Invariance of Symplectic Cohomology under deformations.

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joint work with

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Geometry Seminar, Lisbon, 5 May 2020

Example from Morse theory

(M,g) Riemannian manifold

$[\alpha] \in H^1(M;\mathbb{Z}) \cong [M, S^1]$

$$\begin{tikzcd} t \circlearrowright \widetilde{M} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R} \circlearrowleft \arrow[d] \\ M \arrow[r, "\alpha"] & S^1 \end{tikzcd}$$

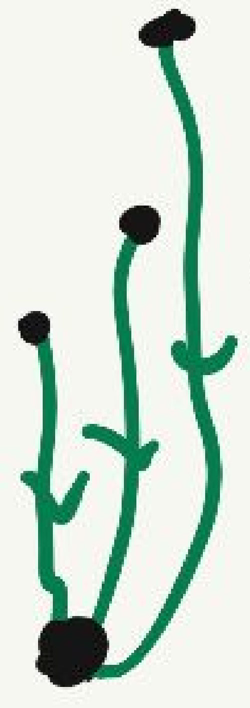
deck transformation $+1$

$\pi^*\alpha = df$

Assume f Morse.

Morse cohomology $H^*(\widetilde{M}, f)$

Chain complex generated by critical points



x

Differential counts $-\nabla f$ trajectories, so must allow infinite sums $\sum a_n x_n$ provided $f(x_n) \to \infty$.

$x_n \in \mathrm{Crit} f$

$a_n \in \mathbb{Z}$

Equivalently: pick lifts $\mathrm{zeros}(\alpha) \to \mathrm{Crit}(f)$

$x \longmapsto \tilde{x}$

$\pi^{-1}x = \left\{ t^n \tilde{x} : n \in \mathbb{Z} \right\} \Rightarrow C^*(\widetilde{M}, f) = \bigoplus_{x \in \mathrm{zeros}(\alpha)} \mathbb{Z}((t)) \cdot \tilde{x}$

Laurent series

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$t \circlearrowright$

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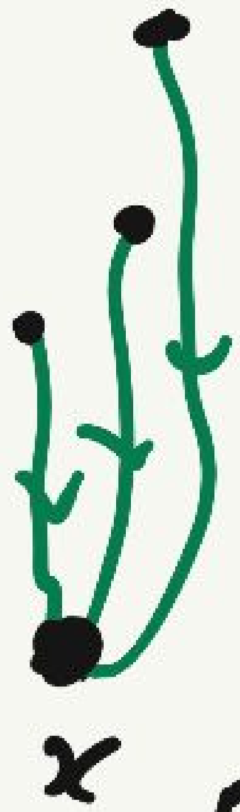

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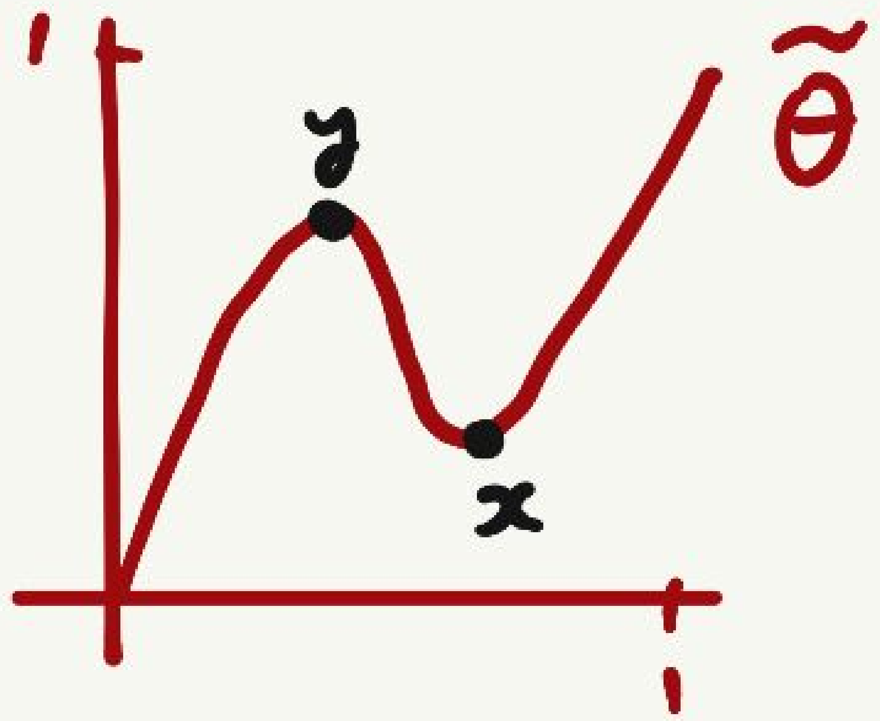
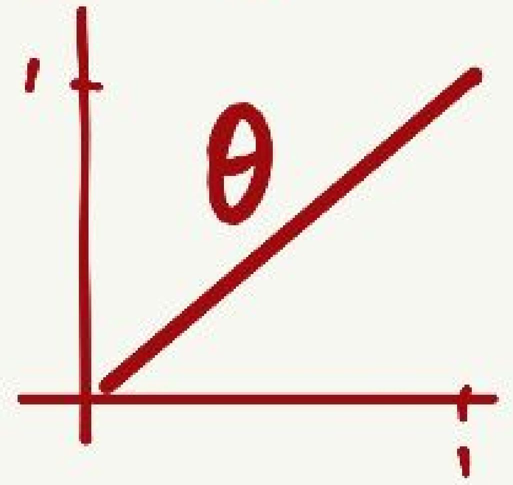
$\pi^{-1}x = \left\{ t^n \tilde{x} : n \in \mathbb{Z} \right\} \Rightarrow C^*(\tilde{M}, f) = \bigoplus_{x \in \mathrm{zeros}(\alpha)} \mathbb{Z}((t)) \cdot \tilde{x}$ ← Laurent series

Example $M = S^1$ $\alpha = d\tilde{\theta}$



(perturbation of angle function



$\tilde{M} = \mathbb{R}$

$\pi^*\theta = (id : \mathbb{R} \to \mathbb{R})$

no critical points

so $H^* = 0$.

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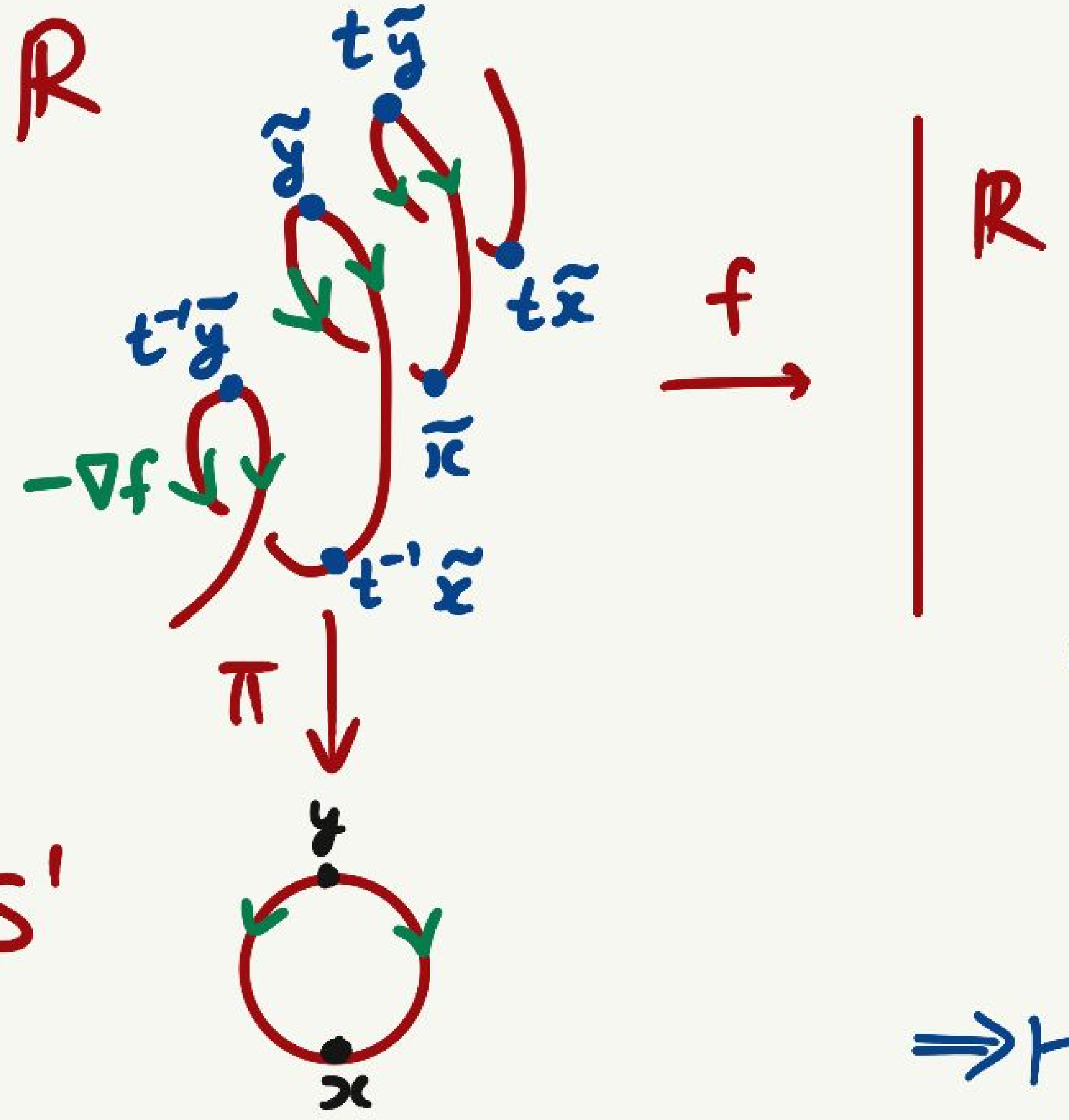


