad (12)$$

If (M, ω) is Fano and $\mathrm{Ric}(\omega)$ is positive, then there is a unique minimum. In the Kähler -Einstein Fano case, where $Ric(\omega) = a\omega$, the point of maximal entropy is the center of mass of P.

Differential entropy of the Gaussian measure γ_{h^k}

There is a second (and much simpler) problem regarding entropies of probability measures on a toric Kähler manifold, or indeed on any polarized Kähler manifold. Associated to any Hermitian metric h on L is a sequence $\{\mathrm{Hilb_k(h)}\}_{\mathrm{k=1}}^{\infty}$ of Hermitian inner products on $H^0(M, L^k)$. In turn the inner product induces a Gaussian measure γ_{h_k} on $H^0(M, L^k)$. If we fix a background metric h_0, or corresponding inner product G_0, then the inner product $\mathrm{Hilb_k}$ is represented by a positive Hermitian matrix P and the Gaussian measure γ_k^h is represented by $\sqrt{\det P} e^{-\langle P^{-1}X, X\rangle}$ on $\mathbb{C}^{N_k}$ where $N_k = \dim_{\mathbb{C}} H^0(M, L^k)$,

Differential entropy of the Gaussian measure γ_{h^k}

When a probability measure μ on $\mathbb{R}^n$ has a density f relative to Lebesgue measure dx, its *differential entropy* is defined by

$$H(fdx) = -\int_{\mathbb{R}^n} f(x) \log f(x) dx.$$

It is well-known that if $f(x) = N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ is a Gaussian, then,

$$h(fdx) = \ln(\sigma\sqrt{2\pi e}).$$

Proposition

Let (L, h, M, ω) be any polarized Kähler manifold, and let γ_k^h be the associated Gaussian measure on $H^0(M, L^k)$. Then $H(\gamma_k^h) = -\log\det \mathrm{Hilb_k(h)}$. The Hermitian metric h for which $H(\gamma_k^h)$ has maximal entropy is the balanced metric.

Wright-Fisher Markov processes

Given a polarized toric Kähler manifold (M, L, ω) with positive Hermitian toric line bundle $(L, h) \to (M, \omega)$ and $N = 1, 2, \ldots,$ the Kähler toric Wright-Fisher Markov chain is defined by the transition matrix $P^{(N)}_{\alpha\beta}$

$$P^{(N)}_{\alpha\beta} = \frac{|s_\alpha(\beta)|^2_{h^N}}{Q_{h^N}(\alpha)\Pi_{h^N}(\beta, \beta)},$$

on the state space $S_N := N\overline{P} \cap \mathbb{Z}^m$.

Theorem

As $N \to \infty$, the Markov chain converges to the diffusion process on $C^2(\overline{P})$ with generator,

$$\mathcal{L}_1 := \sum_{j,k=1}^{m} u^{jk} \frac{\partial^2}{\partial x_j \partial x_k},$$

where u is the symplectic potential of (M, L, ω) and where $\left(u^{jk}\right)$ is the inverse of the Hessian of u.