Non-Commutative Integrable Systems & Their Singularities

- Based on joint works:

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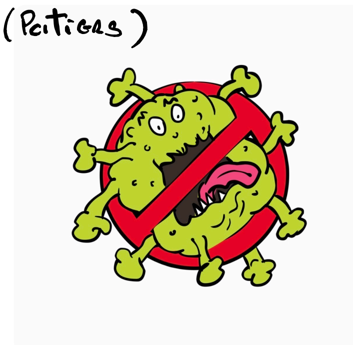

Aim: • Classification of regular NCIS
• Understand singularities of NCIS

Plan:
1) NCIS
2) Lagrangian Fibrations
3) Isotropic Fibrations
4) Singularities

1) NCIS

Def: A **NCIS** on (S^{2m},ω) is a collection $f_1,\dots,f_n \in C^\infty(S)$ such that:

(i) $\{f_i,f_j\}=0$, $1\leq i\leq \underbrace{2m-n}_{=r}$, $1\leq j\leq n$

(ii) $df_1\wedge\dots\wedge df_n \neq 0$ on an open dense set

Rmks:

- r = rank of NCIS $(r\leq n)$
- CIS $\equiv$ NCIS with $r=n=\frac{1}{2}\dim M$
- $p\in S$ is regular point if (ii) holds
- Nekhorošev, Fomenko-Mischenko, Dazord-Delzant (...)

- Many examples: - Motion in a centre force field
  - Kepler Problem
  - Euler-Poisson top
  - Gelfand-Cetlin system

<u>Geometric Version:</u> $\mu = (f_1, \ldots, f_m) : S \to \mathbb{R}^m$

CIS $\longleftrightarrow$ Lagrangian Fibrations

<u>Def</u>: A (regular) <u>isotropic fibration</u> is a submersion $\mu : (S, \omega) \to M$ w/ connected fibers such that:

(i) $\mathcal{F}_\mu$ is isotropic

(ii) $\mathcal{F}_\mu^{\perp\omega}$ is integrable

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<u>Rmk</u>:
- Rank $= \dim M - \dim S = \dim \mathcal{F}_\mu$
- Lagrangian fibration $=$ isotropic fibration of rank $\frac{1}{2} \dim S$
- Regular part of NCIS $\Rightarrow$ isotropic fibration.
- General isotropic fibration: smooth map $\mu : (S, \omega) \to M$ which is regular isotropic fibration on open dense set $U \subset S$.
- NCIS $\longleftrightarrow$ isotropic fibrations

2) Geometry of Lagrangian fibrations

Basic Fact: The base of regular Lagrangian fibration

$\mu:(S,\omega)\longrightarrow M$ w/ compact, connected, fibers has

an integral affine structure on M.

Assume this from now on

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Recall: A $\mathbb{Z}$-affine structure on M is an atlas $\{(U_i,\phi_i)\}$

$$\phi_i\circ\phi_j^{-1} : \underset{\subset\,\mathbb{R}^m}{\phi_i(U_i\cap U_j)} \longrightarrow \underset{\subset\,\mathbb{R}^m}{\phi_j(U_i\cap U_j)}$$

$$x \longmapsto Ax+b \quad \text{w/} \begin{cases} A\in GL_m(\mathbb{Z}) \\ b\in\mathbb{R}^m \end{cases}$$

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$\exists$ 1:1 correspondence:

$$\{\mathbb{Z}\text{-affine str. on } M\} \longleftrightarrow \{\text{integrable lattices } \Lambda\subset T^*M\}$$

$$(U,\phi=(x^1,..,x^m)) \to \Lambda_U = \mathbb{Z}\langle dx^1,..,dx^m\rangle \subset T^*_U M$$

$$\rightsquigarrow \Lambda\subset T^*M \quad \begin{cases} \mathbb{Z}\text{-sub bundle of rank} = \dim M \\ \text{Lagrangian (for } \omega_{can}) \end{cases}$$