$M = S^1$

$\mathbb{Z}((t))\,\tilde{y}$

∂

$\mathbb{Z}((t))\,\tilde{x}$

$$\partial \tilde{x} = \tilde{y} - t\tilde{y} = \underbrace{(1-t)}_{\text{invertible}}\tilde{y}$$

$\Rightarrow H^*(M;\alpha) = 0$

Can we just work in M?

$$C^*(M;\alpha) = \bigoplus_{x\in \mathrm{Zeros}(\alpha)} \mathbb{Z}((t))\cdot x$$

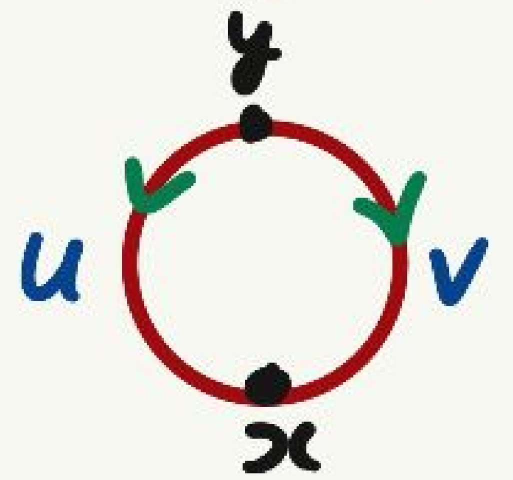
X_α = vector field : $g(\cdot, X_\alpha) = \alpha$

Differential $\quad \partial x = \sum \pm \underbrace{t^{-\int u^*\alpha}} y$

weighted count of X_α-trajectories u

In the example:

$M = S^1$



$\mathbb{Z}((t))\cdot y$

$\mathbb{Z}((t))\cdot x$ $\quad \partial$

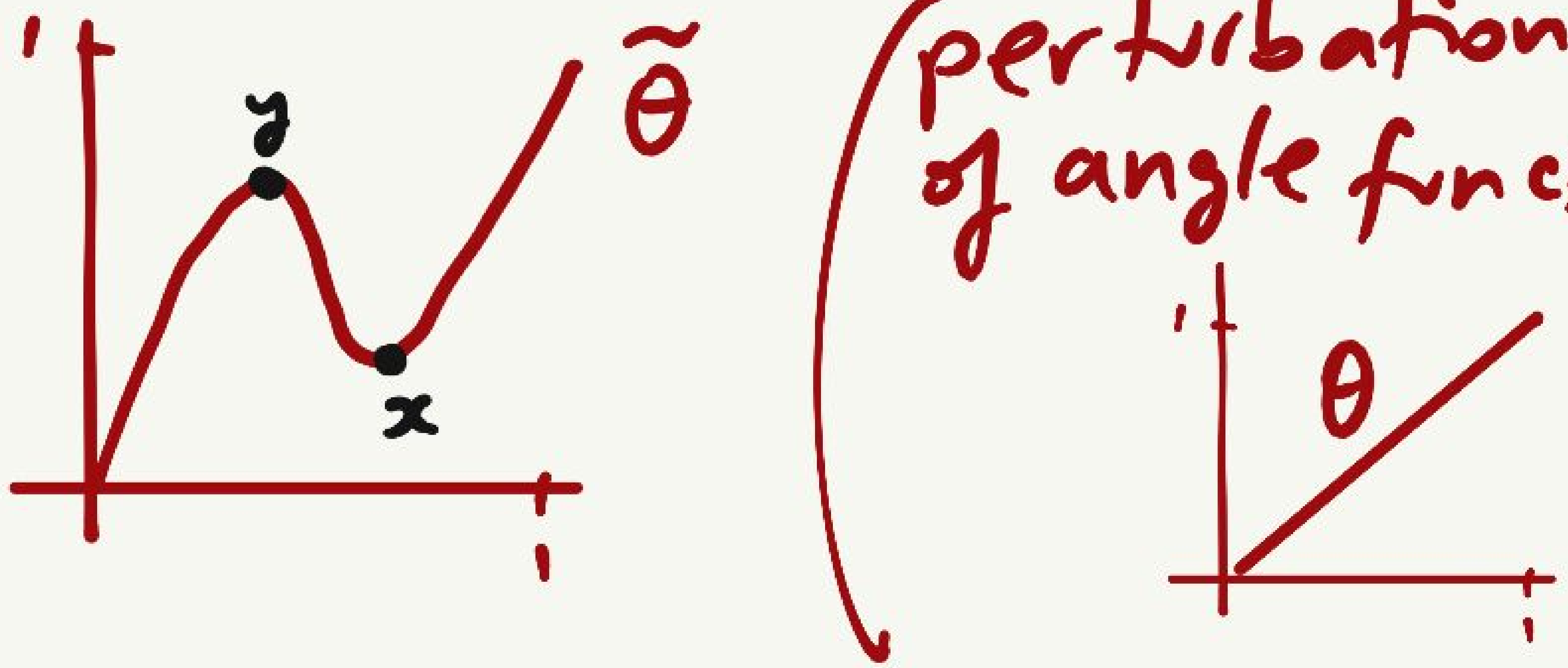
$$\partial x = t^{-\int u^*\alpha} y - t^{-\int v^*\alpha} y$$

$\tilde\theta(y) - \tilde\theta(x)$ $\qquad \tilde\theta(y) + 1 - \tilde\theta(x)$

change of variables $\quad x \mapsto t^{\tilde\theta(x)} x$, $\quad y \mapsto t^{\tilde\theta(y)} y$ $\quad$ gives back: $\partial x = (1-t) y$

Example

$M = S^1 \qquad \alpha = d\tilde\theta$



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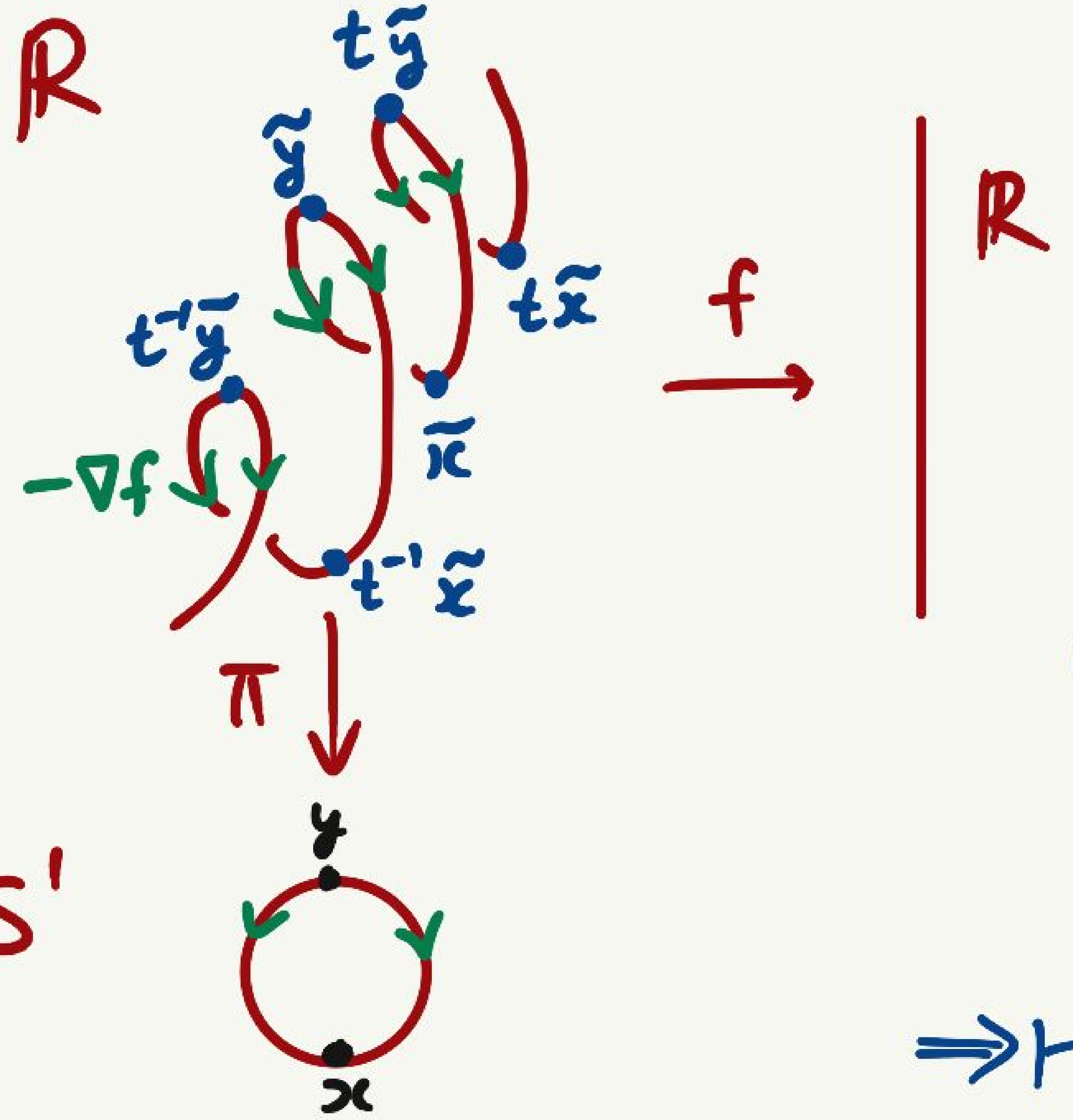

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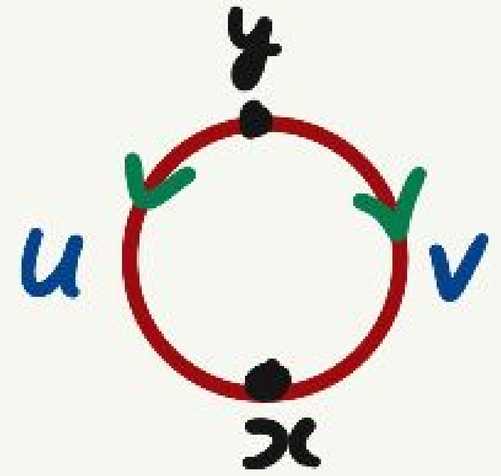

$$\begin{array}{l} \mathbb{Z}((t))\cdot y \\ \uparrow \partial \\ \mathbb{Z}((t))\cdot x \end{array}$$

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Farber, Pajitnov, Ranicki

$H^*(M;\alpha)$ called Novikov cohomology

so Morse cohomology for M for vector field X_α but use local system of coefficients Λ_α for $[\alpha]\in H^1(M;\mathbb{Z})$

Classical invariance theorems:

1) if deform α within its class $[\alpha]\in H^1(M;\mathbb{Z})$ then $H^*(M;\alpha)$ same up to isomorphism.

2) if deformation α_s does not preserve $[\alpha]$?

$$H^*(M;\alpha_1) \cong H^*(M;\alpha_0;\underbrace{\Lambda_{\alpha_1-\alpha_0}}_{\text{need local system}})$$

Not entirely obvious since the flowlines have changed

same flowlines, but infinitesimal twist of coefficients

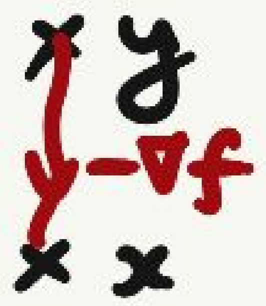
Example $\alpha_0 = d(\text{Morse function } h)$

$$H^*(M;\alpha) \cong \text{Morse cohomology of } h \text{ with } \Lambda_\alpha \cong H^*(M;\Lambda\alpha)$$

e.g. $M=S^1$, h = height function, but insert $t^{-\int u^*\alpha}$.

Floer cohomology

classical (Morse)	Floer
M manifold	$\mathcal{L}M = C^\infty(S^1, M)$ free loop space (M,ω) symplectic mfd
$f: M \to \mathbb{R}$ Morse function	$A_H : \mathcal{L}M \to \mathbb{R}$ Action functional choice of Hamiltonian $H: M \to \mathbb{R}$
Crit f	Crit A_H = 1-periodic Hamiltonian orbits $x'(t) = X_H \qquad (\omega(\cdot, X_H) = dH)$
g Riemannian metric	L^2-metric $\int \omega(\cdot, J\cdot)\,dt$ (choice ω-compatible almost complex structure J)
$-\nabla f$ trajectory $u: \mathbb{R} \to M$ $\partial_s u = -\nabla f$	$-\nabla A_H$-trajectory $u: \mathbb{R} \times S^1 \to M$ $\partial_s u + J\partial_t u = -\nabla H$
Morse chain complex $\partial x = \pm y + \ldots$	Floer chain complex $\partial x = \pm t^{E(u)} y + \ldots$ "energy"



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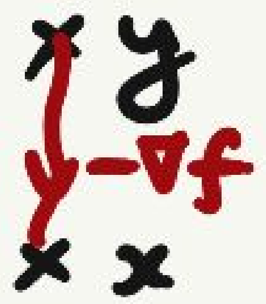
Morse chain complex $\partial x = \pm y + \ldots$ 	Floer chain complex $\partial x = \pm t^{E(u)} y + \ldots$ "energy" 

Floer's invariance theorem 1987:

For M closed, if change H, J

$HF^*(M, \omega, J, H)$ same up to iso.

Moser's argument

If deform ω within its class $[\omega] \in H^2(M;\mathbb{R})$

(M,ω) same up to symplectomorphism.

($\Longrightarrow HF^*$ same up to iso, for M closed)

For closed M:

deform H to C^2-small Morse function f

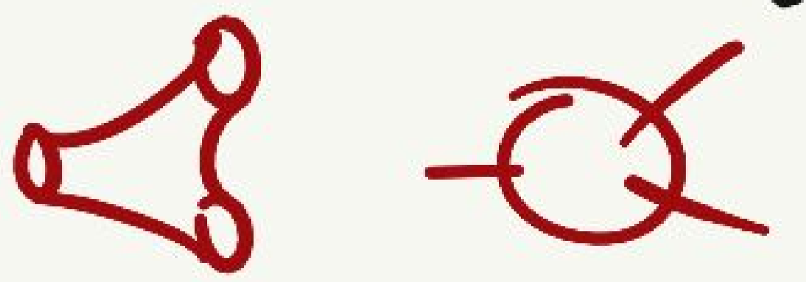
$\Rightarrow$ 1-orbits become critical points of f,
Floer traj. become $-\nabla f$ traj.

$\Rightarrow HF^* \cong H^*(M;\Lambda)$

Novikov ring behaves like $\mathbb{Z}((t))$

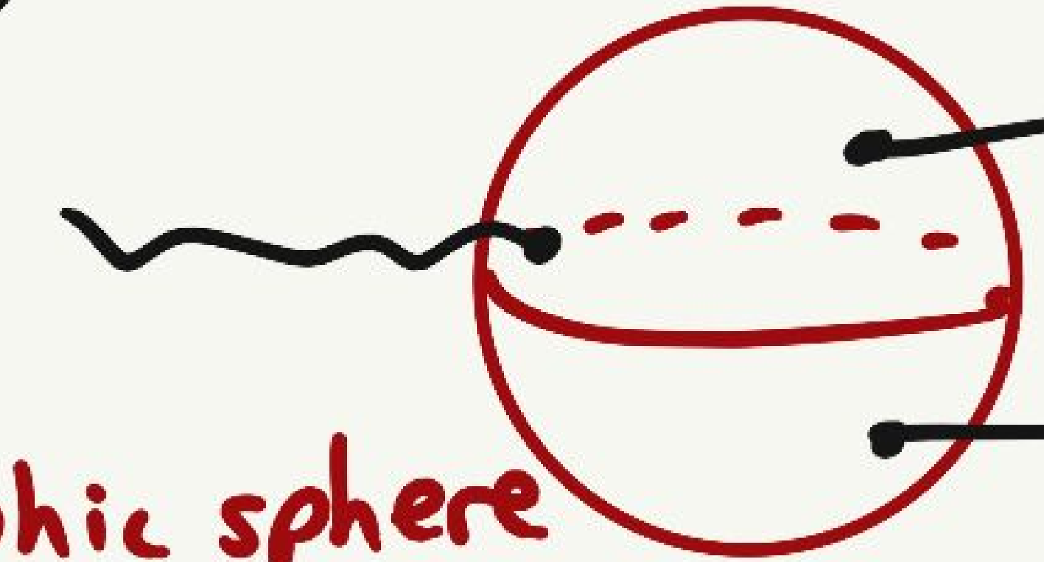
Piunikhin-Salamon-Schwarz 1996 As a ring,