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$\mathbb{Z}$-affine struct of Lagrangian fibration $\mu:(S,\omega)\longrightarrow M$:

$$\Lambda_x := \{\alpha\in T^*_x M : \varphi^1_{X_\alpha} = id\} \Rightarrow \Lambda = \bigcup_{x\in M}\Lambda_x$$

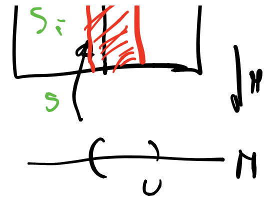
$$i_{X_\alpha}\omega = \mu^*\alpha, \quad X_\alpha\in\mathfrak{X}^1(\mu^{-1}(x))$$

S

Local classification:

$$(T^*_x M, \lambda) \longrightarrow \mu^{-1}(x)$$

$$\alpha_x \longmapsto \varphi^1_{X_{\alpha}}(s(x))$$



$$\left(U\times\mathbb{R}^m/\mathbb{Z}^m, \sum_i da_i\wedge d\theta_i\right) = \left(T^*_U M/\Lambda_0, \omega_{can}\right) \xrightarrow{\ \sim\ } (\mu^{-1}(U), \omega) \subset (S,\omega)$$

$$\mathbb{R}^m/\mathbb{Z}^m \cong \mathbb{T}^m$$

Action-Angle Thm

$$\begin{tikzcd} T^*_U M/\Lambda_0 \arrow[dr, "pr_n"'] & & \mu^{-1}(U) \arrow[dl, "\mu"] \\ & U \arrow[ur, dashed, bend right, "s"'] & \end{tikzcd}$$

s ≡ Lagrangian section

Global Classification:

Thm (Duistermaat) Fix $\Lambda \subset T^*M$ a $\mathbb{Z}$-affine structure:

$$\left\{ \begin{array}{l} \text{Lagrangian Fibrations} \\ \text{inducing } \Lambda \end{array} \right\} \Big/ \text{iso} \longleftrightarrow \underbrace{c \in \check{H}^1(M; (T^*M/\Lambda)_{Lag})}_{\text{Lagrangian-Chern class}}$$

$$\text{——— / ———}$$

. $(T^*M/\Lambda, \omega_{can}) \longrightarrow M \rightsquigarrow (T^*M/\Lambda)_{Lag}$ = sheaf of Lag. sections

$$S_i : U_i \longrightarrow S \text{ Lag. sect.} \Rightarrow \left\{ \begin{array}{l} s_j(x) = \varphi^1_{X_{s_{ij}(x)}}(s_i(x)) \\ \\ s_{ij} : U_i \cap U_j \longrightarrow (T^*M/\Lambda, \omega_{can}) \end{array} \right.$$

$$\Rightarrow \quad c := [s_{ij}]$$

Rmk: . Iso classes of Lagrangian fibration is a vector space

w/ origin $\boxed{(T^*M/\Lambda, \omega_{can}) \longrightarrow M}$

· What about singular Lagrangian fibrations?

- Classification of singularities (local)

  · Singular pt $p_0 \in S$: $d_{p_0}\mu = 0$ $\quad x_0 = \mu(p_0)$

$$a : T^*_{x_0} M \longrightarrow \mathfrak{sp}(T_{p_0}S, \omega_{p_0}), \quad \alpha \longmapsto d_{p_0} X_\alpha$$

$$[\cdot,\cdot] \equiv 0$$

Defn: $p_0 \in S$ is generic if image is a CSA.

Generic singularities are combinations of 3 types:

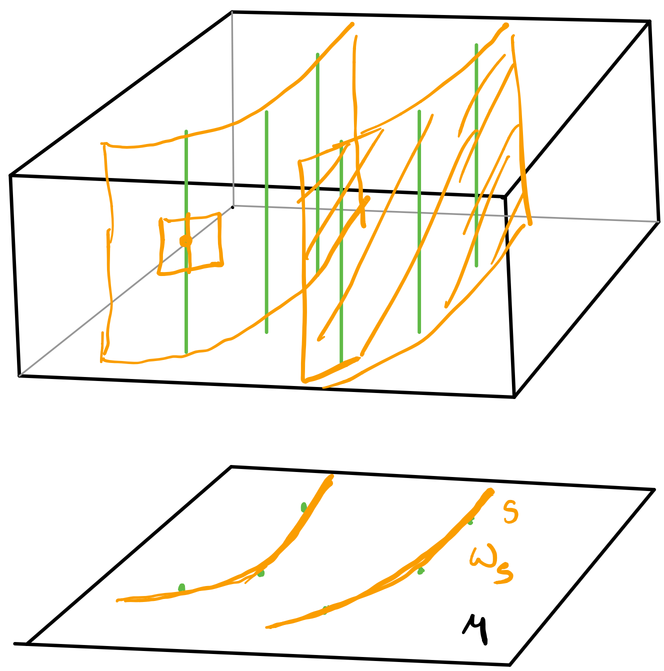
- Elliptic ($\simeq$ Toric action)
- Focus-Focus
- Hyperbolic

Eliasson's Normal Form Theorems

Around a non-degenerate singularity, fibration is equivalent to linear model.

- Classify Lagrangian fibrations w/ specified singularities
  - Toric fibrations (any dimension)
  - semi-toric fibrations (dimension 4)

3) Geometry of Isotropic fibrations



$$\begin{array}{c} (S,\omega) \quad \omega^{-1} \\ \Big\downarrow \mu \;\; \text{Poisson map} \\ (M,\pi) \end{array}$$

$$\pi = \mu_*(\omega^{-1})$$

$\Longrightarrow$ $(M, \mathcal{F}, \omega_{\mathcal{F}})$ - Regular symplectic foliation

$$\Updownarrow \pi$$

(M, π) - Regular Poisson manifold

__Def:__ An __isotropic fibration__ is a Poisson map $\mu : (S,\omega) \to (M,\pi)$ which is a submersion on an open dense set w/ isotropic fibers.

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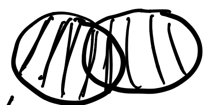
Thm (Dazord-Delzant)

The base (M,π) of isotropic fibration w/ compact connected fibers has a transverse $\mathbb{Z}$-affine structure.

Transverse $\mathbb{Z}$-affine structure:

- Foliation atlas $\{(U_i,\phi_i)\}$ such that

U_i $\quad$ U_j



$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\phi_i"'] \arrow[dr] & \\ \mathbb{R}^q \arrow[rr, "\phi_{ij}"] & & \mathbb{R}^q \end{tikzcd} \qquad \phi_{ij}(x) = Ax+b \quad \begin{cases} A \in GL_q(\mathbb{Z}) \\ b \in \mathbb{R}^q \end{cases}$$

- Integrable lattice $\Lambda \subset \nu^*(\mathcal{F}) = (T\mathcal{F})^0 \subset (T^*M, \omega_{can})$

  $\Lambda \subset \nu^*(\mathcal{F}) = (T\mathcal{F})^0 \subset T^*M$ - $\mathbb{Z}$-subbundle w/ rank $=$ codim $\mathcal{F}$ isotropic (for ω_{can})

Example:

$\mathcal{F} =$ fibers of $p: M \to (B, \Lambda_B)$

$\Lambda = p^*\Lambda_B$, w/ Λ_B $\mathbb{Z}$-affine st. on B

Local classification: (Dazord-Delzant)

$$\alpha_x \longmapsto \varphi^1_{X_\alpha}(s(x))$$

$$\begin{tikzcd} \left(U\times \mathbb{R}^q/\mathbb{Z}^q, \sum_{i=1}^q dy_i \wedge d\theta_i + B\right) = \left(\nu^*_U(\mathcal{F})/\Lambda_U, \omega_{can} + pr_n^* B\right) \arrow[r, "\sim"] \arrow[dr, "pr_n"'] & (\mu^{-1}(U), \omega) \subset (S,\omega) \arrow[d, "\mu"] \\ & U \arrow[u, bend right, dashed, "s"'] \end{tikzcd}$$

$(q =$ codim $\mathcal{F})$

s w/ $\ker s^*\omega \oplus T\mathcal{F} = TM$

Action-angle Thm

$B := s^*\omega \in \Omega^2(M)$

Global classification

Thm (Delzant–Dazord, Crainic–RLF–Martinez)

Given regular (M,π) there exists isotropic fibration $\mu:(S,\omega)\to(M,\pi)$ iff

i) (M,π) is integrable to $(\mathcal{G},\Omega)\rightrightarrows M$

ii) Symplectic gerbe of $(\mathcal{G},\Omega)$ is trivial.