$HF^* \cong QH^*(M,\omega)$



Quantum Cohomology $QH^*(M,\omega)$

$H^*(M;\Lambda)$ but deform cup product:



$$\alpha * \beta = \pm t^{E(u)} \gamma + \cdots$$

"energy" $= \int u^*\omega$

Notice:

1) If deform J, the moduli space of spheres changes. FACT: ring QH^* same up to iso, as long as J tames ω $(\omega(v,Jv)>0\ \forall v\neq 0)$

2) If deform ω by a little, then ω still tames J

$\implies$ deformation theorem

$$QH^*(M,\omega_1) \cong QH^*(M,\omega_0; \underbrace{\Lambda_{\omega_1-\omega_0}})$$

Easy since don't even need to change moduli spaces.

means we insert weights $t^{\int u^*(\omega_1-\omega_0)}$ in counts of spheres

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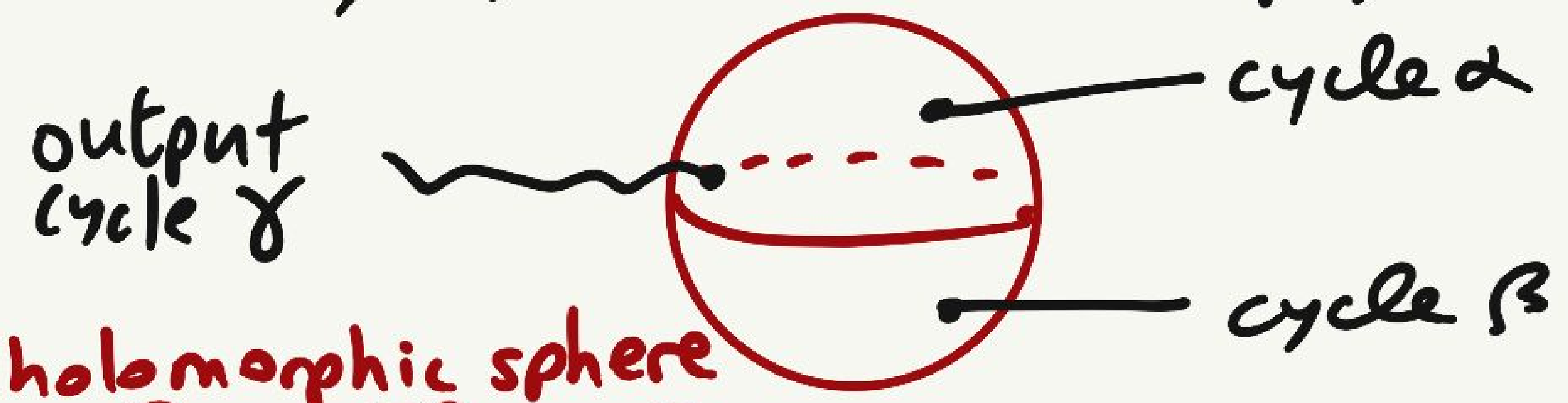

$\partial_s u + J\partial_t u = 0$

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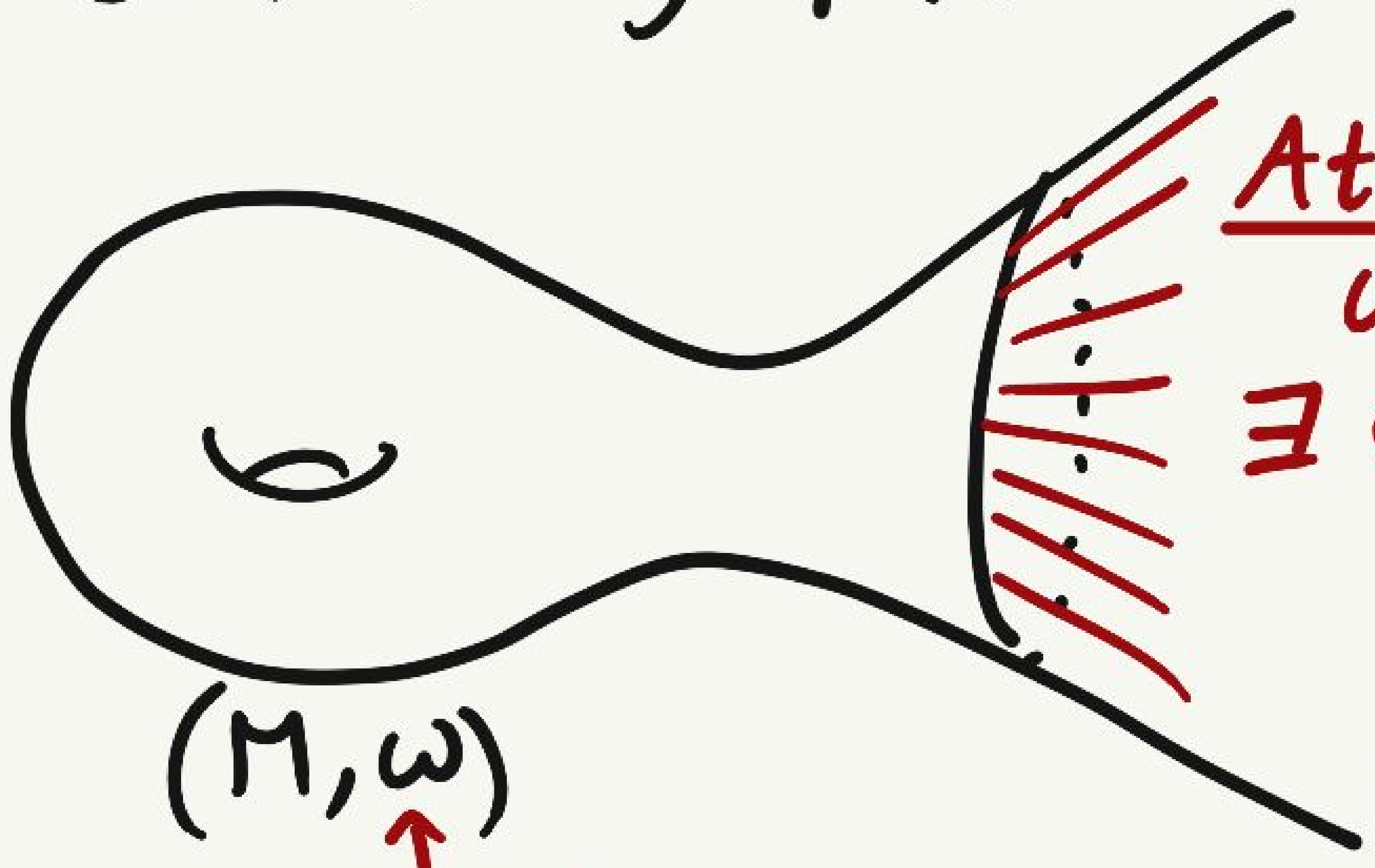
Symplectic Cohomology

Convex symplectic manifold (M,ω)



(M,ω)

possibly not exact

At ∞: require $\omega = d\theta$

$\exists$ exhausting function h (proper, bounded below)

$\theta(X_h) > 0$

($\Longrightarrow \exists$ radial coordinate $R\in[1,\infty)$)

Examples:
- $(T^*N, \omega = d\theta)$ $\quad h = \|p\|^2$
- Magnetic $(T^*\Sigma, \omega = d\theta + \pi^*\sigma)$ — Σ surface, σ closed 2-form on Σ
- Negative complex line bundles e.g. $\mathcal{O}(-k)\to \mathbb{C}P^m$

$$SH^*(M,\omega) = \varinjlim HF^*(M,H)$$

where H linear in R at ∞, and in limit: slope $\to\infty$.

$$QH^*(M,\omega) = HF^*(M, H \text{ with small slope}) \xrightarrow{c^*} SH^*$$

Def Isomorphism of such M: symplectomorph $\varphi:(M,\omega_0,\theta_0)\to(M,\omega_1,\theta_1)$, $\varphi^*\theta_1=\theta_0$ at ∞.

Invariance Thm (R. 2010, based on Seidel 2008)
Such isos preserve SH^*.

Rmk $\Longrightarrow SH^*$ independent of h. determines R

to appear: International Journal of Mathematics

Observation (Benedetti – R. 2018) isos preserve

$$[\omega,\theta]\in H^2(M,\infty)\cong H^2_c(M) \qquad (*)$$

Rmk $\Longrightarrow$ Moser argument that deforms ω within class $[\omega]\in H^2(M)$ may not preserve $(*)$
$\Longrightarrow (*)$ is an obstruction to existence of iso.

Rmk Liouville manifold means $\omega=d\theta$ on M (not just at ∞)

Observation (B.–R.) holds $\Longleftrightarrow (*)=0$.

Not same as just saying ω exact since θ may not extend: call those Quasi-Liouville
e.g. Contreras–Macarini–Paternain 2004:
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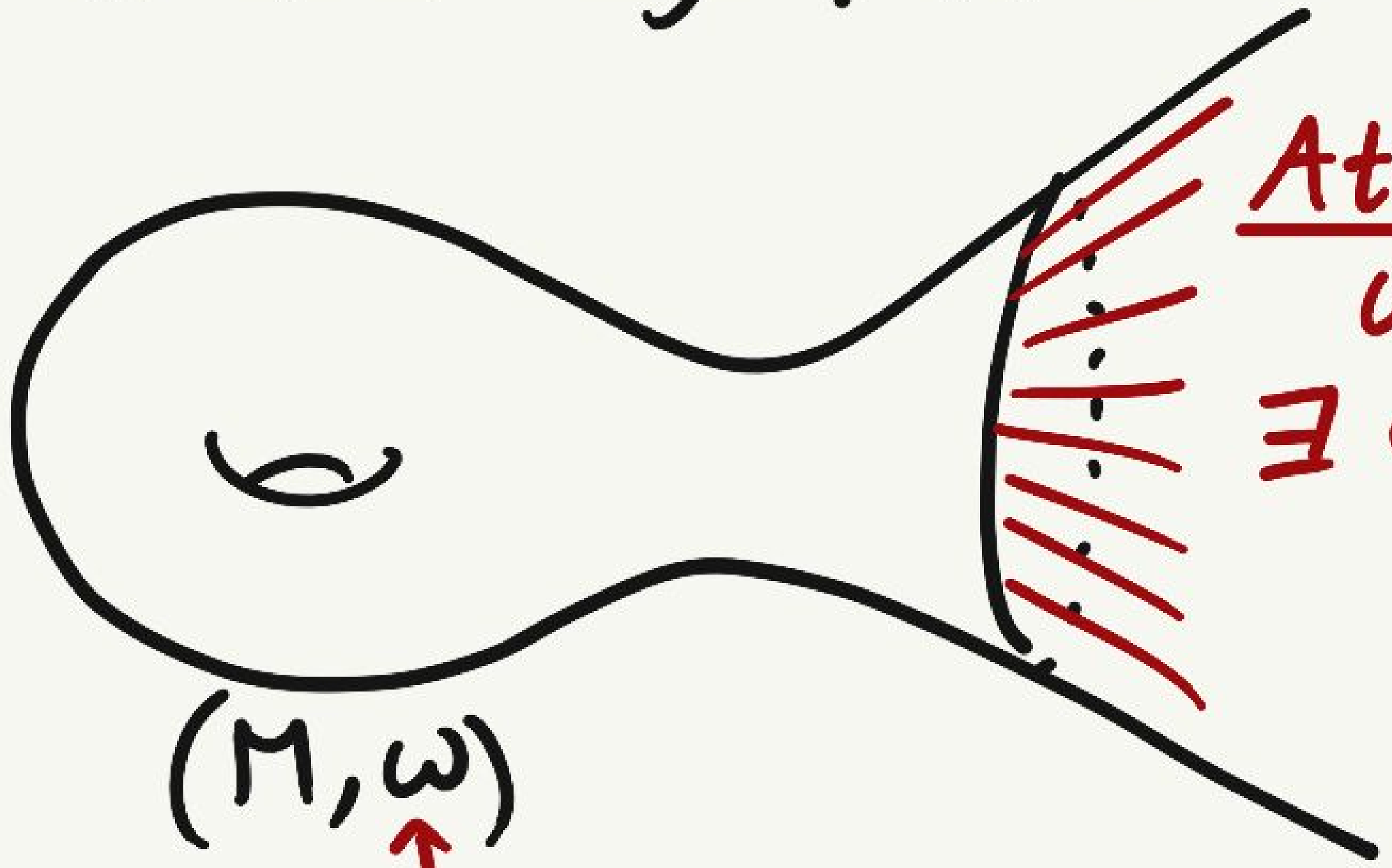

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Invariance under deformations

Def Deformation of (M,ω,θ) means family (M,ω_s,θ_s) convex with exhausting h_s at ∞.
($h: M\times[0,1]\to\mathbb{R}$)

Theorem 1 (B.-R.)
$[\omega_s,\theta_s]\in H^2_c(M)$ constant $\Rightarrow \exists$ iso $(M,\omega_0,\theta_0)\to(M,\omega_1,\theta_1)$ (after "completing at ∞")
$\Rightarrow SH^*$ same

Theorem 2 (B.-R.)
If $[\omega_s,\theta_s]$ not constant, for small $s\geq 0$:

$$\boxed{\begin{array}{ccc} SH^*(M,\omega_s,\theta_s) & \cong & SH^*(M,\omega_0,\theta_0,\underline{\Lambda}_{\omega_s-\omega_0}) \\ c^*\uparrow & & \uparrow c^* \\ QH^*(M,\omega_s) & \cong & QH^*(M,\omega_0,\underline{\Lambda}_{\omega_s-\omega_0}) \end{array}}$$

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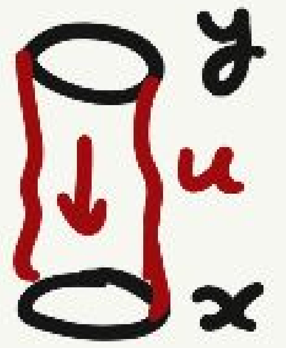
same moduli spaces as for ω_0 but use local system (infinitesimal deform^n)

ring isos

For large $s \geq 0$, Theorem 2 may not hold: even the infinitesimal deformation may not exist.



$$\partial y = \pm\, t^{\underbrace{\int u^*(\omega_s-\omega_0)}_{\text{can be } <0}}\, t^{E(u)}\, x + \dots$$

local system

energy ≥ 0

for convergence need total weights $\to \infty$

Local system $\underline{\Delta}_\omega$ is really on $\mathcal{L}M$

$$H^2(M,\mathbb{R}) \xrightarrow[ev^*]{} H^2(\mathcal{L}M \times S^1,\mathbb{R}) \xrightarrow[\text{project to Künneth summand}]{} H^1(\mathcal{L}M,\mathbb{R})$$

$$[\omega] \xmapsto[\text{transgression}]{} [\tau\omega]$$

If $\tau\omega_s \in \mathbb{R}_{\geq 0}\cdot(\tau\omega_0 + \zeta_0) \in H^1(\mathcal{L}M,\mathbb{R})$

then during the deformation can keep the local system constant $= \underline{\Delta}_{\tau\omega_0+\zeta_0}$

Theorem 3 (B.–R.)

$$SH^*(M,\omega_s,\theta_s,\underline{\Delta}_{\zeta_s}) \cong SH^*(M,\omega_0,\theta_0,\underline{\Delta}_{\zeta_0})$$

$\zeta_s = \tau(\omega_0-\omega_s)+\zeta_0$

Example If M Liouville,

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energy $\geqslant 0$

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<u>Example</u> If M Liouville,

$$SH^*(M,\omega_1,\theta_1) \cong SH^*(M,d\theta_0,\theta_0,\underline{\Delta}_{\tau\omega_1})$$

<u>Application 1</u>

$\pi: T^*S^2 \to S^2$

closed 2-form on S^2

$M = T^*S^2 \qquad \omega_\sigma = d\theta + \pi^*\sigma$

$$SH^*(T^*S^2,\omega_\sigma) \underset{\text{Thm.3}}{\cong} SH^*(T^*S^2,d\theta,\underline{\Delta}_{\tau\pi^*\sigma})$$

$$\underset{\text{R.2009}}{\cong} H_{2-*}(\mathcal{L}S^2;\underline{\Delta}_{\tau\sigma})$$

(based on Viterbo 1999)

$$\underset{\text{R.2009}}{=} \begin{cases} H_{2-*}(\mathcal{L}S^2;\Lambda) & \text{if } [\sigma]=0 \\ 0 & \text{if } [\sigma]\neq 0 \in H^2(S^2;\mathbb{R}) \end{cases}$$

For $\dim N \geqslant 3$, T^*N not convex at ∞, but Groman-Merry 2018 constructed $SH^*(T^*N,\omega_\sigma)$ and showed $\cong SH^*(T^*N,d\theta,\underline{\Delta}_{\tau\pi^*\sigma})$, using the key energy estimate that we used in Theorem 3.