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Rmks: . If $\mathcal{F}$ = fibers of $p: M\to B$, $\Lambda = p^*\Lambda_B$

$$c_2(\mathcal{G},\Omega)\in \check{H}^2(\underset{\underset{M/\mathcal{F}}{\parallel}}{B},(T^*B/\Lambda_B)_{LAG.})$$

Thm (RLF, unpublished)

Given regular (M,π) and integration $(\mathcal{G},\Omega)\rightrightarrows M$

$$\left.\begin{Bmatrix}\text{Isotropic fibrations}\\ \text{inducing }(\mathcal{G},\Omega)\end{Bmatrix}\right/_{\text{iso}} \longleftrightarrow c_1\in \check{H}^1(\mathcal{B};(\nu^*(\mathcal{F})/\Lambda)_{LAG})$$

$\mathcal{G}$ $\qquad (\nu(\mathcal{F})/\underline{\Lambda}^{\omega_{can}})$ $\downarrow\uparrow s$ M

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- $\mathcal{B} = M/\mathcal{F}$ a stack
- $(\nu^*(\mathcal{F})/\Lambda,\omega_{can})\to\mathcal{B}$
- c_1 is a Relative Lagrangian-Chern class

Fix $(S_0,\omega_0)\to(M,\pi)$ inducing $(\mathcal{G},\Omega)$.

For $(S,\omega)\to(M,\pi)$ " " :

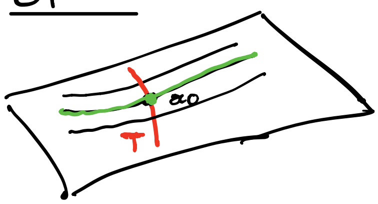
$$c_1(S,S_0)\in\check{H}^1(\mathcal{B},(\nu^*(\mathcal{F})/\Lambda)_{LAG})$$

4) Singularities of isotropic fibrations (w/ D. Sepe)

What are the generic singularities of $\mu : (S, \omega) \to (M, \pi)$?

$p_0 \in S$ singular pt of μ $(d_{x_0}\mu = 0)$ $\Rightarrow$ $\begin{cases} x_0 = \mu(p_0) \text{ is regular pt of } (M,\pi) & (\mathrm{I}) \\ x_0 = \mu(p_0) \text{ is singular pt of } (M,\pi) & (\mathrm{II}) \end{cases}$

Type I:

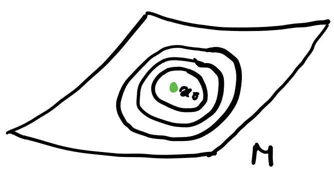


$\Rightarrow$ $(\mu^{-1}(T), \omega|_{\mu^{-1}(T)}) \xrightarrow{\mu} T$, $p_0 \in \mu^{-1}(T)$ Lagrangian Fibration.

Lagrangian w/ generic singularity at p_0

$\rightsquigarrow$ $\begin{cases} \text{- elliptic} \\ \text{- focus-focus} \\ \text{- hyperbolic} \end{cases}$

Type II:



$\pi_{x_0} = 0 \Rightarrow \mathfrak{g}_{x_0} = T^*_{x_0}M$ Lie algebra

Elliptic
- If $\mathfrak{g}_{x_0}$ is compact, semisimple $\Rightarrow M \simeq \mathfrak{g}^*_{x_0}$

  $\rightsquigarrow$ $G\,G\,(S,\omega) \xrightarrow{\mu} \mathfrak{g}^*$ multiplicity free space

- What if $\mathfrak{g}_{x_0}$ is not compact, ss ?

  Cohomological characterization Lie algebra cohomology.

  (on going work w/ D. Sepe).

Thanks for your attention!