Benedetti's PhD 2014

<u>Rmk</u> (Viterbo 1999): if $c^*: QH^* \to SH^*$ is not an iso, then $\exists$ non-constant 1-periodic Ham. orbit

$\Rightarrow$ above: existence theorems for closed magnetic geodesics

Application 1

$\pi : T^*S^2 \to S^2$

closed 2-form on S^2

$M = T^*S^2 \qquad \omega_\sigma = d\theta + \pi^*\sigma$

$$SH^*(T^*S^2, \omega_\sigma) \underset{\text{Thm. 3}}{\cong} SH^*(T^*S^2, d\theta, \Delta_{\tau\pi^*\sigma})$$

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Application 2

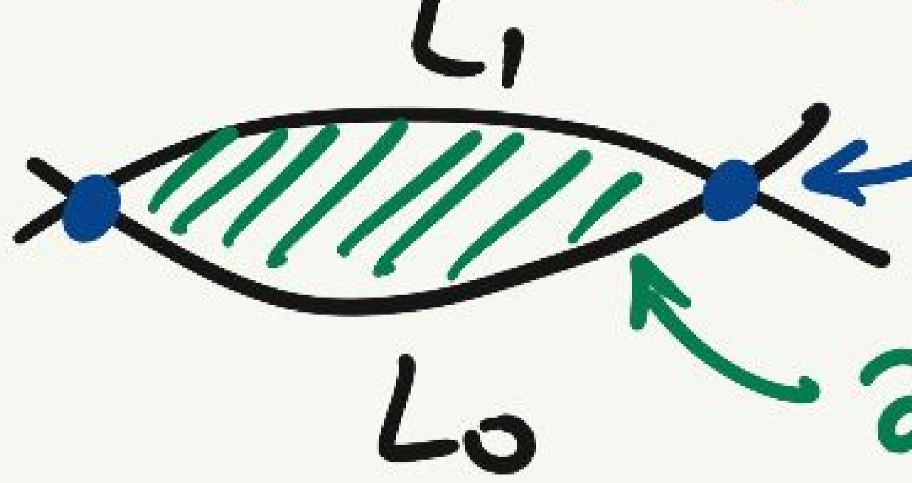
Fukaya category $\mathcal{F}(M)$

objects: Lagrangian submanifolds $L \subseteq M$

locally $\mathbb{R}^n \subseteq \mathbb{C}^n$

$hom(L_0, L_1) = CF^*(L_0, L_1)$

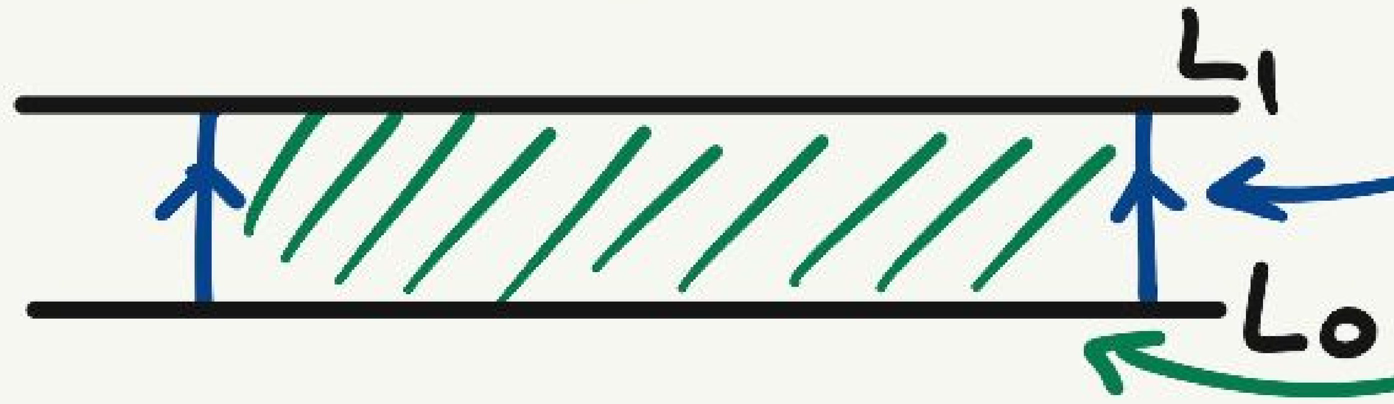
Lagrangian Floer chain complex



intersection points generate CF^* (if $L_0 \pitchfork L_1$ transverse)

∂ counts holomorphic strips

Actually need a Hamiltonian perturbation



time 1 orbits of X_H

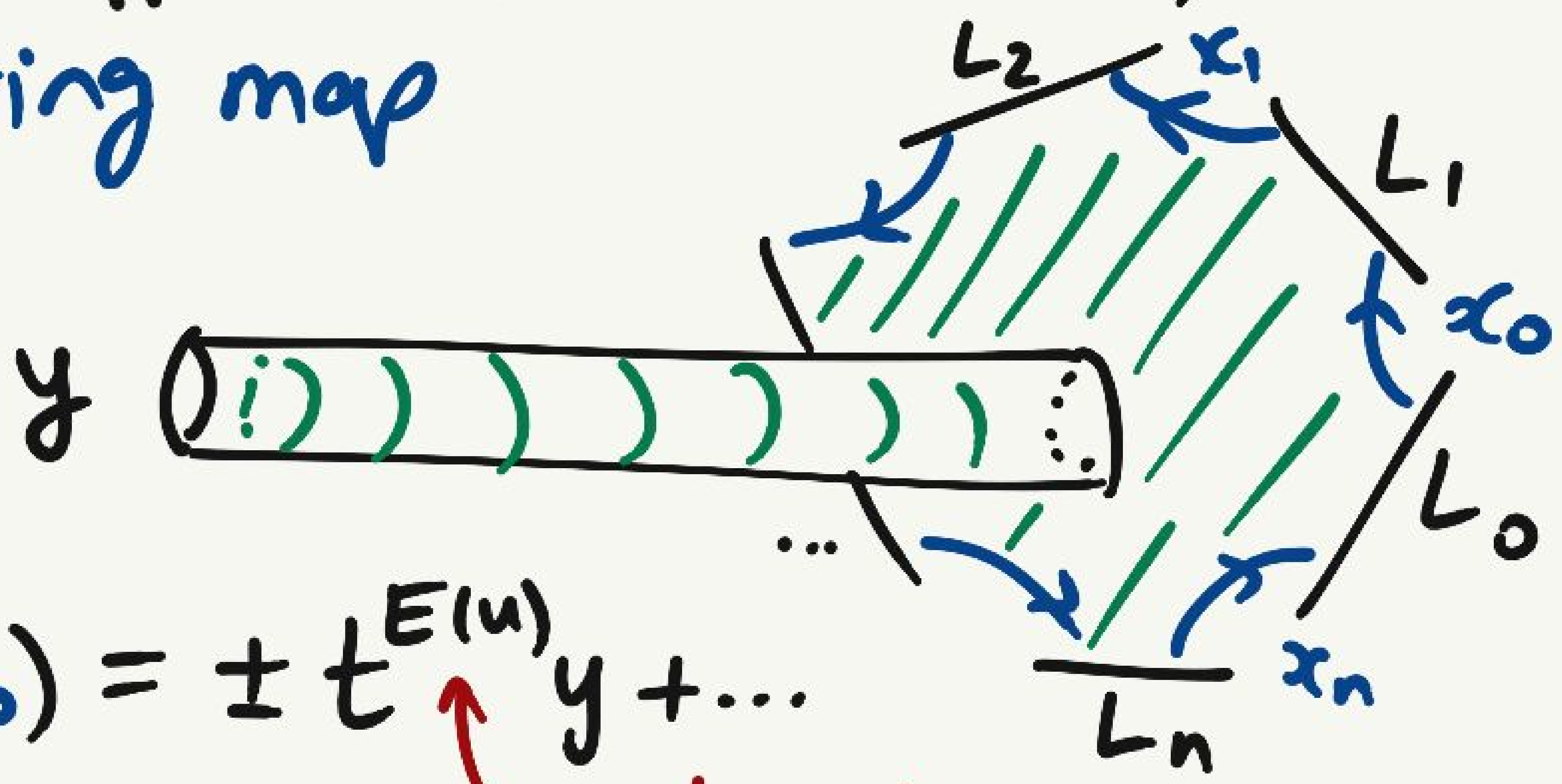
$\partial_s u + J \partial_t u = -\nabla H$

$\Rightarrow$ just the Lagrangian boundary version of $HF^*(M, H)$

If M not closed, define wrapped Fukaya cat. by taking $\varinjlim$ like we did for SH^*.

$OC^*: HH_*(\mathcal{F}_{wrapped}(M)) \longrightarrow SH^*(M)$

Open-closed string map



$OC(x_n \otimes \dots \otimes x_0) = \pm t^{E(u)} y + \dots$

topological energy

Abouzaid 2010 "generation criterion" (for M exact)

If $OC(\alpha) = 1 \in SH^*$ and α involves only $L_0, \dots, L_n$ then $L_0, \dots, L_n$ split-generate category.

R.–Smith 2012 (for M Fano)

$QH^* = \bigoplus QH^*_\lambda \leftarrow$ generalized eigensummands of $c_1(M) * \cdot \circlearrowright QH^*$

carries over to OC which is a module hom over SH^*

$OC = \bigoplus (OC_\lambda : HH_*(\mathcal{F}_{wrapped,\lambda}) \longrightarrow SH^*_\lambda)$

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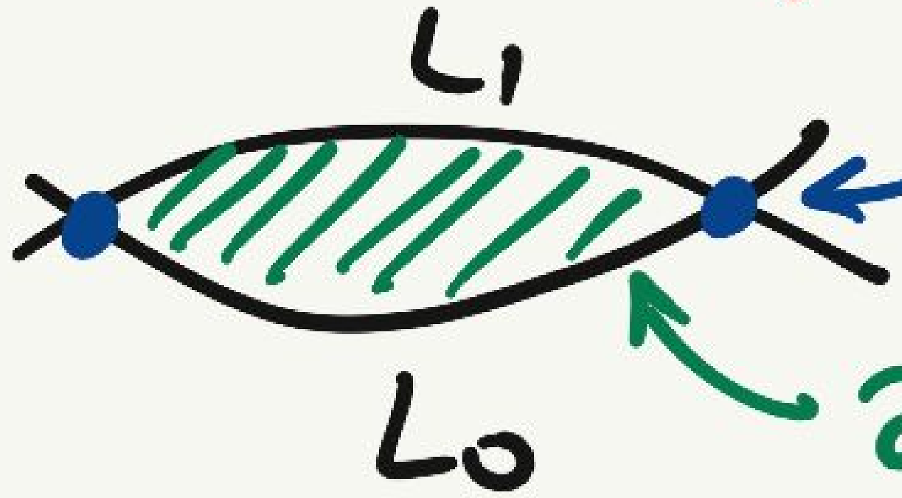

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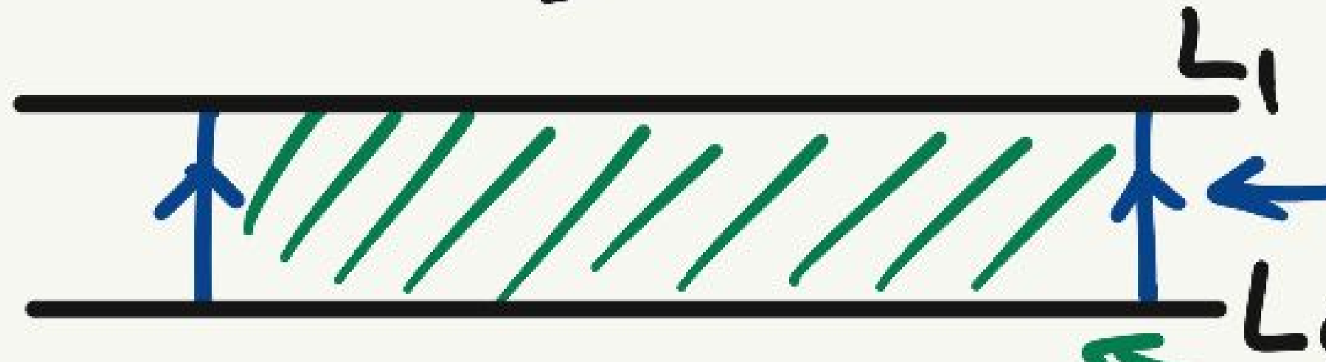

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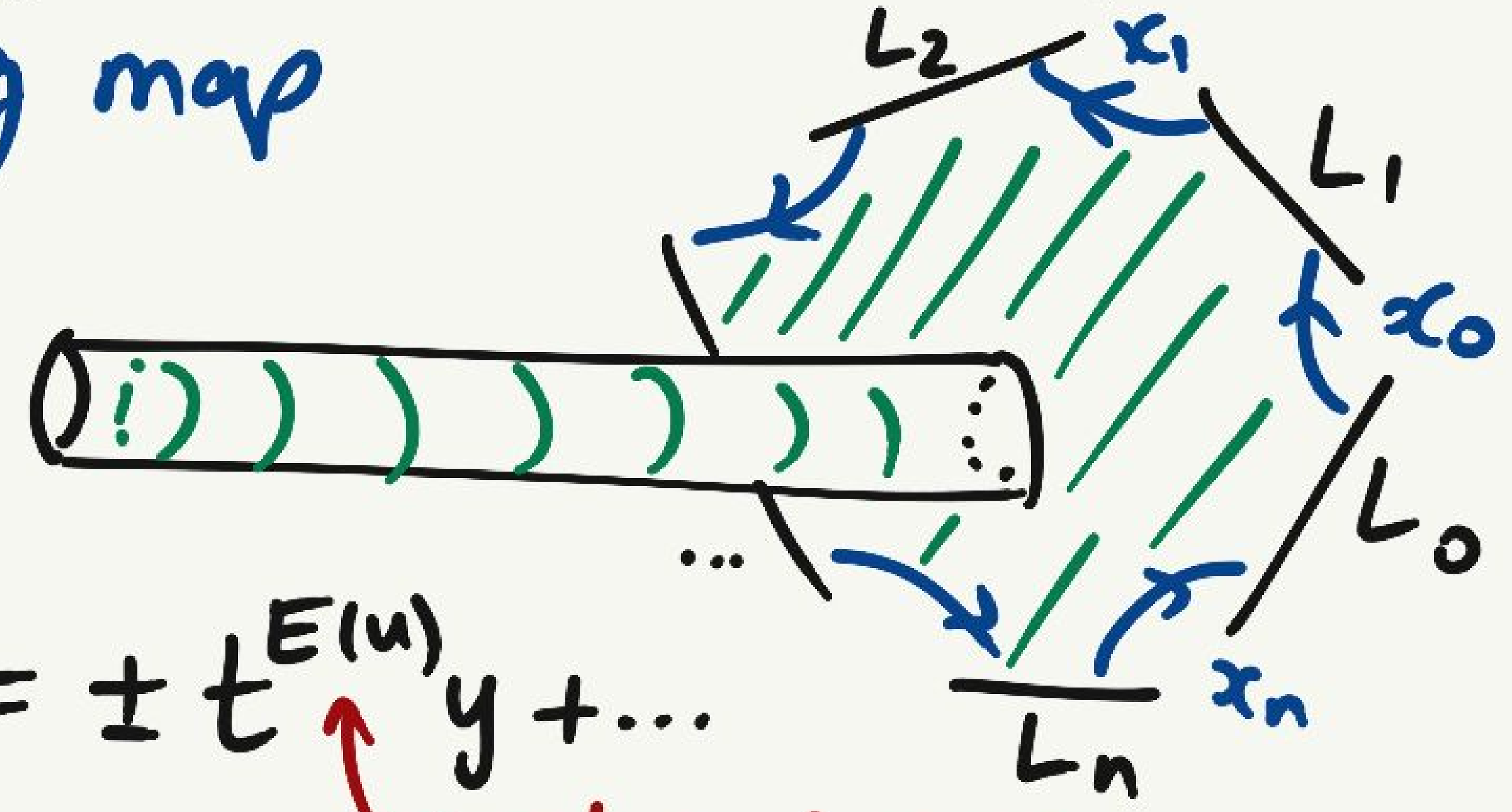

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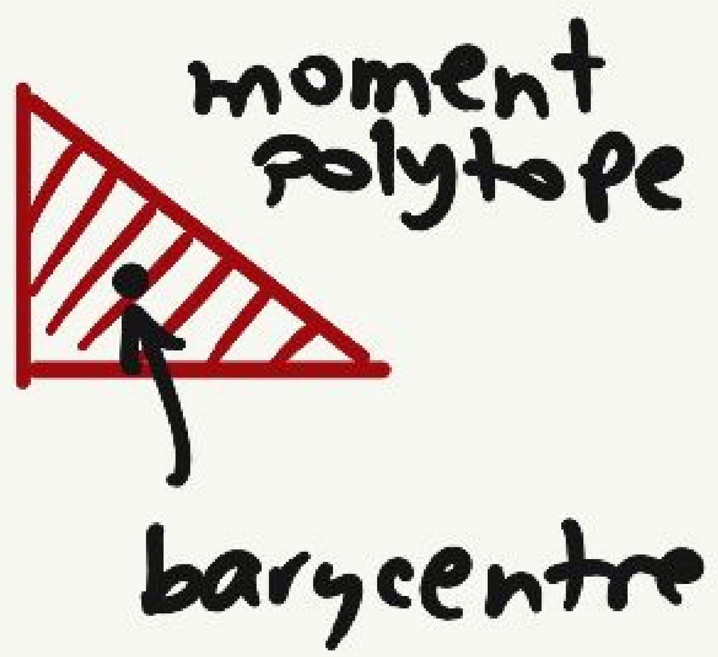
Example $\mathbb{CP}^2$

moment map $\mathbb{CP}^2 \xrightarrow{\pi}$



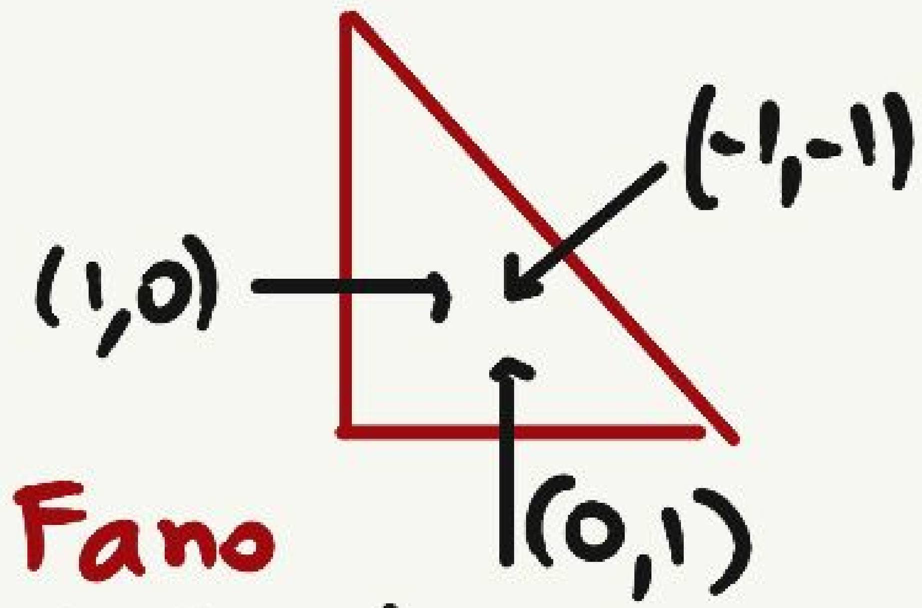
$L = \pi^{-1}(\text{barycentre}) \cong$ torus $S^1 \times S^1$

$c_1 = 3\omega$

$QH^* = \Lambda[x]/(x^3 - t) \cong \Lambda \oplus \Lambda \oplus \Lambda$

$\omega \leftrightarrow x$ eigenspaces for x

Superpotential $W = z_1 + z_2 + t z_1^{-1} z_2^{-1}$



Batyrev 1993 / Givental 1998 : $QH^* \cong Jac(W)$ for M toric Fano

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Above get L with 3 holonomies

$\lambda = W =$ critical value are 3 evals of c_1

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$OC^0 : HF(L_\lambda, L_\lambda) \longrightarrow QH^*_\lambda = \Lambda$ ← field so any non-zero elt is unit

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For non-compact Fano toric manifold (+ assumptions to ensure SH^* exists):

$$\boxed{\begin{aligned} SH^*(M,\omega) &\cong Jac(W) \qquad \text{(not } QH^*!) \\ &\cong QH^*(M,\omega)_{PD[D_i]} \end{aligned}}$$

localize ring at toric divisors

Ostrover-Tyomkin 2009:
If superpotential W is Morse then QH^* is semi-simple i.e. $\oplus$ fields.

$\Rightarrow$ the generation trick for $\mathbb{CP}^2$ essentially works.

Deformation argument: (M,ω) toric Fano

$\Rightarrow \exists$ family of toric Kähler forms ω_F near ω

Moment polytope $\Delta_F = \{y \in \mathbb{R}^n : \langle y, e_i\rangle \geq F(e_i)\}$

$[\omega_F] = \sum -F(e_i)\, PD[D_i]$ — e_i = inward normals to polytope

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monotone ω: $F(e_i)=1$, $[\omega_F]=c_1$

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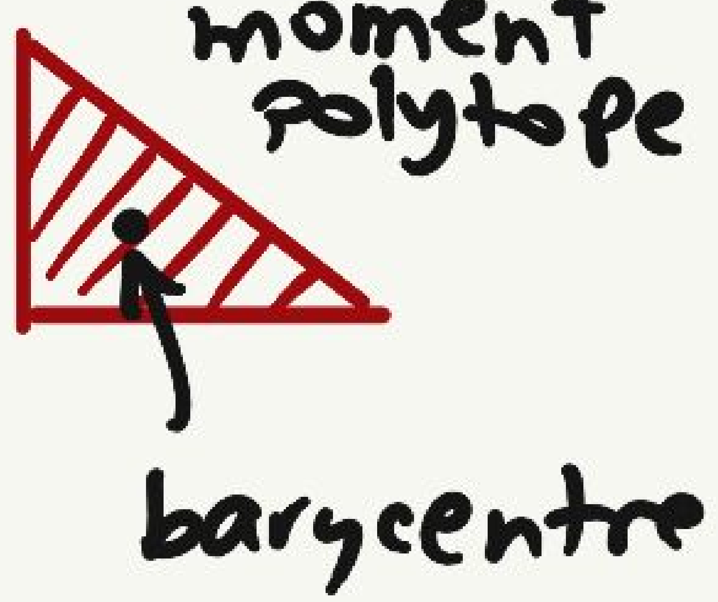

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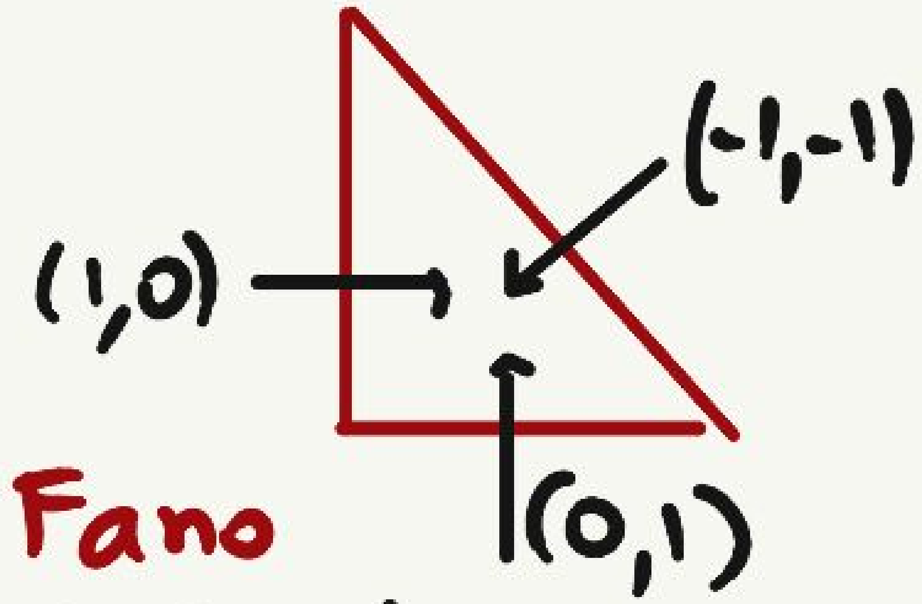

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More precisely: if W Morse but $\exists$ several critical points of W with same eigenvalue λ, then argument of $\mathbb{C}P^2$ has an issue:

$$QH^*_\lambda = \Lambda \oplus \dots \oplus \Lambda$$

e.g. $(1,0,\dots,0)$ not a unit

Deforming ω we can separate these eigenvalues, then do a limiting argument to show that the limit holonomies make the L_λ hit all Λ summands, so (several) L_λ's generate $\mathcal{F}_\lambda$.

OC_λ hits $(1,1,\dots,1)$

So deforming ω can be useful in symplectic topology even if you seek results about the undeformed ω.

Thank you for listening!